# Mathematical modeling of irreversible processes

Pavel Krejčí

## Contents



## Introduction

This is an accompanying text to a series of lectures given at the Faculty of Mathematics of the Technical University in Munich in the summer semester 2010. It is intended to illustrate some particular mathematical aspects of irreversible behavior of materials. Section 1 is devoted to an elementary model of fatigue accumulation and to general rate independent mappings. Analytical properties of simple models of plasticity are investigated in Section 2. The Preisach operators and its variational interpretation form the subject of Section 3. In Section 4 we construct a local solution to a formally reversible problem of wave propagation in nonlinear media, and show that in finite time, reversibility is violated when a singularity due to the intersection of characteristics occur. Thermodynamic admissibility of weak solutions to the wave propagation problem is discussed in Section 5 and 6 on the example of the Riemann problem. In Section 7 we show that shocks do not occur if the constitutive relation has the property of counterclockwise convexity. The system is still hyperbolic in the sense that the speed of propagation is bounded by the speed of the associated linearized system, but, using the language of the Riemann problem, rarefaction waves propagate along the convex/concave branches of the constitutive relation in all directions. Mathematically, this is translated into a second order energy inequality. In Section 8, we show another example, where the counterclockwise convexity of the Preisach operator is used for solving a degenerate parabolic problem motivated by the theory of porous media flow. To conclude, we briefly discuss in Section 9 some questions related to temperature dependent hysteresis and thermodynamic consistency of the corresponding balance laws.

# 1 Simple models for irreversible behavior and rate independent mappings

## 1.1 An elementary model of fatigue

One-dimensional mechanics offers a large variety of examples of material behavior that are in agreement with the common sense and with the everyday experience without any special knowledge in the domain of materials sciences. We start with such examples in order to illustrate the underlying ideas in irreversibility modeling.

Imagine first a solid rod with constant cross section $S$ and referential length $L$, subject to longitudinal loading and unloading. Let $\ell(t)$ be a given time-dependent load, the time interval being e.g. $t \in [0,T]$, and let $u(t)$ be the elongation response to loading. We introduce the quantities $\sigma(t) = \ell(t)/S$ (the *stress*), and $\varepsilon(t) = u(t)/L$ (the *strain*).,

We say that the material is elastic, if there exists a constant $E > 0$ called the elasticity modulus such that

$$\sigma(t) = E\varepsilon(t) \tag{1.1}$$

for all $t \in [0,T]$. This behavior is indeed reversible: the rod returns to the original position whenever $\sigma$ reaches the initial state.

The most common example of irreversibility is brittleness. We assume that the rod remains rigid as long as the stress stays below the so-called fracture limit $h > 0$; at the moment that $|\sigma(t)| = h$, the material breaks, the stress drops to 0 and we lose control on $\varepsilon$. Using the notation

$$|\sigma|_A = \sup\{|\sigma(t)| : t \in A\}$$

for an arbitrary set $A \subset [0,T]$, we describe brittle behavior as follows. Let the damage function $d(t)$ be defined by the formula

$$d(t) = 1 - H\left(h - |\sigma|_{[0,t]}\right), \tag{1.2}$$

where $H$ is the Heaviside function $H(x) = 0$ for $x \le 0$, $H(x) = 1$ for $x > 0$ The constitutive equations can be written in the form

$$\left.\begin{aligned} (1 - d(t))\varepsilon(t) &= 0\,, \\ d(t)\sigma(t) &= 0\,. \end{aligned}\right\} \tag{1.3}$$

Imagine now a parallel composition of two rods: one elastic and one brittle. The strain $\varepsilon(t)$ is the same on both rods, the stress is decomposed into the sum $\sigma(t) = \sigma^e(t) + \sigma^d(t)$ of one elastic component $\sigma^e = E\varepsilon$ carried by the elastic rod, and $\sigma^d$ carried by the brittle rod. We now have

$$\left.\begin{aligned} (1-d(t))\varepsilon(t) &= 0\,, \\ d(t)(\sigma(t) - E\varepsilon(t)) &= 0\,, \end{aligned}\right\} \tag{1.4}$$

where

$$d(t) = 1 - H\left(h - \left|\sigma^d\right|_{[0,t]}\right), \tag{1.5}$$

hence

$$\left.\begin{aligned} \sigma^d(t) &= (1-d(t))\sigma(t)\,, \\ E\varepsilon(t) &= d(t)\sigma(t)\,. \end{aligned}\right\} \tag{1.6}$$

**Lemma 1.1.** *Let $\sigma \in C[0,T]$, $|\sigma(0)| < h$. Then in (1.4)–(1.5), we have for all $t \in [0,T]$ that*

$$H\left(h - \left|\sigma^d\right|_{[0,t]}\right) = H\left(h - |\sigma|_{[0,t]}\right). \tag{1.7}$$

The assumption $|\sigma(0)| < h$ is necessary. Assume that for instance $\sigma(0) \geq h$. If now $d(0) = 0$, then, by (1.6), $\sigma^d(0) = \sigma(0) \geq h$, hence, by (1.5), $d(0) = 1$, which is a contradiction. Similarly, if $d(0) = 1$, then, (1.6) yields $\sigma^d(0) = 0$, hence $d(0) = 1$, which is again a contradiction. Below, we propose a way to overcome this paradox.

*Proof of Lemma 1.1.* Set $t^* = \sup\{t \in [0,T] : |\sigma(t)| < h\}$. For $t \in [0, t^*)$ we have $|\sigma^d(t)| \leq |\sigma(t)| < h$, hence $d(t) = 0$ and $\sigma^d(t) = \sigma(t)$. If $t^* < T$, we thus have $\left|\sigma^d\right|_{[0,t^*]} = |\sigma|_{[0,t^*]} = h$, and $d(t) = 1$ for $t \in [t^*, T]$. ■

By virtue of Lemma 1.1, the constitutive equation can be written as

$$\varepsilon(t) = \frac{1}{E}\left(1 - H\left(h - |\sigma|_{[0,t]}\right)\right)\sigma(t). \tag{1.8}$$

Consider now an arrangement in series of such elastobrittle constructions, parameterized by the fracture limits $h \geq 0$, and assume $\sigma(0) = 0$. For each $h > 0$ we have the constitutive equation

$$\varepsilon_h(t) = \frac{1}{E(h)}\left(1 - H\left(h - |\sigma|_{[0,t]}\right)\right)\sigma(t), \tag{1.9}$$

the case $h = 0$ corresponds to perfect elasticity, $\varepsilon_0(t) = \frac{1}{E(0)}\sigma(t)$. The elongation is now $u_h = L(h)\varepsilon_h$, where $L(h)$ is the reference length of the $h$-th element. We define the total elongation $u$ by the integral

$$u(t) = u_0(t) + \int_0^\infty u_h(t)\,\mathrm{d}h = \sigma(t)\left(\frac{L(0)}{E(0)} + \int_0^{|\sigma|_{[0,t]}} \frac{L(h)}{E(h)}\,\mathrm{d}h\right). \tag{1.10}$$

The total reference length is $L^* = L(0) + \int_0^\infty L(h)\,\mathrm{d}h$. Putting

$$g(x) = \frac{1}{L^*}\left(\frac{L(0)}{E(0)} + \int_0^x \frac{L(h)}{E(h)}\,\mathrm{d}h\right),$$

we can define constitutive operator $G$ by the formula

$$\varepsilon(t) = G[\sigma](t) := g(|\sigma|_{[0,t]})\sigma(t)\,. \tag{1.11}$$

The function $g$ is nondecreasing and positive, $g(0) > 0$, $g'(0) > 0$. Its reciprocal $1/g(|\sigma|_{[0,t]})$ represents the instantaneous elasticity modulus at time $t$, which thus decreases in time in agreement with observations. The explanation that the model (1.9)–(1.10) offers is that the individual rigid connections with larger and larger fracture limits successively break and the material thus becomes softer as a result of material fatigue.

The assumption $\sigma(0) = 0$ that we have made in the derivation of the model is quite restrictive. To remove this condition, we can choose any $\sigma \in C[0,T]$, and imagine a sequence $\{\sigma_n\}$ for $n \in \mathbb{N}$ of functions $\sigma_n \in C[0, T + (1/n)]$ defined as

$$\sigma_n(t) = \begin{cases} n\sigma(0)t & \text{for } t \in [0, 1/n)\,, \\ \sigma(t - (1/n)) & \text{for } t \in [1/n, T + (1/n)]\,. \end{cases}$$

Let

$$\varepsilon(t) = g(|\sigma|_{[0,t]})\sigma(t) \ \text{ for } t \in [0,T]\,, \quad \varepsilon_n(t) = g(|\sigma_n|_{[0,t]})\sigma_n(t) \ \text{ for } t \in [0, T + (1/n)]\,.$$

Then $\varepsilon_n(t) = \varepsilon(t - (1/n))$ for $t \in [1/n, T + (1/n)]$ and for all $n \in \mathbb{N}$. Hence, the constitutive equation (1.11) for inputs with an arbitrary initial condition can be interpreted as a result of an infinitely fast process from zero to the actual initial input value $\sigma(0)$.

We see that the mapping $G : C[0,T] \to C[0,T] : \sigma \mapsto \varepsilon$ given by (1.11) is continuous. We conclude this subsection with the following statement about the Lipschitz continuity of its inverse.

**Proposition 1.2.** *Let $g : [0,\infty) \to (0,\infty)$ be a nondecreasing function. Then the mapping $G$ given by* (1.11) *is invertible, and the inverse $G^{-1} : C[0,T] \to C[0,T]$ to the mapping $G$ satisfies for all $\varepsilon_1, \varepsilon_2 \in C[0,T]$ and all $t \in [0,T]$, the inequality*

$$\left|G^{-1}[\varepsilon_1] - G^{-1}[\varepsilon_2]\right|_{[0,t]} \le \frac{3}{g(0)}\, |\varepsilon_1 - \varepsilon_2|_{[0,t]}\,.$$

*Proof.* Let $\varepsilon, \sigma \in C[0,T]$ be related by Eq. (1.11). Then for all $t \in [0,T]$ we have

$$|\varepsilon|_{[0,t]} = g(|\sigma|_{[0,t]})\, |\sigma|_{[0,t]}\,. \tag{1.12}$$

Furthermore, $\frac{\mathrm{d}}{\mathrm{d}x}(xg(x)) = g(x) + xg'(x) \ge g(0) > 0$ for all $x > 0$, hence the inverse $\varphi$ to the function $x \mapsto xg(x)$ exists and is Lipschitz continuous in $[0,\infty)$. This enables us to rewrite (1.12) as

$$|\sigma|_{[0,t]} = \varphi(|\varepsilon|_{[0,t]})\,, \quad g(|\sigma|_{[0,t]}) = \frac{|\varepsilon|_{[0,t]}}{\varphi(|\varepsilon|_{[0,t]})} \tag{1.13}$$

for all $t \in [0,T]$. Hence, (1.11) can be written in inverse form

$$\sigma(t) = \frac{\varphi(|\varepsilon|_{[0,t]})}{|\varepsilon|_{[0,t]}}\varepsilon(t)\,. \tag{1.14}$$

To check that the $\sigma \mapsto \varepsilon$ mapping thus defined is Lipschitz, consider two inputs $\varepsilon_1, \varepsilon_2$ and the corresponding outputs $\sigma_1, \sigma_2$. For an arbitrary $t \in [0,T]$ we have by triangle inequality

$$\begin{aligned}
|\sigma_1(t) - \sigma_2(t)| &\le \frac{\varphi(|\varepsilon_1|_{[0,t]})}{|\varepsilon_1|_{[0,t]}} |\varepsilon_1(t) - \varepsilon_2(t)| + \left( \frac{\varphi(|\varepsilon_1|_{[0,t]})}{|\varepsilon_1|_{[0,t]}} - \frac{\varphi(|\varepsilon_2|_{[0,t]})}{|\varepsilon_2|_{[0,t]}} \right) |\varepsilon_2|_{[0,t]} \\
&\le \frac{\varphi(|\varepsilon_1|_{[0,t]})}{|\varepsilon_1|_{[0,t]}} \left( |\varepsilon_1(t) - \varepsilon_2(t)| + \left| \, |\varepsilon_1|_{[0,t]} - |\varepsilon_2|_{[0,t]} \right| \right) + |\varphi(|\varepsilon_1|_{[0,t]}) - \varphi(|\varepsilon_2|_{[0,t]})| \\
&\le \frac{3}{g(0)} |\varepsilon_1 - \varepsilon_2|_{[0,t]} ,
\end{aligned}$$

which is the desired inequality. ■

Let us derive the energy balance for processes involving the fatigue constitutive relation $\varepsilon = G[\sigma]$ described by (1.11). The work supplied to the system in a time interval $[t_0, t_1]$ is given by the integral

$$\int_{t_0}^{t_1} \dot{\varepsilon}(t)\sigma(t)\,\mathrm{d}t ,$$

where dot denotes the derivative with respect to time. Part of the work is used for an increase of potential energy $e(t_1) - e(t_0)$, another part is possibly dissipated. For isothermal processes, the dissipation must be nonnegative in agreement with the second principle of thermodynamics. In other words, we look for a potential energy mapping $e(t) = W[\sigma](t)$ such that

$$\sigma(t)\frac{\mathrm{d}}{\mathrm{d}t}G[\sigma](t) - \frac{\mathrm{d}}{\mathrm{d}t}W[\sigma](t) \ge 0 \tag{1.15}$$

in an appropriate sense, say, almost everywhere. The most natural choice is to define $W$ as the integrated potential energies of all elastic components, that is,

$$W[\sigma](t) = \frac{g(|\sigma|_{[0,t]})}{2}\sigma^2(t).$$

The energy balance (1.15) then reads

$$\sigma(t)\frac{\mathrm{d}}{\mathrm{d}t}G[\sigma](t) - \frac{\mathrm{d}}{\mathrm{d}t}W[\sigma](t) = \frac{g'(|\sigma|_{[0,t]})}{2}\sigma^2(t)\frac{\mathrm{d}}{\mathrm{d}t}|\sigma|_{[0,t]} , \tag{1.16}$$

which is obviously nonnegative. We also see that dissipation occurs only when fracture takes place, that is, if $|\sigma(t)| = |\sigma|_{[0,t]}$.

## 1.2 Rate independent mappings

We start with a definition, which slightly differs from the ones in [1] or [10], but which is more suitable for our purposes.

**Definition 1.3.** *A mapping $F : C[0,T] \to C[0,T]$ is said to be* rate independent*, if for every $u \in C[0,T]$ and every nondecreasing mapping $\alpha$ of $[0,T]$ onto $[0,T]$ we have*

$$F[u \circ \alpha](t) = F[u](\alpha(t) \quad \forall t \in [0,T] .$$

Superposition (Nemytskii) operators $w(t) = f(u(t))$ generated by a continuous function $f$ are typical examples of rate independent mappings. As slightly less trivial examples of rate independent mappings, we may mention the operators $G$, $G^{-1}$, $W$ introduced in Subsection 1.1.

One characteristic feature of rate independent mappings is that they can be locally represented by Nemytskii operators. More precisely, we have the following representation theorem. By $\mathrm{conv}\,\{a,b\}$ we mean the closed interval $[\min\{a,b\}, \max\{a,b\}]$.

**Theorem 1.4.** *Let $F : C[0,T] \to C[0,T]$ be a rate independent mapping, and let $u \in C[0,T]$ be monotone (nonincreasing or nondecreasing) in an interval $[t_0, t_1]$. Then there exists a function $f : \mathrm{conv}\,\{u(t_0), u(t_1)\} \to \mathbb{R}$ such that for every $t \in [t_0, t_1]$ we have*

$$F[u](t) = f(u(t))\,.$$

We call the function $f$ a *trajectory of $F$ along $u$*. Different input functions $u$ may generate different trajectories, indeed. The trajectories of $F$ along time periodic inputs constitute the branches of the hysteresis loops.

*Proof of Theorem 1.4.* It is easy to see that if $u$ is constant in an interval $[t_0, t_1]$, then $F[u]$ is constant as well. We can therefore assume that $u(t_0) \neq u(t_1)$. Set

$$u^*(s) = \begin{cases} u(s) & \text{for } s \in [0, t_0) \cup (t_1, T]\,, \\ u(t_0) + \frac{s-t_0}{t_1-t_0}(u(t_1) - u(t_0)) & \text{for } s \in [t_0, t_1]\,, \end{cases}$$

and $w^*(s) = F[u^*](s)$. We now define the time transformation $\alpha$ by the formula

$$\alpha(t) = \begin{cases} t & \text{for } t \in [0, t_0) \cup (t_1, T]\,, \\ t_0 + \frac{u(t)-u(t_0)}{u(t_1)-u(t_0)}(t_1 - t_0) & \text{for } t \in [t_0, t_1]\,. \end{cases}$$

We now claim that the trajectory $f$ is given by the equation

$$f(v) = w^*\left(t_0 + \frac{v - u(t_0)}{u(t_1) - u(t_0)}(t_1 - t_0)\right).$$

Indeed, we have $u = u^* \circ \alpha$, and Definition 1.3 yields $F[u](t) = w^* \circ \alpha(t) = f(u(t))$, which we wanted to prove. ■

**Definition 1.5.** *We say that a rate independent mapping $F : C[0,T] \to C[0,T]$ is*

(i) locally monotone*, if every trajectory $f$ of $F$ along every monotone input $u \in C[0,T]$ is nondecreasing;*

(ii) strictly locally monotone*, if there exists a constant $\gamma > 0$ such that if the derivatives $\dot u(t)$ and $\dot w(t)$ exist for both functions $u$ and $w = F[u]$ at some point $t \in (0,T)$, then*

$$\dot u(t)\dot w(t) \geq \gamma \dot u^2(t)\,.$$

(iii) counterclockwise convex, *if every trajectory of $F$ along every nondecreasing input $u \in C[0,T]$ is convex, and every trajectory along every nonincreasing input is concave;*

(iv) locally counterclockwise convex *with convexity bound $b > 0$, if the convexity property from (ii) holds for every $u \in C[0,T]$ such that $|u|_{[0,T]} \le b$.*

We might define (local) clockwise convexity in a similar way. Note just that if $F$ is invertible and counterclockwise convex, then the inverse $F^{-1}$ is clockwise convex.

The constitutive operator $G : \sigma \mapsto \varepsilon$ defined by (1.11) has very simple trajectories. Let $\sigma$ be nondecreasing in $[t_0, t_1]$. Then either $\sigma(t_1) \le |\sigma|_{[0,t_0]}$, or $\sigma(t_1) > |\sigma|_{[0,t_0]}$. In the latter case there exists $t^* \in [t_0, t_1)$ such that $\sigma(t^*) = |\sigma|_{[0,t_0]}$, and

$$G[\sigma](t) = \begin{cases} g(|\sigma|_{[0,t_0]})\sigma(t) & \text{for } t \in [t_0, t^*], \\ g(\sigma(t))\sigma(t) & \text{for } t \in (t^*, t_1]. \end{cases}$$

The trajectory $f$ along nondecreasing inputs thus always has the form

$$f(v) = g(\max\{v_0, v\})v \tag{1.17}$$

for some $v_0 > 0$ and $v \in [v_1, v_2]$. If we extend formula (1.17) to $\mathbb{R}$, then $f'(v) = g(v_0)$ in $(-\infty, v_0)$, and $f'(v) = g(v) + vg'(v)$ for $v \in (v_0, +\infty)$. Hence, $f'(v_0-) < f'(v_0+)$, and $f''(v) = 2g'(v) + vg''(v)$ in $(v_0, +\infty)$. Assuming e.g. that $g$ is twice continuously differentiable and

$$2g'(v) + vg''(v) \ge 0 \quad \forall v > 0, \tag{1.18}$$

we obtain the property that all trajectories along all nondecreasing inputs are convex. Similarly, if $\sigma$ is nonincreasing in $[t_0, t_1]$, then either $\sigma(t_1) \ge -|\sigma|_{[0,t_0]}$ and $|\sigma|_{[0,t]} = |\sigma|_{[0,t_0]}$ for all $t \in [t_0, t_1]$, or $\sigma(t_1) < -|\sigma|_{[0,t_0]}$, and there exists $t^* \in [t_0, t_1)$ such that $\sigma(t^*) = -|\sigma|_{[0,t_0]}$, and

$$G[\sigma](t) = \begin{cases} g(|\sigma|_{[0,t_0]})\sigma(t) & \text{for } t \in [t_0, t^*], \\ g(-\sigma(t))\sigma(t) & \text{for } t \in (t^*, t_1]. \end{cases}$$

Along nonincreasing inputs, the equation of each trajectories reads

$$f(v) = g(\max\{v_0, -v\})v \tag{1.19}$$

for some $v_0 > 0$. We have $f'(v) = g(v_0)$ for $v > -v_0$, $f'(v) = g(-v) - vg'(-v)$ for $v < -v_0$. Here, we see that $f'(-v_0-) > f'(-v_0+)$ and $f''(v) = -2g'(-v) + vg''(v)$ for $v < -v_0$, hence $f$ is concave provided (1.18) holds. This is equivalent to say that the function $v^2 g'(v)$ is nondecreasing for $v > 0$. If $g$ is a $C^2$-function, then the property is always fulfilled in a neighborhood of $0$. We have thus proved

**Proposition 1.6.** *Let $g : [0,\infty) \to (0,\infty)$ be a nondecreasing $C^2$-function. Then the fatigue constitutive operator $G$ defined by* (1.11) *is strictly locally monotone and locally counterclockwise convex. If moreover condition* (1.18) *holds, then $G$ is counterclockwise convex.*

Eq. (1.18) is certainly satisfied if $g$ itself is convex. There exist also concave bounded functions $g$ with this property, e.g. $g(v) = v/(v+1)$.

The counterclockwise convexity of rate independent operators is a very important property. Theorem 1.7 below states that counterclockwise convex operators admit a higher order energy inequality (1.21). In the next sections, we show how this inequality can be used for deriving higher order a priori estimates for balance equations in hysteretic media, which are in turn necessary for constructing the solution.

**Theorem 1.7.** *Let $F : C[0,T] \to C[0,T]$ be a rate independent, strictly locally monotone, and locally counterclockwise convex operator with convexity bound $b$. Assume that there exists $K \geq 0$ such that every trajectory $f$ along every input $u \in C[0,T]$ such that $|u|_{[0,T]} \leq b$ has the property*

$$\left| \frac{f'(u_1) - f'(u_2)}{u_1 - u_2} \right| \geq K \quad \forall u_1, u_2 : \ -b \leq u_2 < u_1 \leq b\,. \tag{1.20}$$

*Assume that $u \in W^{1,\infty}(0,T)$ is such that $|u|_{[0,t]} \leq b$ and $w := F[u] \in W^{2,1}(0,T)$. Then the inequality*

$$\dot u(t)\ddot w(t) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot w(t)\dot u(t)\right) \geq \frac{K}{2}|\dot u(t)|^3 \tag{1.21}$$

*holds in the sense of distributions on $[0,T]$.*

We call (1.21) the *second order energy inequality* in order to emphasize the formal analogy with (1.15) or (1.16). Note that the "potential" $\frac{1}{2}\dot w(t)\dot u(t)$ as well as the "dissipation" $\frac{K}{2}|\dot u(t)|^3$ are nonnegative.

The fact that (1.21) holds in the sense of distributions can be equivalently stated in the form:

$$\text{The function } \ t^* \mapsto \frac{1}{2}\dot w(t^*)\dot u(t^*) + \int_0^{t^*}\left(\frac{K}{2}|\dot u(t)|^3 - \dot u(t)\ddot w(t)\right)\mathrm{d}t \ \text{ is nonincreasing in } (0,T)\,. \tag{1.22}$$

We see in particular that under the hypotheses of Theorem 1.7, the "potential" $\frac{1}{2}\dot w \dot u$ is a function of bounded variation, hence it admits the right limit at each point $t \in [0,T)$, and the left limit at each point $t \in (0,T]$.

In order to illustrate the motivation for (1.21) to hold, assume first that a trajectory $f$ along $u$ is twice continuously differentiable in an interval. Then the chain rule yields

$$\dot u \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(u) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}f(u)\dot u\right) = \frac{1}{2}\dot u^3 f''(u) \geq \frac{K}{2}|\dot u|^3, \tag{1.23}$$

where we have used the fact that $\dot u$ and $f''(u)$ always have the same sign by hypothesis of counterclockwise convexity.

We cannot expect that the trajectories $f$ be twice continuously differentiable. This is obvious even in the simple case (1.17). A rigorous proof of Theorem 1.7 thus has to be carried out in a different way. We start with an auxiliary result.

**Lemma 1.8.** *Let $u, v \in W^{1,1}(0,T)$ be given, $v(t) \geq 0$ a.e. Let $u$ be monotone (nonincreasing or nondecreasing) in an interval $[s,\tau]$, let $f$ be the trajectory along $u$ of a strictly* *locally monotone, locally counterclockwise convex operator with bound* $b$ *satisfying* (1.20), $b \ge |u|_{[0,\tau]}$. *Then we have*

$$\int_s^\tau \frac{\dot v(t)}{f'(u(t))}\,\mathrm{d}t \ge \left[\frac{v(t)}{f'(u(t))}\right]_s^\tau + K\int_s^\tau \frac{v(t)|\dot u(t)|}{(f'(u(t)))^2}\,\mathrm{d}t\,,$$

*where we set* $[w(t)]_s^\tau = w(\tau-) - w(s+)$ *for an arbitrary function with bounded variation on* $[s,\tau]$.

*Proof of Lemma 1.8.* We have $f'(z) \ge \gamma > 0$ a.e. by strict local monotonicity. Let us approximate $f'$ by $C^1$-functions $f_n$, $n \in \mathbb{N}$, such that $f_n(u(\tau)) = (f' \circ u)(\tau-)$, $f_n(u(s)) = (f' \circ u)(s+)$, $f_n(z) \ge \gamma$, $|f_n'(z)| \ge K$, and such that $f_n(z) \nearrow f'(z)$ for all $z$ in the open interval bounded by $u(s)$ and $u(\tau)$. We have for every $n \in \mathbb{N}$ and a.e. $t \in (s,\tau)$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v(t)}{f_n(u(t))}\right) = \frac{\dot v(t)}{f_n(u(t))} - \frac{v(t)\dot u(t) f_n'(u(t))}{(f_n(u(t)))^2}\,. \tag{1.24}$$

Condition (1.20) and counterclockwise convexity yield that $f_n'(u(t))\dot u(t) \ge K|\dot u(t)|$ a.e., hence

$$\left[\frac{v(t)}{f_n(u(t))}\right]_s^\tau \le \int_s^\tau \frac{\dot v(t)}{f_n(u(t))}\,\mathrm{d}t - K\int_s^\tau \frac{v(t)|\dot u(t)|}{(f_n(u(t)))^2}\,\mathrm{d}t\,. \tag{1.25}$$

Passing to the limit as $n \to \infty$ and using the Lebesgue monotone convergence theorem, we obtain the assertion. ■

*Proof of Theorem 1.7.* Let $[s,\tau] \subset [0,T]$ be any monotonicity interval of $u$, and let $f : \operatorname{conv}\{u(s),u(\tau)\} \to \mathbb{R}$ be the corresponding trajectory. If $u(s) = u(\tau)$, then both $u$ and $w$ are constant in $[s,\tau]$ and the statement is trivial. So, assume that $u(s) \ne u(\tau)$. Set $w(t) = F[u](t) = f(u(t))$ for $t \in [s,\tau]$. Since $f$ is convex or concave, the chain rule

$$\dot w(t) = f'(u(t))\dot u(t) \tag{1.26}$$

holds almost everywhere. We now use Lemma 1.8 with $v(t) := \frac{1}{2}\dot w^2(t)$ and obtain that

$$\left[\frac{1}{2}\dot w(t)\dot u(t)\right]_s^\tau \le \int_s^\tau \dot u(t)\ddot w(t)\,\mathrm{d}t - \frac{K}{2}\int_s^\tau |\dot u(t)|^3\,\mathrm{d}t\,. \tag{1.27}$$

Let now $[t_1,t_2] \subset [0,T]$ be an arbitrary interval. The function $\dot w$ is absolutely continuous by hypothesis, there exist therefore at most countably many intervals $(s_j,\tau_j)$ indexed by elements $j$ of an index set $J \subset \mathbb{N}$, such that

$$N := \{t \in (t_1,t_2) : \dot w(t) \ne 0\} = \bigcup_{j\in J}(s_j,\tau_j)\,. \tag{1.28}$$

By Definition 1.5 (ii) of strict local monotonicity, each $[s_j,\tau_j]$ is a monotonicity interval of $u$, and for a.e. $t \in (t_1,t_2)\setminus N$, we have $\dot u(t) = 0$. Hence,

$$\sum_{j\in J}\left[\frac{1}{2}\dot w(t)\dot u(t)\right]_{s_j}^{\tau_j} \le \int_{t_1}^{t_2} \dot u(t)\ddot w(t)\,\mathrm{d}t - \frac{K}{2}\int_{t_1}^{t_2} |\dot u(t)|^3\,\mathrm{d}t\,. \tag{1.29}$$

Furthermore, since $\dot{w}$ is continuous, we have $\dot{w}(s_j) = \dot{w}(\tau_j) = 0$ for all $j \in J$ but perhaps $s_j = t_1$ or $\tau_j = t_2$. We conclude that

$$\sum_{j\in J}\left[\frac{1}{2}\dot{w}(t)\dot{u}(t)\right]_{t_1}^{t_2} \le \int_{t_1}^{t_2} \dot{u}(t)\ddot{w}(t)\,\mathrm{d}t - \frac{K}{2}\int_{t_1}^{t_2} |\dot{u}(t)|^3\,\mathrm{d}t\,, \tag{1.30}$$

which is precisely (1.22). ■

# 2 Prandtl's plasticity model

Let us imagine that a real elastoplastic material is composed of grains as on Figure 1, each of which exhibits both elastic response to external loading and friction. We denote by $\ell(t)$ a given load depending on time $t$, and by $u(t)$ the elongation at time $t$. If $L > 0$ is a referential length and $S > 0$ is a referential cross section, the *strain* $\varepsilon(t)$ and *stress* $\sigma(t)$ are defined as the ratios $\varepsilon(t) = (L + u(t))/L$, $\sigma(t) = \ell(t)/S$.

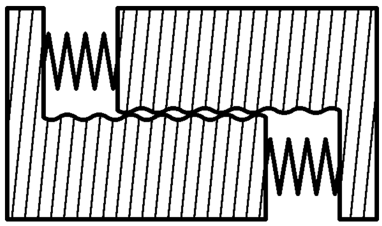

Figure 1: Prandtl's model

We assume that the stress admits the decomposition $\sigma(t) = \sigma^e(t) + \sigma^b(t)$ into an elastic stress component $\sigma^e(t)$ and a *backstress* $\sigma^b(t)$. The backstress is the stress carried by the friction element and cannot exceed in absolute value a threshold $r > 0$ called the *yield point*. As long as the value of the backstress stays away of the limits $-r$ and $r$, the system does not move. Motion occurs when $\sigma^b(t)$ reaches the threshold, and the motion takes place in the stress direction. Formally, the constitutive relations can be written as follows:

$$\sigma(t) = \sigma^e(t) + \sigma^b(t)\,, \tag{2.1}$$

$$\sigma^e(t) = E\varepsilon(t)\,, \tag{2.2}$$

$$|\sigma^b(t)| \le r\,, \tag{2.3}$$

$$|\sigma^b(t)| < r \ \Rightarrow\ \dot{\varepsilon}(t) = 0\,, \tag{2.4}$$

$$\sigma^b(t) = r \ \Rightarrow\ \dot{\varepsilon}(t) \ge 0\,, \tag{2.5}$$

$$\sigma^b(t) = -r \ \Rightarrow\ \dot{\varepsilon}(t) \le 0\,, \tag{2.6}$$

together with an initial condition $\sigma^b_0 \in [-r, r]$.

A straightforward computation shows that (2.3)–(2.6) can be expressed in closed form as the variational inequality

$$|\sigma(t) - \sigma^e(t)| \le r \qquad \forall t \in [0,T]\,, \tag{2.7}$$

$$\dot\sigma^e(t)(\sigma(t) - \sigma^e(t) - x) \ge 0 \qquad \forall |x| \le r \quad \text{a.e.}\,, \tag{2.8}$$

$$\sigma(0) - \sigma^e(0) = \sigma_0^b. \tag{2.9}$$

We state the existence and uniqueness result for solutions to system (2.7)–(2.9) in a more general setting, where the yield point $r$ may additionally depend also on $t$.

**Theorem 2.1.** *Let $\sigma, r \in W^{1,1}(0,T)$ be given, $r(t) > 0$ for all $t \in [0,T]$, and let $x^0 \in [-r(0), r(0)]$ be a given initial condition. Then there exists a unique $\xi \in W^{1,1}(0,T)$ such that*

$$|\sigma(t) - \xi(t)| \le r(t) \qquad \forall t \in [0,T]\,, \tag{2.10}$$

$$\dot\xi(t)(\sigma(t) - \xi(t) - r(t)x) \ge 0 \qquad \forall |x| \le 1 \quad \textit{a.e.}\,, \tag{2.11}$$

$$\sigma(0) - \xi(0) = x^0. \tag{2.12}$$

*Proof.* Assume first that $\sigma, r \in W^{1,\infty}(0,T)$. We construct the solution by time discretization. For a fixed integer $n \in \mathbb{N}$ we define the equidistant partition $t_k = \frac{k}{n}T$, and set $\sigma_k = \sigma(t_k)$, $r_k = r(t_k)$, $k = 0,1,\dots,n$. Let $P_\varrho : \mathbb{R} \to \mathbb{R}$ with a parameter $\varrho > 0$ be the piecewise affine function

$$P_\varrho(x) = \max\{x - \varrho, \min\{0, x + \varrho\}\}\,. \tag{2.13}$$

The relation $z = P_\varrho(x)$ is equivalent to the variational inequality

$$|x - z| \le \varrho\,, \quad z(x - z - y) \ge 0 \quad \forall |y| \le \varrho\,. \tag{2.14}$$

Note also the following elementary properties of $P_\varrho$ that hold for all $\varrho_1, \varrho_2 > 0$ and $x \in \mathbb{R}$:

$$P_{\varrho_1}(P_{\varrho_2}(x)) = P_{\varrho_1+\varrho_2}(x)\,, \tag{2.15}$$

$$|P_{\varrho_1}(x) - P_{\varrho_1+\varrho_2}(x)| \le \varrho_2\,, \tag{2.16}$$

$$P_{\varrho_1}(x) = 0 \ \Rightarrow\ |P_{\varrho_2}(x)| \le |\varrho_1 - \varrho_2|\,. \tag{2.17}$$

We now define a sequence $\xi_k$ by the recurrent formula

$$\xi_0 = \sigma_0 - x^0\,, \quad \xi_k - \xi_{k-1} = P_{r_k}(\sigma_k - \xi_{k-1})\,. \tag{2.18}$$

By virtue of (2.14), this is equivalent to

$$|\sigma_k - \xi_k| \le r_k\,, \quad (\xi_k - \xi_{k-1})(\sigma_k - \xi_k - y_k) \ge 0 \quad \forall |y_k| \le r_k \tag{2.19}$$

for all $k = 1,\dots,n$. We may choose for example $y_k = (I - P_{r_k})(\sigma_{k-1} - \xi_{k-1})$, where $I$ is the identity function, and obtain

$$(\xi_k - \xi_{k-1})((\sigma_k - \xi_k) - (\sigma_{k-1} - \xi_{k-1})) \ge -(\xi_k - \xi_{k-1})P_{r_k}(\sigma_{k-1} - \xi_{k-1}), \tag{2.20}$$

or in other terms,

$$|\xi_k - \xi_{k-1}|^2 \le (\xi_k - \xi_{k-1})(\sigma_k - \sigma_{k-1}) + (\xi_k - \xi_{k-1})P_{r_k}(\sigma_{k-1} - \xi_{k-1}). \tag{2.21}$$

This yields, by (2.17), that

$$|\xi_k - \xi_{k-1}| \le |\sigma_k - \sigma_{k-1}| + |r_k - r_{k-1}| \quad \forall k = 1, \dots, n\,. \tag{2.22}$$

We now define piecewise linear and piecewise affine interpolates

$$\sigma^{(n)}(t) = \sigma_{k-1} + \frac{n}{T}(t - t_{k-1})(\sigma_k - \sigma_{k-1})\,, \tag{2.23}$$

$$r^{(n)}(t) = r_{k-1} + \frac{n}{T}(t - t_{k-1})(r_k - r_{k-1})\,, \tag{2.24}$$

$$\xi^{(n)}(t) = \xi_{k-1} + \frac{n}{T}(t - t_{k-1})(\xi_k - \xi_{k-1})\,, \tag{2.25}$$

$$\bar\sigma^{(n)}(t) = \sigma_k\,, \tag{2.26}$$

$$\bar r^{(n)}(t) = r_k\,, \tag{2.27}$$

$$\bar\xi^{(n)}(t) = \xi_k\,. \tag{2.28}$$

for $t \in [t_{k-1}, t_k)$, continuously extended to $t = T$. As $n \to \infty$, the functions $\sigma^{(n)}$ converge uniformly to $\sigma$, $r^{(n)}$ converge uniformly to $r$, $\dot\sigma^{(n)}$ converge to $\dot\sigma$ and $\dot r^{(n)}$ converge to $\dot r$ strongly in $L^p(0,T)$ for $p < \infty$ and weakly star in $L^\infty(0,T)$. Furthermore, for every $n \in \mathbb{N}$ and $t \in [0,T]$, we have

$$|\sigma^{(n)}(t) - \bar\sigma^{(n)}(t)| \le \frac{C}{n}\,, \quad |r^{(n)}(t) - \bar r^{(n)}(t)| \le \frac{C}{n}$$

with a constant independent of $n$, hence also $\sigma^{(n)}$ converge uniformly to $\sigma$, and $r^{(n)}$ converge uniformly to $r$.

From (2.22) it follows that $\dot\xi^{(n)}$ are uniformly bounded in $L^\infty(0,T)$ and, by (2.18), $\xi^{(n)}(0) = \sigma(0) - x^0$ for all $n$. Hence, there exists a subsequence, that we still label with index $n$, and $\xi \in W^{1,\infty}(0,T)$ such that $\xi^{(n)} \to \xi$ in sup-norm (by Arzelà-Ascoli Theorem), $\bar\xi^{(n)} \to \xi$ in sup-norm, and $\dot\xi^{(n)} \to \dot\xi$ weakly star in $L^\infty(0,T)$. By (2.19), we have for all $n \in \mathbb{N}$ that

$$|\bar\sigma^{(n)}(t) - \bar\xi^{(n)}(t)| \le \bar r^{(n)}(t) \qquad \forall t \in [0,T]\,, \tag{2.29}$$

$$\dot\xi^{(n)}(t)(\bar\sigma^{(n)}(t) - \bar\xi^{(n)}(t) - \bar r^{(n)}(t)x) \ge 0 \qquad \forall |x| \le 1 \quad \text{a.e.}\,, \tag{2.30}$$

$$\sigma^{(n)}(0) - \xi(0) = x^0. \tag{2.31}$$

The above convergences enable us to pass to the limit and check that $\xi$ is the desired solution.

To extend the problem to $W^{1,1}(0,T)$, we derive a Lipschitz bound in $W^{1,1}(0,T)$ for two solutions $\xi_1, \xi_2$, corresponding to two inputs $\sigma_1, r_1, x_1^0$, $\sigma_2, r_2, x_2^0$ as in Theorem 2.1, with the $W^{1,\infty}$-regularity.

Set $x_i(t) = \sigma_i(t) - \xi_i(t)$ for $i = 1, 2$. We have

$$\dot\xi_1(t)\left(\frac{x_1(t)}{r_1(t)} - \frac{x_2(t)}{r_2(t)}\right) \ge 0\,, \quad \dot\xi_2(t)\left(\frac{x_2(t)}{r_2(t)} - \frac{x_1(t)}{r_1(t)}\right) \ge 0 \quad \text{a.e.}\,,$$

hence also

$$\frac{\dot\xi_1(t)}{r_1(t)}\left(\frac{x_1(t)}{r_1(t)} - \frac{x_2(t)}{r_2(t)}\right) \ge 0\,, \quad \frac{\dot\xi_2(t)}{r_2(t)}\left(\frac{x_2(t)}{r_2(t)} - \frac{x_1(t)}{r_1(t)}\right) \ge 0 \quad \text{a.e.}$$

Summing up the two inequalities, we obtain

$$\left(\frac{\dot\xi_1(t)}{r_1(t)} - \frac{\dot\xi_2(t)}{r_2(t)}\right)\left(\frac{x_1(t)}{r_1(t)} - \frac{x_2(t)}{r_2(t)}\right) \ge 0 \quad \text{a.e.} \tag{2.32}$$

On the other hand,

$$\left(\frac{\dot\xi_1(t)}{r_1(t)} - \frac{\dot\xi_2(t)}{r_2(t)}\right) + \left(\frac{\dot x_1(t)}{r_1(t)} - \frac{\dot x_2(t)}{r_2(t)}\right) = \frac{\dot\sigma_1(t)}{r_1(t)} - \frac{\dot\sigma_2(t)}{r_2(t)} \quad \text{a.e.}\,, \tag{2.33}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x_1(t)}{r_1(t)} - \frac{x_2(t)}{r_2(t)}\right) = \left(\frac{\dot x_1(t)}{r_1(t)} - \frac{\dot x_2(t)}{r_2(t)}\right) - \left(\frac{x_1(t)\dot r_1(t)}{r_1^2(t)} - \frac{x_2(t)\dot r_2(t)}{r_2^2(t)}\right) \quad \text{a.e.} \tag{2.34}$$

We now claim that

$$\left|\frac{\dot\xi_1(t)}{r_1(t)} - \frac{\dot\xi_2(t)}{r_2(t)}\right| + \frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{x_1(t)}{r_1(t)} - \frac{x_2(t)}{r_2(t)}\right| \le \left|\frac{\dot\sigma_1(t)}{r_1(t)} - \frac{\dot\sigma_2(t)}{r_2(t)}\right| + \left|\frac{x_1(t)\dot r_1(t)}{r_1^2(t)} - \frac{x_2(t)\dot r_2(t)}{r_2^2(t)}\right| \quad \text{a.e.} \tag{2.35}$$

We define the set

$$A = \left\{t \in (0,T) : \frac{x_1(t)}{r_1(t)} = \frac{x_2(t)}{r_2(t)}\right\}.$$

For a.e. $t \in A$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{x_1(t)}{r_1(t)} - \frac{x_2(t)}{r_2(t)}\right| = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x_1(t)}{r_1(t)} - \frac{x_2(t)}{r_2(t)}\right) = 0$$

hence, by (2.33)–(2.34),

$$\left(\frac{\dot\xi_1(t)}{r_1(t)} - \frac{\dot\xi_2(t)}{r_2(t)}\right) = \left(\frac{\dot\sigma_1(t)}{r_1(t)} - \frac{\dot\sigma_2(t)}{r_2(t)}\right) - \left(\frac{x_1(t)\dot r_1(t)}{r_1^2(t)} - \frac{x_2(t)\dot r_2(t)}{r_2^2(t)}\right), \tag{2.36}$$

and (2.35) follows. For $t \notin A$, we simply test (2.33) by $\operatorname{sign}\big(\frac{x_1(t)}{r_1(t)} - \frac{x_2(t)}{r_2(t)}\big)$, and obtain (2.35) from (2.32) and (2.34).

Let now $\sigma, r$ be in $W^{1,1}(0,T)$. We find sequences $\sigma_j, r_j$ in $W^{1,\infty}(0,T)$, $j \in \mathbb{N}$, converging strongly as $j \to \infty$ in $W^{1,1}(0,T)$ to $\sigma, r$, respectively. It follows from (2.35) that the solutions $\{\xi_j\}$ form a Cauchy sequence in $W^{1,1}(0,T)$. Therefore, the sequence $\{\xi_j\}$ admits a strong limit $\xi \in W^{1,1}(0,T)$, which is thus a solution to Problem (2.10)–(2.12). Uniqueness follows also from (2.35) and from Gronwall's lemma. ■

Theorem 2.1 enables us to define an operator $\mathcal{P}$ which with every element $(\sigma, r, x^0)$ of the domain

$$\mathcal{D}^W := \{(\sigma, r, x^0) \in W^{1,1}(0,T) \times W^{1,1}(0,T) \times \mathbb{R} : r(t) > 0 \text{ for all } t \in [0,T]; |x^0| \le r(0)\}$$

associates the solution $\xi \in W^{1,1}(0,T)$ of Problem (2.10)–(2.12), and we denote $\xi(t) = \mathcal{P}[\sigma, r, x^0](t)$. As an immediate consequence of inequality (2.35), we have

**Corollary 2.2.** *The mapping* $\mathcal{P} : \mathcal{D}^W \to W^{1,1}(0,T) : (\sigma, r, x^0) \mapsto \xi$ *is locally Lipschitz continuous.*

The operator $\mathcal{P}$ admits an extension to a Lipschitz continuous operator $\mathcal{P} : \mathcal{D}^C \to C[0,T]$, where

$$\mathcal{D}^C := \{(\sigma, r, x^0) \in C[0,T] \times C[0,T] \times \mathbb{R} : r(t) \ge 0 \text{ for all } t \in [0,T]; |x^0| \le r(0)\}.$$

We see that in this setting, the yield point $r(t)$ is allowed to vanish on an arbitrarily large set.

The result can be stated as follows.

**Theorem 2.3.** *Let* $(\sigma_i, r_i, x_i^0) \in \mathcal{D}^W$ *be given, and let* $\xi_i = \mathcal{P}[\sigma_i, r_i, x_i^0]$, $i = 1,2$. *Then for every* $t^* \in [0,T]$ *we have*

$$|\xi_1(t^*) - \xi_2(t^*)| \le \max\{|\xi_1(0) - \xi_2(0)|, |\sigma_1 - \sigma_2|_{[0,t^*]} + |r_1 - r_2|_{[0,t^*]}\}, \tag{2.37}$$

*where we set* $|w|_{[0,t]} = \max_{\tau \in [0,t]} |w(\tau)|$ *for* $w \in C[0,T]$ *and* $t \in [0,T]$.

*Proof.* For a fixed $t^* \in [0,T]$ and for any $t \in [0,t^*]$ put

$$V(t) = \max\{(\xi_1(t) - \xi_2(t))^2, (|\sigma_1 - \sigma_2|_{[0,t^*]} + |r_1 - r_2|_{[0,t^*]})^2\},$$

and assume that the set

$$B := \{t \in [0,t^*] : |\xi_1(t) - \xi_2(t)| > |\sigma_1 - \sigma_2|_{[0,t^*]} + |r_1 - r_2|_{[0,t^*]}\} \tag{2.38}$$

is nonempty. In the variational inequality (2.10) for $\xi_1$ we set $r_1(t)x = (I - P_{r_1(t)})(\sigma_2(t) - \xi_2(t))$ (cf. the passage from (2.19) to (2.20)), and obtain for a.e. $t \in (0,T)$ the inequality

$$\dot{\xi}_1(t)(\xi_1(t) - \xi_2(t)) \le |\dot{\xi}_1(t)|(|\sigma_1 - \sigma_2|_{[0,t^*]} + |r_1 - r_2|_{[0,t^*]}\}). \tag{2.39}$$

Interchanging the indices 1 and 2, we have similarly

$$\dot{\xi}_2(t)(\xi_2(t) - \xi_1(t)) \le |\dot{\xi}_2(t)|(|\sigma_1 - \sigma_2|_{[0,t^*]} + |r_1 - r_2|_{[0,t^*]}\}). \tag{2.40}$$

For $t \in B$, this implies in particular that $\dot{\xi}_1(t)(\xi_1(t) - \xi_2(t)) \le 0$, $\dot{\xi}_2(t)(\xi_2(t) - \xi_1(t)) \le 0$, hence $\dot{V}(t) \le 0$. For a.e. $t \in (0,t^*) \setminus B$ we necessarily have $\dot{V}(t) = 0$, hence $V$ is a nonincreasing function, and the estimate (2.37) is proved. The assertion now follows from the density of $\mathcal{D}^W$ in $\mathcal{D}^C$. ∎

In the case that $r(t)$ stays bounded away from zero, the operator $\mathcal{P}:\mathcal{D}^C\to C[0,T]$ admits a Stieltjes integral representation based on a uniform bound for the total output variation. We state the result in the following form.

**Theorem 2.4.** *Let $(\sigma,r,x^0)\in\mathcal{D}^C$, and let $r(t)\ge r_0>0$ for all $t\in[0,T]$. Then $\xi(t)=\mathcal{P}[\sigma,r,x^0](t)$ if and only if $\xi\in BV(0,T)\cap C[0,T]$ and*

$$|\sigma(t)-\xi(t)|\le r(t)\qquad \forall t\in[0,T]\,, \tag{2.41}$$

$$\int_0^T(\sigma(t)-\xi(t)-r(t)\varphi(t))\,\mathrm{d}\xi(t)\ge 0\qquad \forall\varphi\in C[0,T]:|\varphi|_{[0,T]}\le 1 \tag{2.42}$$

$$\sigma(0)-\xi(0)=x^0. \tag{2.43}$$

*Proof.* Let $\sigma_j\in W^{1,1}(0,T)$, $r_j\in W^{1,1}(0,T)$ converge uniformly to $\sigma$ and $r$, respectively, as $j\to\infty$, and let $\xi_j=\mathcal{P}[\sigma_j,r_j,x^0]$ for $j\in\mathbb{N}$. Assume that there exist points $t_0^j<t_1^j<\cdots<t_{m_j}^j$ in $(0,T)$ such that $(-1)^k\dot\xi_j(t_k)>0$. Set $x_j:=\sigma_j-\xi_j$. By (2.11), we have $x_j(t_k^j)=(-1)^k r(t_k^j)$ for all $j$ and $k$. In particular, $|x_j(t_k^j)-x_j(t_{k-1}^j)|\ge 2r_0$. Choosing in Theorem 2.3 $\sigma_1=\sigma_j$, $\sigma_2(t)=\sigma_j(t)$ for $t\in[0,t_{k-1}^j]$, $\sigma_2(t)=\sigma_j(t_{k-1}^j)$ for $t\in(t_{k-1}^j,t_k^j]$, and similarly for $r_1,r_2$, we obtain

$$|\xi_j(t_k^j)-\xi_j(t_{k-1}^j)|\le|\sigma_j(\cdot)-\sigma_j(t_{k-1}^j)|_{[t_{k-1}^j,t_k^j]}+|r_j(\cdot)-r_j(t_{k-1}^j)|_{[t_{k-1}^j,t_k^j]}\,, \tag{2.44}$$

hence

$$|x_j(t_k^j)-x_j(t_{k-1}^j)|\le 2|\sigma_j(\cdot)-\sigma_j(t_{k-1}^j)|_{[t_{k-1}^j,t_k^j]}+|r_j(\cdot)-r_j(t_{k-1}^j)|_{[t_{k-1}^j,t_k^j]}\,. \tag{2.45}$$

Since $\sigma_j,r_j$ converge uniformly to continuous limits, the number of such intervals is uniformly bounded, say,

$$m_j\le M\quad\forall j\in\mathbb{N}.$$

Hence, the functions $\xi_j$ have at most $M$ monotonicity intervals, and their uniform limit $\xi$ has total variation bounded for instance by $2M\,|\xi|_{[0,T]}$. Inequality (2.42) holds for every $j$, letting $j\to\infty$ we check that it is preserved in the limit.

Conversely, let (2.42) hold, and let $t^*\in(0,T]$, $\psi\in C[0,t^*]$ be arbitrary, $|\psi|_{[0,t^*]}\le 1$. We fix some small positive $\delta$, and define

$$\varphi_\delta(t)=\begin{cases}\psi(t) & \text{for } t\in[0,t^*-\delta],\\ \frac{t^*-t}{\delta}\psi(t)+\frac{t+\delta-t^*}{\delta}\frac{1}{r(t)}(\sigma(t)-\xi(t)) & \text{for } t\in(t^*-\delta,t^*),\\ \frac{1}{r(t)}(\sigma(t)-\xi(t)) & \text{for } t\in[t^*,T].\end{cases}$$

We have

$$
\begin{aligned}
0 &\le \int_0^T (\sigma(t) - \xi(t) - r(t)\varphi_\delta(t))\,\mathrm{d}\xi(t) \\
&= \int_0^{t^*} (\sigma(t) - \xi(t) - r(t)\psi(t))\,\mathrm{d}\xi(t) - \int_{t^*-\delta}^{t^*} \frac{t+\delta-t^*}{\delta}(\sigma(t) - \xi(t) - r(t)\psi(t))\,\mathrm{d}\xi(t) \\
&\le \int_0^{t^*} (\sigma(t) - \xi(t) - r(t)\psi(t))\,\mathrm{d}\xi(t) + 2\,|r|_{[0,T]} \operatorname*{Var}_{[t^*-\delta,t^*]} \xi\,. \qquad (2.46)
\end{aligned}
$$

Letting $\delta \to 0+$, we obtain

$$\int_0^{t^*} (\sigma(t) - \xi(t) - r(t)\psi(t))\,\mathrm{d}\xi(t) \ge 0 \qquad (2.47)$$

for every $t^* \in (0,T]$ and every $\psi \in C[0,t^*]$, $|\psi|_{[0,t^*]} \le 1$. Assume that there exist two functions $\xi_1, \xi_2$ satisfying (2.41)–(2.43). Then, choosing $\psi = \sigma - \xi_{3-i}$ in (2.47) for $\xi_i$, $i = 1,2$, and summing up the two inequalities, we obtain

$$\int_0^{t^*} (\xi_1(t) - \xi_2(t))\,\mathrm{d}(\xi_1 - \xi_2)(t) \le 0 \qquad (2.48)$$

for every $t^* \in (0,T]$, hence $\xi_1 = \xi_2$. We have already shown that $\xi = \mathcal{P}[\sigma, r, x^0]$ satisfies (2.41)–(2.43). Hence, it is the unique solution of (2.41)–(2.43). ■

The original problem (2.7)–(2.9) represents an important special case. The variational inequality

$$|\sigma(t) - \xi_r(t)| \le r \qquad \forall t \in [0,T] \qquad (2.49)$$

$$\dot{\xi}_r(t)(\sigma(t) - \xi_r(t) - y) \ge 0 \qquad \forall |y| \le r \quad \text{a.e.} \qquad (2.50)$$

$$\sigma(0) - \xi_r(0) = x^0 \qquad (2.51)$$

for a fixed constant $r > 0$, given input $\sigma \in W^{1,1}(0,T)$, and given initial condition $x^0 \in [-r,r]$, defines the so-called *play operator* $\xi_r(t) = \mathfrak{p}_r[\sigma, x^0](t)$ with threshold $r$. We now briefly derive some useful properties of the play.

**Proposition 2.5.** *Let $\sigma \in C[0,T]$ be monotone (nondecreasing or nonincreasing) in an interval $[t_0,t_1] \subset [0,T]$, and let $\xi_r = \mathfrak{p}_r[\sigma, x^0]$ for some $r$ and $x^0$. Then*

$$\xi_r(t) = \max\{\xi_r(t_0), \sigma(t) - r\} \quad \text{for } t \in [t_0,t_1] \text{ if } \sigma \text{ is nondecreasing in } [t_0,t_1], \qquad (2.52)$$

$$\xi_r(t) = \min\{\xi_r(t_0), \sigma(t) + r\} \quad \text{for } t \in [t_0,t_1] \text{ if } \sigma \text{ is nonincreasing in } [t_0,t_1]. \qquad (2.53)$$

*Proof.* Let $\sigma \in W^{1,1}(0,T)$, and let $\sigma$ be nondecreasing in $[t_0,t_1]$. Let $\hat{\xi}(t)$ be the right hand side of (2.52). It is straightforward to check that the function

$$\tilde{\xi}(t) = \begin{cases} \xi(t) & \text{for } t \in [0,t_0]\,, \\ \hat{\xi}(t) & \text{for } t \in (t_0,t_1] \end{cases}$$

is a solution of (2.49)–(2.51). Since the solution is unique, the statement follows. ∎

It is perhaps interesting to note that (2.52)–(2.53) were originally used in [2] to introduce for the first time the play operator. In this setting, it was first defined on piecewise monotone inputs, see Figure 2, and it was then extended by continuity to the whole space $C[0,T]$.

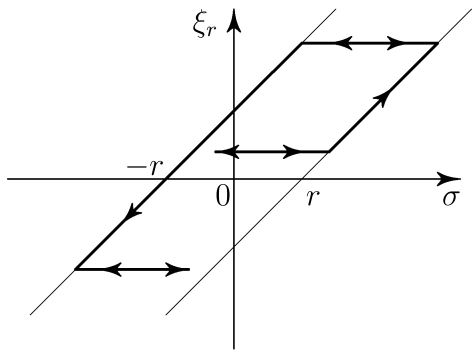


Figure 2: A diagram of the play operator.

**Proposition 2.6** (Hilpert inequality)**.** *Let $\sigma_1, \sigma_2 \in W^{1,1}(0,T)$ and $x_1^0, x_2^0 \in [-r,r]$ be given, and let $\xi_i = \mathfrak{p}[\sigma_i, x_i^0]$, $i = 1,2$. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\xi_1(t) - \xi_2(t))^+ \leq (\dot\xi_1(t) - \dot\xi_2(t))\, H(\sigma_1(t) - \sigma_2(t)) \quad a.e.\,, \tag{2.54}$$

*where $(\cdot)^+$ denotes the positive part and $H$ is the Heaviside function $H(z) = 1$ for $z > 0$, $H(z) = 0$ for $z \leq 0$.*

*Proof.* Choosing $y = \sigma_{3-1} - \xi_{3-i}$ in inequality (2.50) for $\xi_i$, we obtain for a.e. $t$ that

$$\left.\begin{aligned} \dot\xi_1(t)((\sigma_1(t) - \sigma_2(t)) - (\xi_1(t) - \xi_2(t))) \geq 0\,, \\ \dot\xi_2(t)((\sigma_2(t) - \sigma_1(t)) - (\xi_2(t) - \xi_1(t))) \geq 0\,, \end{aligned}\right\} \tag{2.55}$$

hence

$$(\dot\xi_1(t) - \dot\xi_2(t))((\sigma_1(t) - \sigma_2(t)) - (\xi_1(t) - \xi_2(t))) \geq 0 \quad \text{a.e.}$$

Then necessarily

$$(\dot\xi_1(t) - \dot\xi_2(t))(H(\sigma_1(t) - \sigma_2(t)) - H(\xi_1(t) - \xi_2(t))) \geq 0 \quad \text{a.e.}\,,$$

and the assertion follows. ∎

We immediately see the following ordering properties as consequences of the Hilpert inequality:

$$\big(\sigma_1(t) \leq \sigma_2(t)\ \forall t \in [0,T],\ \xi_1(0) \leq \xi_2(0)\big) \ \Rightarrow\ \xi_1(t) \leq \xi_2(t)\ \forall t \in [0,T], \tag{2.56}$$

$$\big(\sigma(t^*) = |\sigma|_{[0,T]}\,,\ \xi(0) \leq \sigma(t^*) - r\big) \ \Rightarrow\ \xi(t) \leq \sigma(t^*) - r\ \forall t \in [0,T]. \tag{2.57}$$

The following relationship between play operators with different thresholds will be used in the sequel.

**Proposition 2.7** (Brokate identity). *For every $\sigma \in C[0,T]$, every $r,s > 0$, and every $|x^0| \le r$, $|y_0| \le s$, we have*

$$\mathfrak{p}_s[\mathfrak{p}_r[\sigma, x^0], y^0](t) = \mathfrak{p}_{s+r}[\sigma, x^0 + y^0](t) \quad \forall t \in [0,T]. \tag{2.58}$$

*Proof.* It suffices to consider $\sigma \in W^{1,1}(0,T)$. Set $\xi = \mathfrak{p}_r[\sigma, x^0]$, $\eta = \mathfrak{p}_s[\xi, y_0]$, $\omega(t) = \mathfrak{p}_{s+r}[\sigma, x^0 + y^0]$. We have for a.e. $t \in (0,T)$ and each $y_1, y_2, y_3 \in [-1,1]$ the inequalities

$$\begin{aligned} \dot\xi(t)(\sigma(t) - \xi(t) - ry_1) &\ge 0\,, \\ \dot\eta(t)(\xi(t) - \eta(t) - sy_2) &\ge 0\,, \\ \dot\omega(t)(\sigma(t) - \omega(t) - (s+r)y_3) &\ge 0\,. \end{aligned}$$

If $\dot\eta(t) \ne 0$, then necessarily $\eta(t) = \xi(t) - r$, and $\dot\eta(t) = \dot\xi(t)$. Hence,

$$\dot\eta(t)(\sigma(t) - \xi(t) - ry_1) \ge 0 \quad \text{a.e.}\,,$$

which entails in turn

$$\dot\eta(t)(\sigma(t) - \eta(t) - ry_1 - sy_2) \ge 0 \quad \text{a.e.}$$

We now choose $y_1, y_2, y_3$ so as to obtain $ry_1 + sy_2 = \sigma(t) - \omega(t)$, $(s+r)y_3 = \sigma(t) - \eta(t)$, which yields

$$(\dot\eta(t) - \dot\omega(t)(\eta(t) - \omega(t)) \le 0 \quad \text{a.e.}$$

By the choice of initial conditions, we have $\eta(0) = \omega(0)$, hence $\eta(t) = \omega(t)$ for all $t$, and the proof is complete. ∎

To conclude this section, we show two examples illustrating the fact that the spaces $C[0,T]$ and $W^{1,1}(0,T)$ are somehow canonical for the play as a typical representative of a rate independent operator. Note that the sup-norm in $C[0,T]$ and the norm

$$\|u\|_{W^{1,1}(0,T)} = |u(0)| + \int_0^T |\dot u(t)|\, \mathrm{d}t$$

in $W^{1,1}(0,T)$ are invariant with respect to increasing homeomorphisms of the interval $[0,T]$, so that they are suitable for characterizing rate independent operators. Partial results on extensions of the play operator to larger spaces than $C[0,T]$ exist, see e.g. [4], but the only framework where the play still behaves in a good way are the so-called *regulated functions*, that is, functions which at each point admit both one-sided limits. The integral representation as in Theorem 2.4 still works with a small modification, provided we interpret the integral in the Kurzweil sense, see [7].

**Example 2.8.** We have seen in the proof of Theorem 2.1 that the play maps $W^{1,\infty}(0,T)$ into $W^{1,\infty}(0,T)$. We now show that the mapping is discontinuous. Set $\sigma(t) = rt$, $\sigma_\varepsilon(t) = (r+\varepsilon)t$ for $t \in [0,1]$. Then, by Proposition 2.5, $\xi(t) = \mathfrak{p}_r[\sigma, 0](t) = 0$, and

$$\xi_\varepsilon(t) = \begin{cases} 0 & \text{for } t \in \left[0, \frac{r}{r+\varepsilon}\right], \\ (r+\varepsilon)t - r & \text{for } t \in \left(\frac{r}{r+\varepsilon}, 1\right]. \end{cases}$$

We see that $\sigma_\varepsilon$ converge strongly to $\sigma$ in $W^{1,\infty}(0,T)$ as $\varepsilon \to 0+$, while $\|\xi_\varepsilon - \xi\|_{W^{1,\infty}(0,T)} \ge r$.

**Example 2.9.** Formula (2.35) for $r_1(t) = r_2(t) = r$ with constant $r$ shows that the play is Lipschitz continuous in $W^{1,1}(0,T)$ with Lipschitz constant $1$. For $1 < p < \infty$, the play operator is still continuous in $W^{1,p}(0,T)$. This follows from the fact that $|\dot\xi(t)| \le |\dot\sigma(t)|$ a.e., which in turn follows e.g. from Proposition 2.5, from the Lipschitz continuity in $W^{1,1}(0,T)$, and from the Lebesgue dominated convergence theorem. We now show that it is not even uniformly continuous in $W^{1,p}(0,T)$ for $p > 1$. For $n \in \mathbb{N}$ set

$$\varrho_n = n^{\frac{1}{p-1}}, \quad \tau_n = n^{-\frac{p}{p-1}},$$

and $\sigma_1^{(n)}(t) = r + \varrho_n \min\{t, \tau_n\}$, $\sigma_2^{(n)} = \sigma_1^{(n)} - \frac{1}{n}$ for $t \in [0,1]$. We choose the initial conditions $\xi_1^{(n)}(0) = \xi_2^{(n)}(0) = 0$, and define $\xi_i(t) = \mathfrak{p}_r[\sigma_i^{(n)}, \sigma_i^{(n)}(0)](t)$ for $i = 1,2$ and $t \in [0,1]$. By Proposition 2.5 we have $\xi_1^{(n)}(t) = \varrho_n \min\{t, \tau_n\}$, $\xi_2^{(n)} \equiv 0$. With the norm in $W^{1,p}(0,1)$

$$\|u\|_{W^{1,p}(0,1)} = |u(0)| + \left(\int_0^1 |\dot u(t)|^p \,\mathrm{d}t\right)^{1/p},$$

we have

$$\|\sigma_1^{(n)} - \sigma_2^{(n)}\|_{W^{1,p}(0,1)} = \frac{1}{n}, \quad \|\xi_1^{(n)} - \xi_2^{(n)}\|_{W^{1,p}(0,1)} = \|\xi_1^{(n)}\|_{W^{1,p}(0,1)} = \left(\int_0^1 |\dot\xi_1^{(n)}(t)|^p \,\mathrm{d}t\right)^{1/p} = 1.$$

We see that the play is not even uniformly continuous in $W^{1,p}(0,T)$.

# 3 The Preisach model

Consider a relay $\varrho_{r,v}$ with input $h(t)$ and parameters $v \in \mathbb{R}$ and $r > 0$, which can take only one of the values $+1$ or $-1$. Switching from $+1$ to $-1$ occurs only if $h(t)$ decreases across $v - r$ and switching from $-1$ to $+1$ occurs only if $h(t)$ increases across $v + r$, see Figure 3.

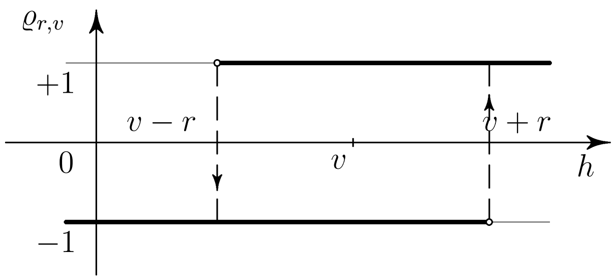


Figure 3: A diagram of the relay with thresholds $v + r$, $v - r$.

A rigorous definition can be stated as follows, see[10]. Let an initial condition $\varrho^0_{r,v} \in \{-1, +1\}$ and a function $h \in C[0,T]$ be given. For each $t \in [0,T]$ we define the set

$$M_{r,v}[h](t) = \{\tau \in [0,t] : |h(\tau) - v| \ge r\}. \tag{3.1}$$

If $M_{r,v}[h](t) \neq \emptyset$, then we put $t_{r,v} = \max M_{r,v}[h](t)$, and

$$\varrho_{r,v}(t) = \begin{cases} \varrho^0_{r,v} & \text{if } M_{r,v}[h](t) = \emptyset\,, \\ +1 & \text{if } h(t_{r,v}) \geq v + r\,, \\ -1 & \text{if } h(t_{r,v}) \leq v - r\,. \end{cases} \tag{3.2}$$

We introduce the Preisach state space $\Lambda$ as

$$\Lambda = \{\lambda \in W^{1,\infty}(0,\infty) : |\lambda'(r)| \leq 1 \ \text{ a.e.}\}, \tag{3.3}$$

fix an element $\lambda \in \Lambda$, and choose the initial condition

$$\varrho^0_{r,v} = \begin{cases} +1 & \text{for } v < \lambda(r)\,, \\ -1 & \text{for } v \geq \lambda(r)\,. \end{cases} \tag{3.4}$$

It is easy to see that the function $\varrho_{r,v}$ belongs to the space $BV_R(0,T)$ of right continuous functions of bounded variation for every $v \in \mathbb{R}$, $r > 0$, and $h \in C[0,T]$.

**Definition 3.1.** *The mapping $R_{r,v} : C[0,T] \times \Lambda \to BV_R(0,T)$ which with each $h \in C[0,T]$ and $\lambda \in \Lambda$ associates the function $\varrho_{r,v}$ defined by* (3.1), (3.2), (3.4), *is called the* relay operator with thresholds $v - r$, $v + r$, *and we write $\varrho_{r,v}(t) = R_{r,v}[h,\lambda](t)$ for $t \in [0,T]$.*

## 3.1 Relay and play operators

With $\lambda \in \Lambda$, we now associate the play operator as in Section 2. More specifically, for a given input function $h \in W^{1,1}(0,T)$, we define $\xi_r \in W^{1,1}(0,T)$ as the solution of the variational inequality

$$|h(t) - \xi_r(t)| \leq r \qquad \forall t \in [0,T] \tag{3.5}$$

$$\dot{\xi}_r(t)(h(t) - \xi_r(t) - x) \geq 0 \qquad \forall |x| \leq r \quad \text{a.e.} \tag{3.6}$$

$$\xi_r(0) = \lambda(r) + P_r(h(0) - \lambda(r)). \tag{3.7}$$

We denote $\xi_r(t) = \mathfrak{p}_r[h,\lambda](t)$ and hope that it will not create any confusion with the previous notation.

**Theorem 3.2.** *Let $h \in C[0,T]$ and $\lambda \in \Lambda$ be arbitrary, and let $\varrho_{r,v}$, $\xi_r$ be as in* (3.1)–(3.7)*. Then for every $t \in [0,T]$ we have*

$$\varrho_{r,v}(t) = \begin{cases} +1 & \textit{if } v < \xi_r(t)\,, \\ -1 & \textit{if } v > \xi_r(t)\,, \end{cases} \tag{3.8}$$

and the function $\lambda_t : (0,\infty) \to \mathbb{R} : r \mapsto \xi_r(t)$ belongs to $\Lambda$.

In other words, the position at time $t$ of the interface between the $+1$ and the $-1$ region in the $(r,v)$-plane is given by the distribution of all play operators at time $t$, which thus

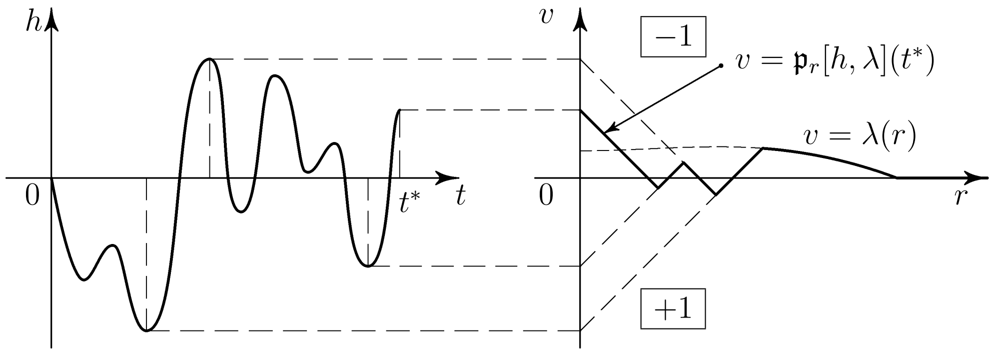


Figure 4: The memory curve $v = \mathfrak{p}_r[h,\lambda](t^*)$.

represent the *memory state of the system* as an element of the Preisach state space $\Lambda$, see Figure 4.

*Proof.* It is sufficient to consider $h \in W^{1,1}(0,T)$. We choose arbitrarily $t \in [0,T]$ and $r > 0$, and assume that

$$v > \xi_r(t)\,, \quad \varrho_{r,v}(t) = +1\,. \tag{3.9}$$

Assume further that $M_{r,v}[h](t) = \emptyset$. Then $\varrho_{r,v}(t) = \varrho^0_{r,v} = +1$, hence $v \le \lambda(r)$. Moreover,

$$v - r < h(\tau) < v + r \tag{3.10}$$

for all $\tau \in [0,t]$. Set

$$V(\tau) = \min\{v, \xi_r(\tau)\}\,. \tag{3.11}$$

For a.e. $\tau \in (0,t)$ we have by (3.7) that $\dot V(\tau)(h(\tau) - V(\tau) + r) \ge 0$. Using (3.10), we have $h(\tau) - V(\tau) + r > v - V(\tau) \ge 0$, hence $V$ is nondecreasing in $[0,t]$. By hypothesis (3.9), we conclude that

$$\xi_r(0) = \lambda(r) + P_r(h(0) - \lambda(r)) < v \le \lambda(r)\,. \tag{3.12}$$

This implies that $P_r(h(0)-\lambda(r)) = h(0)-\lambda(r)+r < 0$, which yields in turn that $h(0) < v-r$, which contradicts (3.10).

So, we see that $M_{r,v}[h](t) \neq \emptyset$. Since $h(t) \le \xi_r(t) + r < v + r$, $M_{r,v}[h](t)$ does not contain the point $t$ and $t_0 := \max M_{r,v}[h](t) < t$, $h(t_0) \ge v + r$, $\xi_r(t_0) \ge h(t_0) - r \ge v$. For all $\tau \in [t_0,t]$, inequalities (3.10) hold, and we easily check again that the function $V$ defined by (3.11) is nondecreasing in $[t_0,t]$. On the other hand, we have $V(t_0) = v$, $V(t) < v$, which is a contradiction.

The argument in the case

$$v < \xi_r(t)\,, \quad \varrho_{r,v}(t) = -1 \tag{3.13}$$

is similar.

It remains to prove that the function $r \mapsto \xi_r(t)$ belongs to $\Lambda$ for every $t \in [0,T]$. This is obvious for $t = 0$ by virtue of (3.7), and we set $\lambda_0(r) := \lambda(r) + P_r(h(0) - \lambda(r)$, $\lambda_0 \in \Lambda$. For all $r > 0$ and $t \in [0,T]$, we have $\xi_r(t) = \mathfrak{p}_r[h,\lambda_0](t)$. We now refer to Proposition 2.7, which states that $\mathfrak{p}_{r+s}[h,\lambda_0](t) = \mathfrak{p}_s[\xi_r,\tilde\lambda_0](t)$ with $\tilde\lambda_0(s) = \lambda_0(r+s) - \lambda_0(r)$ for all $s > 0$. Hence, by (3.5),

$$|\xi_{r+s}(t) - \xi_r(t)| = |\mathfrak{p}_s[\xi_r,\tilde\lambda_0](t) - \xi_r(t)| \le s\,,$$

which implies that the curve $r \mapsto \mathfrak{p}_r[h,\lambda](t)$ representing the interface in the Preisach plane belongs to $\Lambda$ for all $\lambda \in \Lambda$, $h \in C[0,T]$, and $t \in [0,T]$. This completes the proof. ∎

## 3.2 Preisach operator

Preisach's idea is to represent a ferromagnetic body as a collection of dipoles that behave, under a given external time dependent magnetic field $h(t)$, like relays with all possible parameters $r, v$. The value $\varrho_{r,v}(t)$ represents the magnetization of each elementary dipole at time $t$. The bulk magnetization $m(t)$ is then obtained by averaging over all dipoles with a given nonnegative density function $\mu \in L^1((0,\infty)\times\mathbb{R}) \cap L^\infty((0,\infty)\times\mathbb{R})$, that is,

$$m(t) = \frac{1}{2}\int_0^\infty \int_{-\infty}^\infty \varrho_{r,v}(t)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r\,. \tag{3.14}$$

The reason for writing the factor $1/2$ in (3.14) will be clear below. We denote

$$m_0 = \frac{1}{2}\int_0^\infty \int_{-\infty}^0 \mu(r,v)\,\mathrm{d}v\,\mathrm{d}r - \frac{1}{2}\int_0^\infty \int_0^\infty \mu(r,v)\,\mathrm{d}v\,\mathrm{d}r\,, \tag{3.15}$$

and

$$g(r,v) = \int_0^v \mu(r,v')\,\mathrm{d}v' \qquad \text{for } (r,v) \in (0,\infty)\times\mathbb{R}\,. \tag{3.16}$$

It follows from Theorem 3.2 that (3.15) can equivalently be written as

$$m(t) = m_0 + \int_0^\infty g(r,\xi_r(t))\,\mathrm{d}r\,. \tag{3.17}$$

We thus can introduce the Preisach operator $F : C[0,T]\times\Lambda \to C[0,T]$ by the formula

$$F[h,\lambda](t) = \int_0^\infty g(r,\mathfrak{p}_r[h,\lambda](t))\,\mathrm{d}r \qquad \text{for } h \in C[0,T]\,,\ \lambda\in\Lambda\,,\ \text{and } t\in[0,T]. \tag{3.18}$$

By virtue of the Lipschitz continuity of the play operator and of the function $g$ with respect to $v$, we easily obtain

**Corollary 3.3.** *The Preisach operator $F : C[0,T]\times\Lambda \to C[0,T]$ is Lipschitz continuous in the sense that*

$$|F[h_1,\lambda_1](t) - F[h_2,\lambda_2](t)| \le \|\mu\|_{L^\infty} \max\{\|\lambda_1-\lambda_2\|_{L^\infty}, |h_1-h_2|_{[0,t]}\} \tag{3.19}$$

*for all $(h_1,\lambda_1),(h_2,\lambda_2) \in C[0,T]\times\Lambda$ and all $t\in[0,T]$.*

It is interesting to note that an infinite combination of *discontinuous* relay operators in (3.14) leads here to a *Lipschitz continuous* mapping in the space of continuous functions.

The "play" formalism in (3.18) has another advantage: the energy balance can easily be derived. Indeed, if we denote by

$$m_r(t) = g(r, \xi_r(t))\,, \quad \xi_r(t) = \mathfrak{p}_r[h, \lambda](t)$$

the contribution to the magnetization at memory level $r$, we obtain, as a consequence of (3.7), that

$$\dot m_r(t)(h(t) - \xi_r(t) - x) \ge 0 \quad \forall |x| \le r \quad \text{a.e.}\,, \tag{3.20}$$

or, equivalently,

$$\dot m_r(t)h(t) - \mu(r, \xi_r(t))\xi_r(t)\dot\xi_r(t) = r|\dot m_r(t)| \ \text{ a.e.} \tag{3.21}$$

Set

$$G(r, v) = \int_0^v \mu(r, v')v' \,\mathrm{d}v' \qquad \text{for } (r, v) \in (0, \infty) \times \mathbb{R}\,. \tag{3.22}$$

Then (3.21) can be written as the energy equality

$$\dot m_r(t)h(t) - \frac{\partial}{\partial t} G(r, \xi_r(t)) = r|\dot m_r(t)| \quad \text{a.e.}\,, \tag{3.23}$$

where $\dot m_r(t)h(t)$ is the power supplied to the system, one part of which is stored as potential energy increment $\frac{\partial}{\partial t}G(r, \xi_r(t))$, and another (always positive) part $r|\dot m_r(t)|$ is dissipated. The energy equality for the bulk magnetization reads

$$\dot m(t)h(t) - \frac{\mathrm{d}}{\mathrm{d}t} U[h, \lambda](t) = r \left| \frac{\mathrm{d}}{\mathrm{d}t} D[h, \lambda](t) \right| \quad \text{a.e.}\,, \tag{3.24}$$

where the *potential operator* $U$ and the *dissipation operator* $D$ associated with $F$ have the form

$$U[h, \lambda](t) = \int_0^\infty G(r, \mathfrak{p}_r[h, \lambda](t)) \,\mathrm{d}r\,, \quad D[h, \lambda](t) = \int_0^\infty r g(r, \mathfrak{p}_r[h, \lambda](t)) \,\mathrm{d}r\,. \tag{3.25}$$

Notice that in the passage from (3.23) to (3.24), we have interchanged on the right hand side the absolute value and the derivative. This is legal, as all nonzero derivatives of $\xi_r(t)$ corresponding to one input $h(t)$ have the same sign, namely the sign of $\dot h(t)$.

In the section on general rate independent operators, we have shown the importance of convexity of trajectories for rate independent operators. We have the following local result.

**Theorem 3.4.** *Let* $\varrho > 0$ *be given, and let* $D_\varrho$ *be the triangle* $D_\varrho = \{(r, v) \in (0, \infty) \times \mathbb{R} : r + |v| < \varrho\}$. *Assume that there exist positive numbers* $A, C, K$ *such that for a.e.* $(r, v) \in D_\varrho$ *we have*

$$\mu(r, v) \ \ge \ A\,, \tag{3.26}$$

$$\left|\frac{\partial \mu}{\partial v}(r, v)\right| \ \le \ C \tag{3.27}$$

$$\frac{A}{2} - \varrho C \ = \ K\,. \tag{3.28}$$

*Let $\lambda \in \Lambda$ be such that $\lambda(r) = 0$ for $r \geq \varrho$, and let $F$ be the operator* (3.18)*. Let the input $h \in C[0,T]$ be monotone (nondecreasing or nonincreasing) in an interval $[t_0, t_1]$, and let $|h|_{[0,T]} \leq \varrho$. Then the trajectory $f$ corresponding to $F[h,\lambda]\big|_{[t_0,t_1]}$ is increasing and convex if $h$ is nondecreasing, increasing and concave if $h$ is nonincreasing in $[t_0, t_1]$.*

*Proof.* We carry out the proof only for $h$ nondecreasing in $[t_0, t_1]$, the other case is analogous. put $\bar\lambda(r) = \mathfrak{p}_r[h,\lambda](t_0)$ for $r > 0$. Then $\bar\lambda \in \Lambda$, and by Proposition 2.5 we have for all $t \in [t_0, t_1]$ that

$$F[h,\lambda](t) = \int_0^\infty g(r, \max\{\bar\lambda(r), h(t) - r\})\,\mathrm{d}r\,. \tag{3.29}$$

The function $(I + \bar\lambda) : r \mapsto r + \bar\lambda(r)$ is a nondecreasing mapping of $(0,\infty)$ onto $(h(t_0), \infty)$, as $\bar\lambda \in \Lambda$ and $\bar\lambda(0+) = h(t_0)$ by (3.5). There exists therefore a unique left continuous inverse $S = (I + \bar\lambda)^{-1}$, and we have

$$S(h_1) - S(h_2) \geq \frac{1}{2}(h_1 - h_2) \quad \forall h_1 > h_2\,.$$

Hence,

$$\begin{aligned} F[h,\lambda](t) &= F[h,\lambda](t_0) + \int_0^{S(h(t))} (g(r, h(t) - r) - g(r, \bar\lambda(r)))\,\mathrm{d}r \\ &= F[h,\lambda](t_0) + \int_0^{S(h(t))} \int_{\bar\lambda(r)}^{h(t)-r} \mu(r,v)\,\mathrm{d}v\,\mathrm{d}r\,. \end{aligned} \tag{3.30}$$

The trajectory $f$ is thus given by the formula

$$f(h) = f(h(t_0)) + \int_0^{S(h)} \int_{\bar\lambda(r)}^{h-r} \mu(r,v)\,\mathrm{d}v\,\mathrm{d}r = f(h(t_0)) + \int_0^{S(h)} \int_{(I+\bar\lambda)(r)}^{h} \mu(r, v' - r)\,\mathrm{d}v'\,\mathrm{d}r\,, \tag{3.31}$$

or, by Fubini's theorem,

$$f(h) = f(h(t_0)) + \int_{h(t_0)}^{h} \int_0^{S(v')} \mu(r, v' - r)\,\mathrm{d}r\,\mathrm{d}v'\,. \tag{3.32}$$

It follows from Hilpert's inequality that $\bar\lambda(r) = 0$ for $r \geq \varrho$, hence $S(h) \leq \varrho$ for $h \leq \varrho$. The integration domain $0 < r < S(h)$, $\bar\lambda(r) < v < h - r$ is thus contained in $D_\varrho$. Furthermore, $f$ is absolutely continuous, and

$$f'(h) = \int_0^{S(h)} \mu(r, h - r)\,\mathrm{d}r \geq 0 \quad \text{a.e.} \tag{3.33}$$

For $h(t_0) \leq h_1 < h_2 \leq h(t_1)$ we have

$$f'(h_2) - f'(h_1) = \int_{S(h_1)}^{S(h_2)} \mu_2 - r)\,\mathrm{d}r + \int_0^{S(h_1)} (\mu(r, h_2 - r) - \mu(r, h_1 - r))\,\mathrm{d}r\,. \tag{3.34}$$

By hypotheses (3.26), (3.27), we have

$$\int_{S(h_1)}^{S(h_2)} \mu(r, h_2 - r)\,\mathrm{d}r \geq A(S(h_2) - S(h_1)) \geq \frac{A}{2}(h_2 - h_1)\,,$$

$$\left|\int_0^{S(h_1)} (\mu(r, h_2 - r) - \mu(r, h_1 - r))\,\mathrm{d}r\right| \leq C S(h_1)(h_2 - h_1) \leq C\varrho(h_2 - h_1)\,,$$

which yields

$$\frac{f'(h_2) - f'(h_1)}{h_2 - h_1} \geq \frac{A}{2} - \varrho C = K > 0\,,$$

hence $f$ is strictly convex. The case that $h$ is nonincreasing is similar. ■

A locally convex Preisach operator can be extended to a globally convex one by the following construction.

**Corollary 3.5.** *Let* $\mu$ *and* $\lambda$ *satisfy the hypotheses of Theorem 3.4. Put*

$$\mu_\varrho(r, v) = \begin{cases} \mu(r, v) & \text{for } (r, v) \in D_\varrho\,, \\ \mu(r, \varrho - r) & \text{for } r < \varrho, v \geq \varrho - r\,, \\ \mu(r, -\varrho + r) & \text{for } r < \varrho, v \leq -\varrho + r\,, \\ \mu(\varrho, 0) & \text{for } r \geq \varrho\,, \end{cases} \tag{3.35}$$

*and* $g_\varrho(r, v) = \int_0^v \mu_\varrho(r, v')\,\mathrm{d}v'$. *Then the Preisach operator*

$$F\varrho[h, \lambda] = \int_0^\infty g_\varrho(r, \mathfrak{p}_r[h, \lambda])\,\mathrm{d}r$$

*Then the convexity/concavity property of the trajectories is fulfilled for all inputs* $h \in C[0, T]$ *without any restriction of the amplitude.*

We conclude this section by a variant of the Hilpert inequality, cf. (2.54).

**Proposition 3.6** (Hilpert inequality for the Preisach operator)**.** *Let the functions* $h_1, h_2 \in W^{1,1}(0, T)$ *and* $\lambda_1, \lambda_2 \in \Lambda$ *be given, let* $\xi_r^i = \mathfrak{p}_r[h_i, \lambda_i]$, $i = 1, 2$, *let* $\mu \in L^1((0, \infty) \times \mathbb{R}) \cap L^\infty((0, \infty) \times \mathbb{R})$ *be a given nonnegative density function, and let* $F$ *be the associated Preisach operator as in* (3.16), (3.18). *Then the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \big(g(r, \xi_r^1(t)) - g(r, \xi_r^2(t))\big)^+\,\mathrm{d}r \leq \frac{\mathrm{d}}{\mathrm{d}t}\big(F[h_1, \lambda_1](t) - F[h_2, \lambda_2](t)\big)\,H(h_1(t) - h_2(t)), \tag{3.36}$$

*where* $H$ *is again the Heaviside function, holds almost everywhere in* $(0, T)$.

*Proof.* We multiply the first inequality in (2.55) by $\mu(r, \xi_r^1)$, the second one by $\mu(r, \xi_r^2)$, and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(g(r, \xi_r^1(t)) - g(r, \xi_r^2(t))\big)\big(H(h_1(t) - h_2(t)) - H(\xi_r^1(t) - \xi_r^2(t))\big) \geq 0 \quad \text{a.e.}$$

We have indeed

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(g(r,\xi_r^1(t)) - g(r,\xi_r^2(t)))H(\xi_r^1(t) - \xi_r^2(t)) \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}(g(r,\xi_r^1(t)) - g(r,\xi_r^2(t)))H(g(r,\xi_r^1(t)) - g(r,\xi_r^2(t))) \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}(g(r,\xi_r^1(t)) - g(r,\xi_r^2(t)))^+ \quad \text{a.e.}\,,
\end{aligned}$$

and it suffices to integrate over $r$ to obtain the assertion. ∎

# 4 The method of characteristics

We recall here a classical method for the construction of local solutions to the system

$$v_t = c^2(\varepsilon)\varepsilon_x\,, \tag{4.1}$$

$$\varepsilon_t = v_x \tag{4.2}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$, with initial conditions

$$v(x,0) = v_0(x)\,, \tag{4.3}$$

$$\varepsilon(x,0) = \varepsilon_0(x)\,, \tag{4.4}$$

for $x \in \mathbb{R}$, under the following hypotheses.

**Hypothesis 4.1.** *The data of Problem* (4.1)–(4.4) *have the following properties.*

(i) *The function $c$ is Lipschitz continuous in $\mathbb{R}$ with constant $L_c > 0$, and there exist constants $c_1, c_2$ such that $0 < c_1 \le c(\varepsilon) \le c_2$ for all $\varepsilon \in \mathbb{R}$;*

(ii) *The functions $v_0, \varepsilon_0$ are Lipschitz continuous in $\mathbb{R}$ with Lipschitz constant $L_I$.*

For a given function $\varepsilon : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ and a given point $(x,t) \in \mathbb{R} \times (0,\infty)$, consider for $\tau \in (0,t)$ the ODEs

$$\dot\xi(\tau) = c(\varepsilon(\xi(\tau),\tau))\,, \qquad \xi(t) = x\,, \tag{4.5}$$

$$\dot\eta(\tau) = -c(\varepsilon(\eta(\tau),\tau))\,, \qquad \eta(t) = x\,, \tag{4.6}$$

where the dot denotes derivative with respect to $\tau$. The "initial condition" is given at time $t$ and $\tau$ goes backwards from $t$ to $0$. The trajectories of $\xi$ and $\eta$ are called the *characteristics*.

For $\varepsilon \in \mathbb{R}$ we set $C(\varepsilon) = \int_0^\varepsilon c(z)\,\mathrm{d}z$, and define the functions (the so-called *Riemann invariants*)

$$\varphi(v,\varepsilon) = v - C(\varepsilon)\,, \tag{4.7}$$

$$\psi(v,\varepsilon) = v + C(\varepsilon) \tag{4.8}$$

for $(v,\varepsilon) \in \mathbb{R}^2$.

If $v,\varepsilon$ are smooth solutions to (4.1)–(4.2) and if $\xi,\eta$ satisfy (4.5)–(4.6), then (for simplicity, we omit the arguments on the right hand side)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\varphi(v(\xi(\tau),\tau),\varepsilon(\xi(\tau),\tau)) \;=\; v_x\dot\xi + v_t + c(\varepsilon)(\varepsilon_x\dot\xi + \varepsilon_t) = 0\,, \tag{4.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\psi(v(\eta(\tau),\tau),\varepsilon(\eta(\tau),\tau)) \;=\; v_x\dot\eta + v_t - c(\varepsilon)(\varepsilon_x\dot\eta + \varepsilon_t) = 0\,. \tag{4.10}$$

We see that each of the Riemann invariants is constant along the corresponding characteristic. This enables us to represent the value of the solution at the point $(x,t)$ by the formula

$$v(x,t) \;=\; \frac{1}{2}(\varphi(v_0(\xi(0)),\varepsilon_0(\xi(0))) + \psi(v_0(\eta(0)),\varepsilon_0(\eta(0))))\,, \tag{4.11}$$

$$\varepsilon(x,t) \;=\; C^{-1}\left(\frac{1}{2}(-\varphi(v_0(\xi(0)),\varepsilon_0(\xi(0))) + \psi(v_0(\eta(0)),\varepsilon_0(\eta(0))))\right)\,, \tag{4.12}$$

where $C^{-1}$ is the inverse function to $C$.

The method of solving Problem (4.1)–(4.4) called *method of characteristics* is based on this representation. Given $T>0$ and a function $\hat\varepsilon \in W^{1,\infty}(\mathbb{R}\times(0,T))$, we solve for each $(x,t)\in\mathbb{R}\times(0,T)$ the system

$$\dot\xi(\tau) = c(\hat\varepsilon(\xi(\tau),\tau))\,, \qquad \xi(t) = x\,, \tag{4.13}$$

$$\dot\eta(\tau) = -c(\hat\varepsilon(\eta(\tau),\tau))\,, \qquad \eta(t) = x\,, \tag{4.14}$$

and define $v,\varepsilon$ by the formulas (4.11)–(4.12). A fixed point of the mapping which with $\hat\varepsilon$ associates $\varepsilon$ is called a *characteristic solution* of Problem (4.1)–(4.4). The above computation shows that every smooth solution is a characteristic solution. We prove here the following Theorem.

**Theorem 4.2.** *Let Hypothesis 4.1 hold. Then there exists $T>0$ and a unique characteristic solution to Problem* (4.1)–(4.4) *on* $\mathbb{R}\times(0,T)$.

We will see that the characteristic solution cannot in general exist globally for all times. We postpone the proof of Theorem 4.2 and make a preliminary step.

For any $T>0$ and any increasing function $\beta:[0,T]\to(0,\infty)$, we define the sets

$$K_{T,\beta} = \{\varepsilon \in W^{1,\infty}(\mathbb{R}\times(0,T)) : \varepsilon(x,0)=\varepsilon_0(x), |\varepsilon_x(x,t)| \le \beta(t), |\varepsilon_t(x,t)| \le c_2\beta(t) \;\text{ a.e.}\}, \tag{4.15}$$

and prove the following statement.

**Proposition 4.3.** *Under Hypothesis 4.1, there exist $T>0$ and a continuous positive increasing function $\beta$ on $[0,T]$ such that the function $\varepsilon$ given by* (4.12), (4.13)–(4.14) *belongs to $K_{T,\beta}$ for every $\hat\varepsilon \in K_{T,\beta}$.*

*Proof of Proposition 4.3.* Let $\beta$ be an arbitrary increasing function in an arbitrary interval $[0,T]$ for the moment, and let $\hat\varepsilon \in K_{T,\beta}$ be arbitrarily chosen. We choose arbitrary points $x_1,x_2\in\mathbb{R}$ and $t\in(0,T)$, and define the characteristics

$$\dot\xi_i(\tau) = c(\hat\varepsilon(\xi_i(\tau),\tau))\,, \qquad \xi_i(t) = x_i\,, \tag{4.16}$$

$$\dot\eta_i(\tau) = -c(\hat\varepsilon(\eta_i(\tau),\tau))\,, \qquad \eta_i(t) = x_i\,, \tag{4.17}$$

for $i = 1, 2$. By Hypothesis 4.1 we have for $\tau \in (0, T)$ that

$$|\dot{\xi}_1(\tau) - \dot{\xi}_2(\tau)| \le L_c \beta(\tau) |\xi_1(\tau) - \xi_2(\tau)|. \tag{4.18}$$

Integrating from $\tau$ to $t$, we obtain

$$|\xi_1(\tau) - \xi_2(\tau)| \le |x_1 - x_2| + L_c \int_\tau^t \beta(s) |\xi_1(s) - \xi_2(s)| \,\mathrm{d}s. \tag{4.19}$$

We now use the Gronwall argument to estimate the difference $\xi_1 - \xi_2$. For $\tau \in [0, t]$ set $B(\tau) = \int_0^\tau \beta(s)\,\mathrm{d}s$. Then

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathrm{e}^{L_c B(\tau)} \int_\tau^t \beta(s)|\xi_1(s) - \xi_2(s)|\,\mathrm{d}s\right) \\ &= \beta(\tau)\,\mathrm{e}^{L_c B(\tau)}\left(L_c \int_\tau^t \beta(s)|\xi_1(s) - \xi_2(s)|\,\mathrm{d}s - |\xi_1(\tau) - \xi_2(\tau)|\right) \\ &\ge -|x_1 - x_2|\beta(\tau)\,\mathrm{e}^{L_c B(\tau)} \\ &= -\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{|x_1 - x_2|}{L_c}\,\mathrm{e}^{L_c B(\tau)}\right). \end{aligned} \tag{4.20}$$

Integrating from $\tau$ to $t$ we obtain

$$|x_1 - x_2| + L_c \int_\tau^t \beta(s)|\xi_1(s) - \xi_2(s)|\,\mathrm{d}s \le \mathrm{e}^{L_c(B(t) - B(\tau))}|x_1 - x_2|, \tag{4.21}$$

and combining (4.19) with (4.21) we obtain

$$|\xi_1(\tau) - \xi_2(\tau)| \le \mathrm{e}^{L_c(B(t) - B(\tau))}|x_1 - x_2| \quad \forall \tau \in [0, t]. \tag{4.22}$$

A similar argument yields

$$|\eta_1(\tau) - \eta_2(\tau)| \le \mathrm{e}^{L_c(B(t) - B(\tau))}|x_1 - x_2| \quad \forall \tau \in [0, t]. \tag{4.23}$$

We now use formula (4.12) to estimate $\varepsilon(x_1, t) - \varepsilon(x_2, t)$. The functions $C, C^{-1}$ are both Lipschitz continuous with constants $c_2, 1/c_1$, respectively. Hence, by Hypothesis 4.1,

$$|\varepsilon(x_1, t) - \varepsilon(x_2, t)| \le L_I \frac{1 + c_2}{c_1}\,\mathrm{e}^{L_c B(t)}|x_1 - x_2|. \tag{4.24}$$

We now choose $B$ to be the solution of the differential equation

$$\beta(t) = L_I \frac{1 + c_2}{c_1}\,\mathrm{e}^{L_c B(t)}, \quad B(0) = 0, \tag{4.25}$$

that is,

$$\beta(t) = \frac{1}{L_c}\frac{1}{\kappa - t}, \quad \kappa = \frac{c_1}{L_c L_I (1 + c_2)}. \tag{4.26}$$

Choosing any $T < \kappa$, we see that $|\varepsilon_x(x,t)| \le \beta(t)$ for a.e. $(x,t) \in \mathbb{R} \times (0,T)$. We now similarly prove that

$$|\varepsilon_t(x,t)| \le c_2\beta(t) \qquad \text{for} \quad \text{a.e. } (x,t) \in \mathbb{R} \times (0,T). \tag{4.27}$$

Indeed, for $x \in \mathbb{R}$ and $0 < t_2 < t_1 < T$ we define the characteristics by the equations

$$\dot{\xi}_i(\tau) = c(\hat{\varepsilon}(\xi_i(\tau),\tau))\,, \qquad \xi_i(t_i) = x\,, \tag{4.28}$$

$$\dot{\eta}_i(\tau) = -c(\hat{\varepsilon}(\eta_i(\tau),\tau))\,, \qquad \eta_i(t_i) = x\,. \tag{4.29}$$

By (4.22)–(4.23), we have

$$|\xi_1(0) - \xi_2(0)| \le \mathrm{e}^{L_c B(t_2)}|\xi_1(t_2) - x| = \mathrm{e}^{L_c B(t_2)}|\xi_1(t_2) - \xi_1(t_1)| \le c_2\,\mathrm{e}^{L_c B(t_2)}|t_1 - t_2|\,, \tag{4.30}$$

and similarly

$$|\eta_1(0) - \eta_2(0)| \le c_2\,\mathrm{e}^{L_c B(t_2)}|t_1 - t_2|\,, \tag{4.31}$$

and (4.27) follows from formula (4.12) as above. This completes the proof of Proposition 4.3. ■

Example 4.4 below shows a characteristic solution that blows up at a time close to $\kappa$. This is due to the fact that the characteristics carrying different values of the Riemann invariants may intersect at finite time.

We first finish the proof of Theorem 4.2.

*Proof of Theorem 4.2.* We show that with the choice (4.26) for $\beta$ and with $T < \kappa$ sufficiently small, the mapping defined by (4.11)–(4.14), which with $\hat{\varepsilon}$ associates $\varepsilon$, is a contraction on $K_{T,\beta}$. We fix $x$ and $t$, and as in (4.13)–(4.14), set for $\tau \in [0,t]$ and $i = 1,2$

$$\dot{\xi}_i(\tau) = c(\hat{\varepsilon}_i(\xi_i(\tau),\tau))\,, \qquad \xi_i(t) = x\,, \tag{4.32}$$

$$\dot{\eta}_i(\tau) = -c(\hat{\varepsilon}_i(\eta_i(\tau),\tau))\,, \qquad \eta_i(t) = x\,. \tag{4.33}$$

We endow $K_{T,\beta}$ with the supremum norm $\|\varepsilon\| = \sup_{(x,t)\in\mathbb{R}\times(0,T)} |\varepsilon(x,t)|$. Similarly as in (4.18) we have

$$|\dot{\xi}_1(\tau) - \dot{\xi}_2(\tau)| \le L_c\left(\|\hat{\varepsilon}_1 - \hat{\varepsilon}_2\| + \beta(\tau)|\xi_1(\tau) - \xi_2(\tau)|\right), \tag{4.34}$$

hence

$$|\xi_1(\tau) - \xi_2(\tau)| \le L_c\left((t-\tau)\|\hat{\varepsilon}_1 - \hat{\varepsilon}_2\| + \int_\tau^t \beta(s)|\xi_1(s) - \xi_2(s)|\,\mathrm{d}s\right). \tag{4.35}$$

We proceed as in the proof of Proposition 4.3 and obtain by Gronwall's argument

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathrm{e}^{L_c B(\tau)}\left((t-\tau)\|\hat{\varepsilon}_1 - \hat{\varepsilon}_2\| + \int_\tau^t \beta(s)|\xi_1(s) - \xi_2(s)|\,\mathrm{d}s\right)\right) \ge -\|\hat{\varepsilon}_1 - \hat{\varepsilon}_2\|\,\mathrm{e}^{L_c B(\tau)}. \tag{4.36}$$

Hence,

$$|\xi_1(\tau) - \xi_2(\tau)| \le L_c\|\hat{\varepsilon}_1 - \hat{\varepsilon}_2\| \int_\tau^t \mathrm{e}^{L_c(B(s)-B(\tau))}\,\mathrm{d}s, \tag{4.37}$$

and similarly,

$$|\eta_1(\tau) - \eta_2(\tau)| \leq L_c \|\hat{\varepsilon}_1 - \hat{\varepsilon}_2\| \int_\tau^t \mathrm{e}^{L_c(B(s)-B(\tau))}\,\mathrm{d}s. \tag{4.38}$$

By (4.12) we have

$$\varepsilon_i(x,t) = C^{-1}\left(\frac{1}{2}(-\varphi(v_0(\xi_i(0)), \varepsilon_0(\xi_i(0))) + \psi(v_0(\eta_i(0)), \varepsilon_0(\eta_i(0))))\right). \tag{4.39}$$

We now argue as in (4.24) and obtain

$$|\varepsilon_1(x,t) - \varepsilon_2(x,t)| \leq \left(L_I \frac{1+c_2}{c_1} \int_0^t \mathrm{e}^{L_c B(s)}\,\mathrm{d}s\right) \|\hat{\varepsilon}_1 - \hat{\varepsilon}_2\|. \tag{4.40}$$

The coefficient in front of $\|\hat{\varepsilon}_1 - \hat{\varepsilon}_2\|$ is less than one if $t$ is sufficiently small. Hence, the mapping $\hat{\varepsilon} \mapsto \varepsilon$ is a contraction on $K_{T,\beta}$ with respect to the norm $\|\cdot\|$ provided $T$ is sufficiently small. Since $K_{T,\beta}$ is closed with respect to the sup-norm, the Banach contraction principle yields the existence and uniqueness of a characteristic solution to Problem (4.1)–(4.4). ■

**Example 4.4.** Consider the system (4.1)–(4.4) under Hypothesis 4.1 with data satisfying the following conditions:

$$\varepsilon_0(x) = \varepsilon_{00} + L_I x \qquad \text{for } x \in [x_1, x_2], \tag{4.41}$$

$$v_0(x) = C(\varepsilon_0(x)) \qquad \text{for } x \in \mathbb{R}, \tag{4.42}$$

$$c(z) = c_0 + L_c z \qquad \text{for } z \in [z_1, z_2], \tag{4.43}$$

where $[x_1, x_2]$ is an arbitrarily fixed interval and $z_i = \varepsilon_0(x_i)$, $i = 1, 2$. For

$$t < t_0 := \frac{1}{L_I L_c} \tag{4.44}$$

we can implicitly define, by means of the contraction argument, the functions

$$\varepsilon(x,t) = \varepsilon_0(x + tc(\varepsilon(x,t))), \tag{4.45}$$

$$v(x,t) = C(\varepsilon(x,t)), \tag{4.46}$$

and check by straightforward differentiation that they solve Problem (4.1)–(4.4). For $y \in [x_1, x_2]$ we now define the family of straight lines

$$\lambda_y = \{(x,t) \in \mathbb{R}^2 : x = y - tc(\varepsilon_0(y))\}. \tag{4.47}$$

Along each $\lambda_y$, the function $\varepsilon$ given by (4.45) has the value

$$\varepsilon(x,t) = \varepsilon_0(y - tc(\varepsilon_0(y)) + tc(\varepsilon(x,t))), \tag{4.48}$$

hence

$$|\varepsilon(x,t) - \varepsilon_0(y)| \leq t L_I L_c |\varepsilon(x,t) - \varepsilon_0(y)|. \tag{4.49}$$

For $t < t_0$ and $(x,t) \in \lambda_y$ we thus necessarily have $\varepsilon(x,t) = \varepsilon_0(y) = \varepsilon_{00} + L_I y$. On the other hand, for $(x,t) \in \lambda_y$ and $t \to t_0-$, we have

$$x \to x_0 := \frac{c_0 + L_c \varepsilon_{00}}{L_I L_c}.$$

Hence, all straight lines $\lambda_y$ are the characteristics given by the equations

$$\dot\eta(\tau) = -c(\varepsilon(\eta(\tau),\tau)), \quad \eta(t_0) = x_0 . \tag{4.50}$$

They all intersect in the point $(x_0, t_0)$, while the solution $\varepsilon(x,t)$ carries different values along each characteristic. Hence, the characteristic solution cannot be extended to the whole time interval $[0, t_0]$.

# 5 The Riemann problem

We have seen in Example 4.4 that continuous solutions to Problem (4.1)–(4.4) cannot exist for all times even if the data are smooth. We set

$$g(\varepsilon) = \int_0^\varepsilon c^2(z)\,\mathrm{d}z , \tag{5.1}$$

and state equations (4.1)–(4.2) in the form

$$v_t = g(\varepsilon)_x , \tag{5.2}$$
$$\varepsilon_t = v_x . \tag{5.3}$$

To illustrate the complicated character of the system (4.1)–(4.4) after the discontinuity (a *shock*) appears, assume first that the first discontinuity point $(x_0, t_0)$ is isolated and that the left and right limits exist at this point. The equations (5.2)–(5.3) are invariant with respect to the shift $x \mapsto x - x_0$, $t \mapsto t - t_0$, we can therefore assume that the initial data $v_0, \varepsilon_0$ have a jump at the origin. We now "zoom in" the neighborhood of the origin by the transform

$$F_R(x,t) = \left(\frac{x}{R}, \frac{t}{R}\right)$$

and let formally $R$ tend to $\infty$. Assume that for a generic function $\theta$ of $x$ and $t$ the limit

$$\theta^*(x,t) = \lim_{R\to\infty} \theta\left(\frac{x}{R}, \frac{t}{R}\right) \tag{5.4}$$

exists. Then obviously $\theta^*$ is invariant with respect to $F_R$ (we say that it is *self-similar*), and we have

$$\theta^*(x,t) = \theta^*\left(\frac{x}{t}, 1\right) =: \bar\theta\left(\frac{x}{t}\right) \qquad \text{for } t > 0 . \tag{5.5}$$

System (5.2)–(5.3) is invariant with respect to $F_R$ for every $R > 0$, and the question about special self-similar solutions of the form (5.5) is justified.

Consider first a general inequality

$$\theta^*_t - \chi^*_x \geq 0 \quad \text{in the sense of distributions} \tag{5.6}$$

between two self-similar functions $\theta^*, \chi^*$. The weak formulation of (5.6) reads

$$\int_{-\infty}^{\infty}\int_0^{\infty} (-\theta^*\,\varrho_t + \chi^*\,\varrho_x)(x,t)\ \mathrm{d}t\ \mathrm{d}x\ \geq\ 0 \tag{5.7}$$

for all Lipschitz continuous nonnegative test functions $\varrho$ with compact support in $\mathbb{R}\times(t_0,\infty)$. In terms of $\bar\theta, \bar\chi$ from (5.5), we have the following characterization of weak solutions to (5.6). Here and in the sequel, prime denotes derivative with respect to $z = x/t$.

**Lemma 5.1.** *Let $\theta^*, \chi^*$ be as in (5.5). Then inequality (5.7) holds if and only if*

$$(z\bar\theta(z) + \bar\chi(z))' \leq \bar\theta(z) \tag{5.8}$$

*in the sense of distributions, that is, the function*

$$z\bar\theta(z) + \bar\chi(z) - \int_0^z \bar\theta(s)\,\mathrm{d}s \tag{5.9}$$

*is nonincreasing in $\mathbb{R}$.*

As an easy consequence of Lemma 5.1, we have

**Corollary 5.2.** *We have equality in* (5.6)*, that is,*

$$\theta^*_t - \chi^*_x = 0 \quad a.e. \tag{5.10}$$

*if and only if the function $z \mapsto z\bar\theta(z) + \bar\chi(z)$ is absolutely continuous and the identity*

$$(z\bar\theta(z) + \bar\chi(z))' = \bar\theta(z) \tag{5.11}$$

*holds almost everywhere in $\mathbb{R}$.*

*Proof of Lemma 5.1.* Let (5.8) hold, and let $\varrho$ be an arbitrary admissible test function. In the left hand side of (5.7), we substitute $z = x/t$, and put

$$\gamma(z) = \int_0^{\infty} \varrho(zt,t)\ \mathrm{d}t \tag{5.12}$$

for $z \in \mathbb{R}$. Using the identity

$$0 = \int_0^{\infty} \frac{\partial}{\partial t}(t\varrho(zt,t))\,\mathrm{d}t = \gamma(z) + z\gamma'(z) + \int_0^{\infty} t\varrho_t(zt,t)\,\mathrm{d}t\,,$$

we obtain the integral

$$\int_{-\infty}^{\infty} (\bar\theta(z)\,(z\gamma(z))' + \bar\chi(z)\,\gamma'(z))\ \mathrm{d}z\,, \tag{5.13}$$

which is greater or equal to 0 by virtue of (5.8), and (5.7) follows. Conversely, for an arbitrary test function $\gamma$ with compact support in $\mathbb{R}$ we can find $\varrho$ such that (5.12) holds. Indeed, it suffices to put

$$\varrho(x,t) = \gamma\left(\frac{x}{t}\right)\mu(t)$$

with a nonnegative function $\mu$ with compact support in $(0,\infty)$ such that $\int_0^\infty \mu(t)\,\mathrm{d}t = 1$. By the same substitution, we thus conclude that (5.7) implies (5.8). ■

Assume that $v^*, \varepsilon^*$ are now self-similar solutions to (5.2)–(5.3). By (5.4), the initial conditions for $v^*, \varepsilon^*$ read

$$\left\{\begin{array}{ll} v^*(x,0) = v_0(0+) =: V_+ \ \text{ for } x > 0\,, & v^*(x,0) = v_0(0-) =: V_- \ \text{ for } x < 0\,, \\ \varepsilon^*(x,0) = \varepsilon_0(0+) =: E_+ \ \text{ for } x > 0\,, & \varepsilon^*(x,0) = \varepsilon_0(0-) =: E_- \ \text{ for } x < 0\,. \end{array}\right. \tag{5.14}$$

This constitutes the so-called *Riemann problem*. By virtue of Corollary 5.2, we rewrite the problem in terms of the self-similar variable $z$ in the form

$$\left\{\begin{array}{rl} (z\bar{v}(z) + g(\bar{\varepsilon}(z)))' & = \bar{v}(z) \ \text{ a.e. }, \\ (z\bar{\varepsilon}(z) + \bar{v}(z))' & = \bar{\varepsilon}(z) \ \text{ a.e. }, \\ \bar{v}(\pm\infty) & = V_\pm\,, \\ \bar{\varepsilon}(\pm\infty) & = E_\pm\,. \end{array}\right. \tag{5.15}$$

It is convenient to eliminate the unknown function $\bar{v}$ and state the problem only for $\bar{\varepsilon}$.

**Lemma 5.3.** *An equivalent formulation of Problem* (5.15) *reads*

$$\left\{\begin{array}{rl} (z^2\bar{\varepsilon}(z) - g(\bar{\varepsilon}(z)))' & = 2z\bar{\varepsilon}(z) \ \textit{ a.e. }, \\ \bar{\varepsilon}(\pm\infty) & = E_\pm\,, \\ \int_{-\infty}^{\infty}(\bar{\varepsilon}(z) - P(z))\,\mathrm{d}z & = V_+ - V_-\,, \end{array}\right. \tag{5.16}$$

*where* $P(z) = E_+$ *for* $z > 0$, $P(z) = E_-$ *for* $z < 0$.

*Proof.* Let $\bar{v}, \bar{\varepsilon}$ satisfy (5.15). We have $z^2\bar{\varepsilon}(z) - g(\bar{\varepsilon}(z)) = z(z\bar{\varepsilon}(z) + \bar{v}(z)) - (z\bar{v}(z) + g(\bar{\varepsilon}(z)))$, which is an absolutely continuous function, and the first equation of (5.16) easily follows. We now prove that under Hypothesis 4.1 (i) we have

$$\bar{\varepsilon}(z) = E_+ \ \text{ for } z > c_2\,, \quad \bar{\varepsilon}(z) = E_- \ \text{ for } z < -c_2\,. \tag{5.17}$$

Indeed, choosing $z > R > c_2$, we obtain by integrating the first equation of (5.16) from $R$ to $z$ the identity

$$z^2(\bar{\varepsilon}(z) - \bar{\varepsilon}(R)) = g(\bar{\varepsilon}(z)) - g(\bar{\varepsilon}(R)) + \int_R^z 2s(\bar{\varepsilon}(s) - \bar{\varepsilon}(R))\,\mathrm{d}s\,. \tag{5.18}$$

Hence,

$$(R^2 - c_2^2)|\bar{\varepsilon}(z) - \bar{\varepsilon}(R)| \le \int_R^z 2s|\bar{\varepsilon}(s) - \bar{\varepsilon}(R)|\,\mathrm{d}s\,, \tag{5.19}$$

and by Gronwall's argument we conclude that $\bar\varepsilon$ is constant in $[R,\infty)$ for all $R > c_2$. The argument for $z < -c_2$ is similar. We thus see that (5.17) holds. For any $R > c_2$ we further have, by virtue of (5.18), that

$$\int_{-R}^{R} (\bar\varepsilon(z) - P(z))\,\mathrm{d}z = \int_{-R}^{R} ((z\bar\varepsilon(z) + \bar v(z))' - P(z))\,\mathrm{d}z = \bar v(R) - \bar v(-R)\,, \tag{5.20}$$

and it suffices to let $R$ tend to infinity to obtain (5.16). Conversely, assuming that $\bar\varepsilon$ is a solution to (5.16), it suffices to set

$$\bar v(z) = -z(\bar\varepsilon(z) - P(z)) + \int_{-\infty}^{z} (\bar\varepsilon(s) - P(s))\,\mathrm{d}s + V_-\,, \tag{5.21}$$

and check that (5.15) is satisfied. ■

Problem (5.16) is an ODE of the first order with three "boundary conditions" and looks like an overdetermined problem. We will see that this is not the case. The reason is that the solution may exhibit jumps. However, the condition that the function $z \mapsto z^2\bar\varepsilon(z) - g(\bar\varepsilon(z))$ is absolutely continuous imposes a restriction. If $\bar\varepsilon(z_0-) = \varepsilon_- \neq \bar\varepsilon(z_0+) = \varepsilon_+$ at some point $z_0$, then necessarily

$$z_0^2 = \frac{g(\varepsilon_+) - g(\varepsilon_-)}{\varepsilon_+ - \varepsilon_-}\,. \tag{5.22}$$

This condition is known under the name *Rankine-Hugoniot condition*. It describes the relation between the speed of shock propagation $z_0 = x/t$ and the height of the shock.

We state the existence result in the following form.

**Proposition 5.4.** *Let Hypothesis 4.1 hold. Then*

(i) *for every $E_\pm, V_\pm \in \mathbb{R}$, there exists a piecewise constant solution to* (5.16)*, and the solution is unique if $g$ is linear;*

(ii) *if $g$ is nonlinear, then there exist infinitely many distinct piecewise constant solutions to* (5.16) *even if $V_\pm = E_\pm = 0$.*

Indeed, the case $V_\pm = E_\pm = 0$ always admits the natural solution $\bar\varepsilon \equiv 0$. Proposition 5.4 (ii) thus cannot be considered as a satisfactory answer to the problem, and we will come back to this question in the sequel.

*Proof.* (i) Given $E_0 \in \mathbb{R}$, we define $\bar\varepsilon$ as

$$\bar\varepsilon(z) = \begin{cases} E_- & \text{for } z \in (-\infty, z_-)\,, \\ E_0 & \text{for } z \in (z_-, z_+)\,, \\ E_+ & \text{for } z \in (z_+, \infty)\,, \end{cases} \tag{5.23}$$

where we set

$$z_- = -\sqrt{\frac{g(E_0) - g(E_-)}{E_0 - E_-}}\,, \quad z_+ = \sqrt{\frac{g(E_0) - g(E_+)}{E_0 - E_+}}\,, \tag{5.24}$$

with the convention $(g(E_0) - g(E_-))/(E_0 - E_-) = g'(E_-)$ if $E_0 = E_-$ etc. We see that the function $z \mapsto z^2\bar{\varepsilon}(z) - g(\varepsilon(z))$ is absolutely continuous in $\mathbb{R}$, cf. (5.22), and the first equation of (5.16) is satisfied almost everywhere. The second condition is trivially fulfilled. It only remains to check that $E_0$ can be chosen in such a way that the third condition in (5.16) hold, that is,

$$(E_0 - E_-)\sqrt{\frac{g(E_0) - g(E_-)}{E_0 - E_-}} + (E_0 - E_+)\sqrt{\frac{g(E_0) - g(E_+)}{E_0 - E_+}} = V_+ - V_- \,. \tag{5.25}$$

The left hand side of (5.25) is a continuous functions which tends to $-\infty$ as $E_0$ tends to $-\infty$ and to $+\infty$ as $E_0$ tends to $+\infty$, hence such an $E_0$ does exist. The uniqueness of this solution if $g$ is linear easily follows from the Rankine-Hugoniot condition (5.22).

(ii) The construction is slightly more involved in this case. Assume for definiteness that $g(0) = 0$ (this changes nothing in the statement of the problem). We first prove that there exist $q < 0 < p$ such that the following alternative holds:

$$(\mathrm{A}_-) \ \text{either} \ \frac{g(p) - g(q)}{p - q} > \frac{g(p) - g(r)}{p - r} \qquad \forall r \in \ (q, 0]\,,$$

$$(\mathrm{A}_+) \ \text{or} \ \frac{g(p) - g(q)}{p - q} > \frac{g(s) - g(q)}{s - q} \qquad \forall s \in \ [0, p)\,.$$

Indeed, assume that for every $q < 0 < p$, both sets

$$A_+(p,q) \ = \ \left\{ s \in [0,p); \ \frac{g(p) - g(q)}{p - q} \le \frac{g(s) - g(q)}{s - q} \right\}, \tag{5.26}$$

$$A_-(p,q) \ = \ \left\{ r \in (q,0]; \ \frac{g(p) - g(q)}{p - q} \le \frac{g(p) - g(r)}{p - r} \right\} \tag{5.27}$$

are non-empty. Put $\bar{r} := \max A_-(p,q)$, $\bar{s} := \min A_+(p,q)$, and assume for instance $\bar{r} < 0$. By hypothesis, the set $A_-(p,\bar{r})$ is a non-empty subset of $A_-(p,q)$ which contradicts the definition of $\bar{r}$. We therefore have $\bar{r} = 0$, and similarly $\bar{s} = 0$. The inequalities $\frac{g(p)-g(q)}{p-q} \le \frac{g(p)}{p}$, $\frac{g(p)-g(q)}{p-q} \le \frac{g(q)}{q}$ combined with the elementary identity

$$\frac{g(p) - g(q)}{p - q} - \frac{g(q)}{q} = \frac{p}{q}\left(\frac{g(p) - g(q)}{p - q} - \frac{g(p)}{p}\right) \tag{5.28}$$

yield

$$\frac{g(p)}{p} = \frac{g(q)}{q} \quad \forall q < 0 < p\,.$$

This is only possible if $g$ is linear, which is a contradiction.

In the case $(\mathrm{A}_-)$ with fixed values of $p > 0 > q$, we choose some $r \in (q, 0)$ which will be specified later, and put

$$z_1 = -\sqrt{\frac{g(q)}{q}}, \ z_2 = -\sqrt{\frac{g(p) - g(q)}{p - q}}, \ z_3 = -\sqrt{\frac{g(p) - g(r)}{p - r}}, \ z_4 = \sqrt{\frac{g(r)}{r}}. \tag{5.29}$$

By $(A_-)$, we have $z_1 < z_2 < z_3 < z_4$, and we may put

$$\bar{\varepsilon}(z) = \begin{cases} 0 & \text{for } z < z_1, \\ q & \text{for } z \in (z_1, z_2), \\ p & \text{for } z \in (z_2, z_3), \\ r & \text{for } z \in (z_3, z_4), \\ 0 & \text{for } z > z_4. \end{cases} \tag{5.30}$$

The Rankine-Hugoniot condition (5.22) is satisfied, hence $\bar{\varepsilon}$ will be a solution to Problem (5.16) provided

$$\int_{-\infty}^{\infty} \bar{\varepsilon}(z)\,\mathrm{d}z = -\sqrt{g(q)q} + \sqrt{(g(p)-g(q))(p-q)} - \sqrt{(g(p)-g(r))(p-r)} - \sqrt{g(r)r} = 0. \tag{5.31}$$

This can be achieved by a suitable choice of $r$. Indeed, let us denote by $h(r)$ the middle term in (5.31). We have $h(0) > 0$, $h(q) < 0$, hence a suitable $r \in (q, 0)$ satisfies (5.31).

Case $(A_+)$ is similar. We define for some $s \in (0, p)$

$$z_1 = -\sqrt{\frac{g(p)}{p}},\ z_2 = -\sqrt{\frac{g(p)-g(q)}{p-q}},\ z_3 = -\sqrt{\frac{g(s)-g(q)}{s-q}},\ z_4 = \sqrt{\frac{g(s)}{s}}. \tag{5.32}$$

We have again $z_1 < z_2 < z_3 < z_4$, and it suffices to put

$$\bar{\varepsilon}(z) = \begin{cases} 0 & \text{for } z < z_1, \\ p & \text{for } z \in (z_1, z_2), \\ q & \text{for } z \in (z_2, z_3), \\ s & \text{for } z \in (z_3, z_4), \\ 0 & \text{for } z > z_4. \end{cases} \tag{5.33}$$

The counterpart of (5.31) reads

$$\int_{-\infty}^{\infty} \bar{\varepsilon}(z)\,\mathrm{d}z = \sqrt{g(p)p} - \sqrt{(g(p)-g(q))(p-q)} + \sqrt{(g(s)-g(q))(s-q)} + \sqrt{g(s)s} = 0. \tag{5.34}$$

Let now $\tilde{h}(s)$ be the middle term in (5.34). We have $\tilde{h}(0) < 0$, $\tilde{h}(p) > 0$, hence $\bar{\varepsilon}$ is a nontrivial (and in fact non-physical) solution to (5.16).

It remains to check that there exist in fact infinitely many solutions of the above form. In case $(A_-)$, for every

$$\lambda \in \left(\frac{g(p)}{p}, \frac{g(p)-g(q)}{p-q}\right)$$

we repeat the same construction, replacing $q$ by

$$q_\lambda = \max\left\{u \in (q, 0) : \frac{g(p)-g(u)}{p-u} = \gamma\right\}.$$

Case $(A_+)$ is again similar. This completes the proof of Proposition 5.4. ■

# 6 Energy balance

We associate with System (5.2)– (5.3) the functions

$$\mathcal{E}(v,\varepsilon) = \frac{1}{2}v^2 + E(\varepsilon), \quad \mathcal{F}(v,\varepsilon) = vg(\varepsilon)\,, \tag{6.1}$$

called the *energy density* and *energy flux density*, respectively, where $E$ is the antiderivative of $g$, that is,

$$E(\varepsilon) = \int_0^\varepsilon g(u)\,\mathrm{d}u\,. \tag{6.2}$$

Smooth solutions of System (5.2)– (5.3) conserve the total energy in the sense

$$\frac{\partial}{\partial t}\mathcal{E}(v,\varepsilon) - \frac{\partial}{\partial x}\mathcal{F}(v,\varepsilon) = 0. \tag{6.3}$$

For weak solutions, one cannot ensure that the energy be preserved for all times if $g$ is nonlinear. Instead of (6.3), according to the $2^{\text{nd}}$ Principle of Thermodynamics we only require that the energy dissipation rate given by the left hand side of (6.3) be nonnegative, i.e.

$$\frac{\partial}{\partial t}\mathcal{E}(v,\varepsilon) - \frac{\partial}{\partial x}\mathcal{F}(v,\varepsilon) \geq 0 \qquad \text{in the sense of distributions.} \tag{6.4}$$

For self-similar solutions we can rewrite condition (6.4) in the following way.

**Proposition 6.1.** *A solution* $\bar{\varepsilon}$ *of* (5.16) *satisfies condition* (6.4) *if and only if the function*

$$z \mapsto E(\bar{\varepsilon}(z)) - \bar{\varepsilon}(z)g(\bar{\varepsilon}(z)) + \frac{z^2}{2}\bar{\varepsilon}^2(z) - \int_0^z s\bar{\varepsilon}^2(s)\,\mathrm{d}s \tag{6.5}$$

*is nondecreasing for* $z > 0$ *and nonincreasing for* $z < 0$.

*Proof.* Condition (6.5) can be equivalently written as

$$z\left(E(\bar{\varepsilon}(z)) - \bar{\varepsilon}(z)g(\bar{\varepsilon}(z)) + \frac{z^2}{2}\bar{\varepsilon}^2(z)\right)' \leq z^2\bar{\varepsilon}^2(z) \tag{6.6}$$

in the sense of distributions. On the other hand, by Lemma 5.1 we have

$$\left(z\left(\frac{1}{2}\bar{v}^2(z) + E(\bar{\varepsilon}(z))\right) + \bar{v}(z)g(\bar{\varepsilon}(z))\right)' \leq \left(\frac{1}{2}\bar{v}^2 + E(\bar{\varepsilon}(z))\right) \tag{6.7}$$

in the sense of distributions. We now use the identity

$$\frac{1}{2}z\bar{v}^2(z) + \bar{v}(z)g(\bar{\varepsilon}(z)) = \frac{z^3}{2}\bar{\varepsilon}^2(z) - z\bar{\varepsilon}(z)g(\bar{\varepsilon}(z)) + V(z)\,, \tag{6.8}$$

where we set

$$V(z) := (\bar{v}(z) + z\bar{\varepsilon}(z))\Big((g(\bar{\varepsilon}(z)) + z\bar{v}(z)) - \frac{z}{2}(\bar{v}(z) + z\bar{\varepsilon}(z))\Big)\,. \tag{6.9}$$

By (5.15), $V$ is absolutely continuous and

$$V'(z) = \frac{1}{2}\bar{v}^2(z) + \bar{\varepsilon}(z)g(\bar{\varepsilon}(z)) - \frac{3z^2}{2}(\bar{\varepsilon}^2(z) \qquad \text{a.e.} \tag{6.10}$$

Substituting (6.8)–(6.10) into (6.7) we obtain

$$\left(zE(\bar{\varepsilon}(z)) + \frac{z^3}{2}\bar{\varepsilon}^2(z) - z\bar{\varepsilon}(z)g(\bar{\varepsilon}(z))\right)' \leq E(\bar{\varepsilon}(z)) + \frac{z^2}{2}\bar{\varepsilon}^2(z) - \bar{\varepsilon}(z)g(\bar{\varepsilon}(z)) + z^2\bar{\varepsilon}^2(z)\,, \tag{6.11}$$

and (6.6) follows. ∎

In the remainder of this section, we restrict ourselves to solutions which belong to the space of *regulated functions*, that is, functions $u : \mathbb{R} \to \mathbb{R}$ that at each point $z \in \mathbb{R}$ admit both one-sided limits $u(z-), u(z+)$.

Condition (6.5) has a clear geometrical meaning. It states that if $\varepsilon_- := \bar{\varepsilon}(z_0-) \neq \varepsilon_+ := \bar{\varepsilon}(z_0+)$ at some point $z_0$, then

*The signed area between the segment with endpoints* $(\varepsilon_-, g(\varepsilon_-))$ *and* $(\varepsilon_+, g(\varepsilon_+))$, *and the part of the graph of* $g$ *between* $\varepsilon_-$ *and* $\varepsilon_+$ *is nonnegative.* (6.12)

Indeed, let $z_0 > 0$ be such a discontinuity point. As a counterpart to (5.22), w have by (6.5) that

$$(E(\varepsilon_+) - \varepsilon_+ g(\varepsilon_+)) - (E(\varepsilon_-) - \varepsilon_- g(\varepsilon_-)) + \frac{z_0^2}{2}(\varepsilon_+^2 - \varepsilon_-^2) \geq 0. \tag{6.13}$$

Combining (6.13) with (5.22) we obtain

$$(E(\varepsilon_+) - (E(\varepsilon_-) - \frac{1}{2}(g(\varepsilon_+) + g(\varepsilon_-))(\varepsilon_+ - \varepsilon_-) \geq 0, \tag{6.14}$$

which is precisely (6.12). Similarly, for $z_0 < 0$ the condition reads

$$(E(\varepsilon_+) - (E(\varepsilon_-) - \frac{1}{2}(g(\varepsilon_+) + g(\varepsilon_-))(\varepsilon_+ - \varepsilon_-) \leq 0. \tag{6.15}$$

No discontinuity can arise at $z_0 = 0$ by virtue of (5.22), indeed.

The dissipation condition (6.5) eliminates all non-physical solutions to the Riemann problem *if and only if* $g$ *has no inflection points*. We show here the "if" part which is easy. Examples of multiple solutions around the inflection point are quite technical and we refer the reader to [6]. We focus here on the following special case.

**Proposition 6.2.** *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a strictly convex continuously differentiable function. Then for every* $E_\pm, V_\pm \in \mathbb{R}$, *System* (5.16) *admits a unique regulated solution satisfying the dissipation condition* (6.5).

*Proof.* We choose an arbitrary real number $E_0$, solve the equation separately on $(-\infty, 0]$ and $[0, \infty)$ with boundary conditions $\bar{\varepsilon}(\pm\infty) = E_\pm$, $\bar{\varepsilon}(0) = E_0$, and show that there exists a unique $E_0$ such that the third condition in (5.16) is satisfied. Consider first the interval $[0, \infty)$ and assume that $\bar{\varepsilon}$ has a jump $\varepsilon_- := \bar{\varepsilon}(z_0-) \neq \varepsilon_+ := \bar{\varepsilon}(z_0+)$ at some point $z_0$. The segment connecting $(\varepsilon_-, g(\varepsilon_-))$ with $(\varepsilon_+, g(\varepsilon_+))$ is above the graph of $g$, since $g$ is convex. Hence, by (6.14), $\varepsilon_+ < \varepsilon_-$. In other words, the solution can *only jump down for* $z_0 > 0$. Furthermore, by (5.22), we have

$$g'(\varepsilon-) > z_0^2 > g'(\varepsilon_+)\,. \tag{6.16}$$

We find $\delta > 0$ and $z_1 < z_0 < z_2$ such that

$$\frac{g(\varepsilon_-) - g(\bar\varepsilon(z))}{\varepsilon_- - \bar\varepsilon(z)} > z_0^2 + \delta \quad \text{for } z \in (z_1, z_0)\,, \tag{6.17}$$

and

$$\frac{g(\varepsilon_+) - g(\bar\varepsilon(z))}{\varepsilon_+ - \bar\varepsilon(z)} < z_0^2 - \delta \quad \text{for } z \in (z_0, z_2)\,. \tag{6.18}$$

We now argue as in (5.18). In the identities

$$z^2(\bar\varepsilon(z) - \varepsilon_+) - (g(\bar\varepsilon(z)) - g(\varepsilon_+)) = \int_{z_0}^{z} 2s(\bar\varepsilon(s) - \varepsilon_+)\,\mathrm{d}s \qquad \text{for } z \in (z_0, z_2), \tag{6.19}$$

$$z^2(\varepsilon_- - \bar\varepsilon(z)) - (g(\varepsilon_-) - g(\bar\varepsilon(z))) = \int_{z}^{z_0} 2s(\varepsilon_- - \bar\varepsilon(s))\,\mathrm{d}s \qquad \text{for } z \in (z_1, z_0) \tag{6.20}$$

we use (6.17)–(6.18) and obtain

$$\delta|\bar\varepsilon(z) - \varepsilon_+| \le \int_{z_0}^{z} 2s|\bar\varepsilon(s) - \varepsilon_+|\,\mathrm{d}s \qquad \text{for } z \in (z_0, z_2), \tag{6.21}$$

$$\delta|\varepsilon_- - \bar\varepsilon(z)| \le \int_{z}^{z_0} 2s|\varepsilon_- - \bar\varepsilon(s)|\,\mathrm{d}s \qquad \text{for } z \in (z_1, z_0). \tag{6.22}$$

Hence, by Gronwall's argument, $\bar\varepsilon(z) = \varepsilon_-$ in $(z_1, z_0)$ and $\bar\varepsilon(z) = \varepsilon_+$ in $(z_0, z_2)$. We now choose maximal intervals $(z_1, z_0)$, $(z_0, z_2)$ with the properties (6.17)–(6.18). If $z_1 > 0$, then $\bar\varepsilon(z_1-) =: \varepsilon_* > \varepsilon_- > \varepsilon_+$ and

$$\frac{g(\varepsilon_*) - g(\varepsilon_-)}{\varepsilon_* - \varepsilon_-} = z_1^2 < z_0^2 = \frac{g(\varepsilon_-) - g(\varepsilon_+)}{\varepsilon_- - \varepsilon_+}\,, \tag{6.23}$$

which contradicts the convexity of $g$. We obtain the same contradiction assuming that $z_2 < \infty$. Hence, $\bar\varepsilon$ is constant in $(0, z_0)$ and $(z_0, \infty)$, and we have

$$\bar\varepsilon(z) = \begin{cases} E_0 & \text{for } 0 < z < z_0 := \sqrt{\frac{g(E_0) - g(E_+)}{E_0 - E_+}}, \\ E_+ & \text{for } z > z_0 \end{cases} \tag{6.24}$$

We see that if $\bar\varepsilon$ has a jump in $(0, \infty)$, then the jump points downwards and the solution is uniquely determined by the boundary conditions $E_0 > E_+$. If $E_0 < E_+$, a discontinuous solution therefore cannot exist. Smooth solutions are obtained by direct differentiation in (5.16), which yields

$$(z^2 - g'(\bar\varepsilon(z)))\bar\varepsilon'(z) = 0, \tag{6.25}$$

hence the solution is composed of *constant states* $\bar\varepsilon'(z) = 0$ and *rarefaction waves* $z^2 = g'(\bar\varepsilon(z))$. More specifically,

$$\bar\varepsilon(z) = \begin{cases} E_0 & \text{for } 0 < z < z_1 := \sqrt{g'(E_0)}, \\ (g')^{-1}(z^2) & \text{for } z_1 < z < z_2 := \sqrt{g'(E_+)}, \\ E_+ & \text{for } z > z_2. \end{cases} \tag{6.26}$$

The solution is obviously constant if $E_0 = E_+$. Note that we use in a substantial way the fact that $g$ is *strictly convex*.

In the interval $(-\infty, 0)$, the situation is opposite. We have a discontinuity (shock) if $E_0 > E_-$ and a rarefaction wave if $E_0 < E_-$, that is,

$$\bar\varepsilon(z) = \begin{cases} E_- & \text{for } z < z_0 := -\sqrt{\frac{g(E_0)-g(E_-)}{E_0-E_-}}, \\ E_0 & \text{for } 0 > z > z_0 \end{cases} \tag{6.27}$$

if $E_0 > E_-$, and

$$\bar\varepsilon(z) = \begin{cases} E_- & \text{for } z < z_1 := -\sqrt{g'(E_-)}, \\ (g')^{-1}(z^2) & \text{for } z_1 < z < z_2 := -\sqrt{g'(E_0)}, \\ E_0 & \text{for } 0 > z > z_2 \end{cases} \tag{6.28}$$

if $E_0 < E_-$.

It remains to check that $E_0$ can be chosen in a unique way to fulfill (5.16). In the situation of (6.28), for example, we have, by substituting $z = -\sqrt{g'(u)}$, that

$$\int_{z_1}^0 \bar\varepsilon(z)\,\mathrm{d}z = \int_{z_1}^{z_2} (g')^{-1}(z^2)\,\mathrm{d}z - z_2 E_0 = z_1 E_- - \int_{E_0}^{E_-} \sqrt{g'(u)}\,\mathrm{d}u\,. \tag{6.29}$$

This yields

$$\int_{-\infty}^0 (\bar\varepsilon(z) - P(z))\,\mathrm{d}z = \begin{cases} (E_0 - E_-)\sqrt{\frac{g(E_0)-g(E_-)}{E_0-E_-}} & \text{if } E_0 > E_-, \\ -\int_{E_0}^{E_-} \sqrt{g'(u)}\,\mathrm{d}u & \text{if } E_0 \le E_-. \end{cases} \tag{6.30}$$

Similarly,

$$\int_0^\infty (\bar\varepsilon(z) - P(z))\,\mathrm{d}z = \begin{cases} (E_0 - E_+)\sqrt{\frac{g(E_0)-g(E_+)}{E_0-E_+}} & \text{if } E_0 > E_+, \\ -\int_{E_0}^{E_+} \sqrt{g'(u)}\,\mathrm{d}u & \text{if } E_0 \le E_+. \end{cases} \tag{6.31}$$

We see that the function which with $E_0$ associates $\int_{-\infty}^\infty (\bar\varepsilon(z) - P(z))\,\mathrm{d}z$ is continuous, strictly increasing, and unbounded from above and from below, hence (5.16) has a unique solution. ∎

A uniqueness criterion for general functions $g$ is stated in [6]. It is based on a "maximal dissipation condition" and coincides with the vanishing viscosity criterion.

# 7 Wave propagation in hysteretic media

We have seen in the previous section that waves in nonlinear media always develop singularities in finite time. If hysteresis is present, the system still preserves the property of bounded speed of propagation, which characterizes hyperbolicity, but singularities do not occur due to specific dissipation properties of hysteresis operators.

Let us start with an easy ODE problem.

**Proposition 7.1.** *Consider a mapping $S : C([0,T];X) \to C([0,T];X)$, where $X$ is a Banach space. Assume that there exists a function $\ell \in L^1(0,T)$ such that for all $u_1, u_2 \in C([0,T];X)$ and $t \in [0,T]$ we have*

$$|S[u_1](t) - S[u_2](t)| \le \ell(t)\,|u_1 - u_2|_{[0,t]}\,. \tag{7.1}$$

*Then the differential equation*

$$\dot u(t) = S[u](t)\,, \quad u(0) = u^0 \tag{7.2}$$

*has a unique solution $u \in C^1([0,T];X)$ for every $u^0 \in X$.*

*Proof.* We proceed by the Banach contraction principle. Let $\Phi : C([0,T];X) \to C([0,T];X)$ be the mapping

$$\Phi[u](t) = u^0 + \int_0^t S[u](\tau)\,\mathrm{d}\tau\,, \tag{7.3}$$

and let $L(t) = \int_0^t \ell(\tau)\,\mathrm{d}\tau$. We define in $C([0,T];X)$ the norm

$$|||u||| = \sup_{t\in[0,T]} \left(\mathrm{e}^{-L(t)}\,|u|_{[0,t]}\right). \tag{7.4}$$

For $u_1, u_2 \in C([0,T];X)$, we have by (7.1) that

$$\begin{aligned} |\Phi[u_1](t) - \Phi[u_2](t)| &\le \int_0^t |S[u_1](\tau) - S[u_2](\tau)|\,\mathrm{d}\tau \le \int_0^t \ell(\tau)\,|u_1 - u_2|_{[0,\tau]}\,\mathrm{d}\tau \\ &\le |||u_1 - u_2|||\int_0^t \ell(\tau)\,\mathrm{e}^{L(\tau)}\,\mathrm{d}\tau\,. \end{aligned} \tag{7.5}$$

hence,

$$\mathrm{e}^{-L(t)}\,|\Phi[u_1] - \Phi[u_2]|_{[0,t]} \le |||u_1 - u_2|||\int_0^t \ell(\tau)\,\mathrm{e}^{L(\tau)-L(t)}\,\mathrm{d}\tau = \left(1 - \mathrm{e}^{-L(t)}\right)|||u_1 - u_2|||\,. \tag{7.6}$$

We see that $\Phi$ is a contraction with respect to the norm $|||\cdot|||$, which we wanted to prove. ∎

As a model problem to study wave propagation in hysteretic media, we consider for $(x,t) \in (0,1)\times(0,T)$ the system

$$\varrho v_t = \sigma_x\,, \tag{7.7}$$

$$\varepsilon_t = v_x\,, \tag{7.8}$$

$$\varepsilon(x,t) = P[\sigma(x,\cdot)](t)\,, \tag{7.9}$$

where $\varrho > 0$, $\ell > 0$, and $T > 0$ are fixed constants, and $P : W^{1,2}(0,T) \to W^{1,2}(0,T)$ is a constitutive operator whose properties are specified below in Hypothesis 7.3. We prescribe boundary conditions

$$v(0,t) = 0\,, \tag{7.10}$$

$$\sigma(1,t) = \sigma_*(t)\,, \tag{7.11}$$

where $\sigma_1$ is a given time dependent external load, and initial conditions

$$v(x,0) = v_0(x)\,, \tag{7.12}$$

$$\sigma(x,0) = \sigma_0(x)\,. \tag{7.13}$$

Before we state and prove the existence result, let us show that under natural assumptions on the operator $F$, system (7.7)–(7.9) is hyperbolic in the sense of bounded propagation of waves.

**Proposition 7.2.** *Let there exist a potential energy operator $V$ such that*

(i) *there exists a constant $c > 0$ such that $V[\sigma](t) \geq \frac{c}{2}\sigma^2(t)$,*

(ii) *the energy inequality $\sigma(t)\frac{\mathrm{d}}{\mathrm{d}t}P[\sigma](t) \geq \frac{\mathrm{d}}{\mathrm{d}t}V[\sigma](t)$ holds*

*for every $\sigma \in W^{1,2}(0,T)$ and a.e. $t \in (0,T)$. Let $[a,b] \subset \mathbb{R}$ be an interval such that that the initial conditions vanish in $[a,b]$. Then every solution satisfying system (7.7)–(7.9) almost everywhere vanishes in the whole triangle*

$$\Omega = \left\{(x,t) \in \mathbb{R} \times (0,\infty) : a + \frac{t}{\sqrt{c\varrho}} < x < b - \frac{t}{\sqrt{c\varrho}}\right\}. \tag{7.14}$$

Proposition 7.2 states that that the speed of propagation does not exceed $1/\sqrt{c\varrho}$. Note that conditions (i), (ii) in Proposition 7.2 are satisfied if $P = G$ is the fatigue operator (1.11). The energy inequality is derived in (1.16) and $c = g(0)$. If $P[\sigma] = c\sigma + F[\sigma,\lambda]$, where $F$ is the Preisach operator (3.18) with a fixed initial memory $\lambda$, then $V[\sigma] = \frac{c}{2}\sigma^2 + U[\sigma,\lambda]$, where $U$ is given by (3.25), and we see again that the hypotheses of Proposition 7.2 hold.

*Proof of Proposition 7.2.* For $T < \frac{1}{2}\sqrt{c\varrho}(b-a)$ set $\Omega_T = \{(x,t) \in \Omega : t < T\}$. We easily check that the function $L(x,t) = E_t(x,t) - Q_x(x,t)$, where

$$E(x,t) := \frac{\varrho}{2}v^2 + V[\sigma]\,, \quad Q(x,t) = v\sigma\,,$$

is nonnegative on $\Omega$ by virtue of assumption (ii). Let us denote by $Z(x,t)$ the vector $(-Q(x,t), E(x,t))$. We integrate $L(x,t)$ over $\Omega_T$ with a fixed $T$ and obtain, by Gauss Theorem, that

$$\begin{aligned} 0 &\geq \int_{\Omega_T} L(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Gamma_0} (n_0 \cdot Z)\,\mathrm{d}S + \int_{\Gamma_T} (n_T \cdot Z)\,\mathrm{d}S + \int_{\Gamma_+} (n_+ \cdot Z)\,\mathrm{d}S + \int_{\Gamma_-} (n_- \cdot Z)\,\mathrm{d}S\,, \end{aligned}$$

where $\Gamma_0 = \{(x,0) : x \in (a,b)\}$, $\Gamma_T = \{(x,T) : x \in (a+T/\sqrt{c\varrho}, b-T/\sqrt{c\varrho})\}$, $\Gamma_+ = \{(x,t) \in \partial\Omega : x = a + t/\sqrt{c\varrho}, t \in (0,T)\}$, $\Gamma_- = \{(x,t) \in \partial\Omega : x = b - t/\sqrt{c\varrho}, t \in (0,T)\}$, and $n_q$ is the unit outward normal to $\Gamma_q$, $q \in \{0, T, +, -\}$. We have $n_0 = (0,-1)$, $n_T = (0,1)$, $n_+ = \sqrt{1/(1+c\varrho)^2}(-\sqrt{c\varrho}, 1)$, $n_- = \sqrt{1/(1+c\varrho)^2}(\sqrt{c\varrho}, 1)$.
By hypothesis, we have $E(x,t) \ge \frac{\varrho}{2}v^2 + \frac{c}{2}\sigma^2 \ge \sqrt{c\varrho}Q(x,t)$ for all $(x,t) \in \bar\Omega$, hence

$$\int_{\Gamma_\pm} (n_\pm \cdot Z)\,\mathrm{d}S \ge 0\,.$$

Furthermore, since the initial conditions vanish, the integral over $\Gamma_0$ is zero, and we obtain

$$\int_{\Gamma_T} \left(\frac{\varrho}{2}v^2 + \frac{c}{2}\sigma^2\right)(x,T)\,\mathrm{d}x \le 0\,.$$

The level $T$ has been chosen arbitrarily. We conclude that $v = \sigma \equiv 0$ in $\Omega$, and the proof is complete. ∎

The existence of solutions to system (7.7)–(7.13) will be proved under the following hypotheses.

**Hypothesis 7.3.** *The mapping $P : W^{1,2}(0,T) \to W^{1,2}(0,T)$ is rate independent, and*

(i) *admits a locally Lipschitz continuous extension to $P : C[0,T] \to C[0,T]$;*

(ii) *its inverse $P^{-1} : C[0,T] \to C[0,T]$ satisfies the Lipschitz condition (7.1) with $\ell$ independent of $t$;*

(iii) *there exists $b > 0$ such that $P$ is strictly locally monotone and locally counterclockwise convex with bound $b$.*

Consider any $\sigma \in W^{1,2}(0,T)$, $h > 0$, and $\tau \in (0,T)$. Put $\sigma_1(t) = \sigma(t)$ for $t \le \tau$, $\sigma_1(t) = \sigma(\tau)$ for $t > \tau$, $\sigma_2(t) = \sigma(t)$ for $t \le \tau - h$, $\sigma_2(t) = \sigma(\tau - h)$ for $t > \tau - h$. Letting $h \to 0+$, we obtain from the hypotheses that

$$|dot\sigma(t)| \le \ell|\dot\varepsilon(t)|\,, \quad |\dot\varepsilon(t)| \le C(|\sigma|_{[0,t]})|\dot\sigma(t)| \ \text{ a.e.}\,, \tag{7.15}$$

where $\varepsilon = P[\sigma]$, and $C(|\sigma|_{[0,t]})$ is a nondecreasing function of $|\sigma|_{[0,t]}$.
Here again, the conditions are satisfied if $P = G$ is the fatigue operator (1.11). For operators of the form $P[\sigma] = c\sigma + F[\sigma, \lambda]$, where $F$ is the Preisach operator (3.18) with a fixed initial memory $\lambda$, conditions (i) and (iii) are proved in Section 3. The proof of Lipschitz continuity of the inverse is more involved, see [1, Theorem 2.11.20] or [3, Theorem II.3.17]. The existence result reads as follows.

**Theorem 7.4.** *Let Hypothesis 7.3 be satisfied, let $\sigma_0, v_0 \in W^{1,2}(0,1)$ and $\sigma_* \in W^{2,2}(0,T)$ be given such that $|\sigma_0(x)| < b/2$, and let the compatibility condition $\sigma_0(1) = \sigma_*(0)$ hold. Then there exists $\delta > 0$ such that if*

$$\int_0^1 |v_0'(x)|^2\,\mathrm{d}x + \int_0^1 |\sigma_0'(x)|^2\,\mathrm{d}x + \int_0^T |\ddot\sigma_*(t)|^2\,\mathrm{d}t < \delta^2 b^2\,, \tag{7.16}$$

*then system (7.7)–(7.13) admits a solution such that $v_t, v_x, \sigma_t, \sigma_x \in L^\infty(0,T; L^2(0,1))$.*

The smallness of the data guarantees that the solution does nor leave the convexity domain of $P$. If $P$ is globally convex, that is, $b = \infty$, then no smallness is required.

*Proof.* We discretize the system in space. For a fixed $n \in \mathbb{N}$ and $j = 0, 1, \dots, n$ set

$$v_j^0 = v_0(j/n)\,, \quad \sigma_j^0 = \sigma_0(j/n)\,, \tag{7.17}$$

and consider for $t \in [0,T]$ and $j = 1, 2, \dots, n-1$ the system

$$\begin{aligned} \varrho \dot{v}_j(t) &= n(\sigma_{j+1}(t) - \sigma_j(t))\,, && (7.18)\\ \dot{\varepsilon}_j(t) &= n(v_j(t) - v_{j-1}(t))\,, && (7.19)\\ \varepsilon_j(t) &= P[\sigma_j](t)\,, && (7.20) \end{aligned}$$

with initial conditions $v_j(0) = v_j^0$, $\sigma_j(0) = \sigma_j^0$, and "boundary conditions" $v_0(t) = 0$, $\sigma_n(t) = \sigma_*(t)$. Using the Lipschitz continuity of $P^{-1}$ and Proposition 7.1, we see that for every $n \in \mathbb{N}$, system (7.18)–(7.20) with the prescribed data has a unique solution such that $\ddot{v}_j, \ddot{\varepsilon}_j$ belong to $L^2(0,T)$, and the identities

$$\begin{aligned} \varrho \ddot{v}_j(t) &= n(\dot{\sigma}_{j+1}(t) - \dot{\sigma}_j(t))\,, && (7.21)\\ \ddot{\varepsilon}_j(t) &= n(\dot{v}_j(t) - \dot{v}_{j-1}(t))\,, && (7.22) \end{aligned}$$

hold a.e. in $(0,T)$ for $j = 1, \dots, n-1$. We now test (7.21) by $\dot{v}(t)$, (7.22) by $\dot{\sigma}(t)$, and sum over $j = 1, \dots, n-1$ to obtain

$$\frac{1}{n}\sum_{j=1}^{n-1}(\varrho \ddot{v}_j(t)\dot{v}_j(t) + \ddot{\varepsilon}_j(t)\dot{\sigma}_j(t)) = \dot{\sigma}_*(t)\dot{v}_{n-1}(t)\,. \tag{7.23}$$

For every $n \in \mathbb{N}$ there exists $T_n > 0$ such that $|\sigma_j(t)| < b$ for $j = 1, \dots, n$ and $t \in [0, T_n]$. In this interval, the operator $P$ is counterclockwise convex, and integrating (7.23) from 0 to some $t^* \le T_n$, we obtain from (1.22) that

$$\frac{1}{n}\sum_{j=1}^{n-1}\left(\frac{\varrho}{2}\dot{v}_j^2(t^*) + \frac{1}{2}\dot{\varepsilon}_j(t^*)\dot{\sigma}_j(t^*)\right) \le \frac{1}{n}\sum_{j=1}^{n-1}\left(\frac{\varrho}{2}\dot{v}_j^2(0) + \frac{1}{2}\dot{\varepsilon}_j(0)\dot{\sigma}_j(0+)\right) + \int_0^{t^*} \dot{\sigma}_*(t)\dot{v}_{n-1}(t)\,\mathrm{d}t\,. \tag{7.24}$$

We have by (7.15) that $(\dot{\sigma}_j(t))^2 \le \ell\dot{\varepsilon}_j(t)\dot{\sigma}_j(t) \le \ell^2\dot{\varepsilon}_j^2(t)$ a.e., hence

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{n-1}\left(\frac{\varrho}{2}\dot{v}_j^2(t^*) + \frac{1}{2\ell}\dot{\sigma}_j^2(t^*)\right) &\le n\sum_{j=1}^{n-1}\left(\frac{1}{2\varrho}(\sigma_{j+1}^0 - \sigma_j^0)^2 + \frac{\ell}{2}(v_j^0 - v_{j-1}^0)^2\right)\\ &\quad + \int_0^{t^*} \dot{\sigma}_*(t)\dot{v}_{n-1}(t)\,\mathrm{d}t\,,\\ &\le \int_0^1 \left(\frac{1}{2\varrho}|\sigma_0'(x)|^2 + \frac{\ell}{2}|v_0'(x)|^2\right)\mathrm{d}x + \int_0^{t^*} \dot{\sigma}_*(t)\dot{v}_{n-1}(t)\,\mathrm{d}t\,. \qquad (7.25) \end{aligned}$$

On $[0,T_n]$, again by (7.15), there exists $C_b$ such that $|\dot\varepsilon_j(t)| \le C_b|\dot\sigma_j(t)|$ a.e. Hence,

$$
\begin{aligned}
n\sum_{j=1}^{n-1}&\left(\frac{1}{2\varrho}(\sigma_{j+1}-\sigma_j)^2(t^*)+\frac{1}{2C_b}(v_j-v_{j-1})^2(t^*)\right)\\
&\le \int_0^1\left(\frac{1}{2\varrho}|\sigma_0'(x)|^2+\frac{\ell}{2}|v_0'(x)|^2\right)\mathrm{d}x+\int_0^{t^*}\dot\sigma_*(t)\dot v_{n-1}(t)\,\mathrm{d}t\,. \qquad (7.26)
\end{aligned}
$$

Assume now that (7.16) holds with some $\delta$ that will be determined later. We denote by $C_1, C_2, \dots$ constants independent of $\delta$.

The last term in (7.26) can be estimated using (7.16) and integration by parts with respect to $t$ by

$$
\int_0^{t^*}\dot\sigma_*(t)\dot v_{n-1}(t)\,\mathrm{d}t \le C_1\delta\,|\dot v_{n-1}|_{[0,t^*]}\,. \qquad (7.27)
$$

This and (7.26) yield

$$
n\sum_{j=1}^{n-1}(v_j-v_{j-1})^2(t^*) \le C_2\delta\left(\delta+\left|\sum_{j=1}^{n-1}|v_j-v_{j-1}|\right|_{[0,t^*]}\right). \qquad (7.28)
$$

Since $t^*\in[0,T_n]$ is arbitrary, we obtain

$$
\left|n\sum_{j=1}^{n-1}(v_j-v_{j-1})^2\right|_{[0,t^*]} \le C_3\delta\left(\delta+\left(\left|n\sum_{j=1}^{n-1}(v_j-v_{j-1})^2\right|_{[0,t^*]}\right)^{1/2}\right), \qquad (7.29)
$$

Hence, setting $v_n = v_{n-1}+(1/n)\frac{\mathrm{d}}{\mathrm{d}t}P^{-1}[\sigma_*]$ in agreement with (7.19)–(7.20), we have

$$
\left|n\sum_{j=1}^{n}(v_j-v_{j-1})^2\right|_{[0,t^*]} \le C_4\delta^2\,. \qquad (7.30)
$$

From (7.27)–(7.28), and putting $\sigma_0=\sigma_1$, which is compatible with (7.18)), we derive the bound

$$
\left|n\sum_{j=1}^{n}(\sigma_j-\sigma_{j-1})^2\right|_{[0,t^*]} \le C_5\delta^2\,. \qquad (7.31)
$$

Furthermore,

$$
\left|\frac{1}{n}\sum_{j=1}^{n}\left(\dot\sigma_j^2+\dot v_{j-1}^2\right)\right|_{[0,t^*]} \le C_6\delta^2\,. \qquad (7.32)
$$

We now define piecewise linear and piecewise constant interpolates

$$
\begin{aligned}
v^{(n)}(x,t) &= v_{j-1}(t)+n(x-(j-1)/n)(v_j(t)-v_{j-1}(t))\,, && (7.33)\\
\sigma^{(n)}(x,t) &= \sigma_{j-1}(t)+n(x-(j-1)/n)(\sigma_j(t)-\sigma_{j-1}(t))\,, && (7.34)\\
\bar v^{(n)}(x,t) &= v_{j-1}(t)\,, && (7.35)\\
\bar\sigma^{(n)}(x,t) &= \sigma_j(t) && (7.36)
\end{aligned}
$$

for $x \in [(j-1)/n, j/n)$, continuously extended to $x = 1$. By virtue of the above estimates, the functions $v_x^{(n)}, v_t^{(n)}, \sigma_x^{(n)}, \sigma_t^{(n)}$ are bounded above in $L^\infty(0, T_n; L^2(0,1))$ by $C_7\delta$. In particular, $|\sigma^{(n)}(x,t)| \le b/2 + C_8\delta$. Choosing $\delta < b/(2C_8)$, we can take $T_n = T$ and obtain bounded sequences $v_x^{(n)}, v_t^{(n)}, \sigma_x^{(n)}, \sigma_t^{(n)}$ in $L^\infty(0, T_n; L^2(0,1))$. By compact embedding into $C([0,1]\times[0,T])$, we can choose subsequences, still labeled by $n$, such that $v_x^{(n)}, v_t^{(n)}, \sigma_x^{(n)}, \sigma_t^{(n)}$ converge weakly-star in $L^\infty(0,T; L^2(0,1))$ to $v_x, v_t, \sigma_x, \sigma_t$, respectively, and $v^{(n)}$ converge to $v$, $\sigma^{(n)}$ converge to $\sigma$ strongly in $C([0,1]\times[0,T])$ as $n \to \infty$. The approximations satisfy the system

$$\varrho \bar{v}_t^{(n)} = \sigma_x^{(n)}, \tag{7.37}$$

$$\bar{\varepsilon}_t^{(n)} = v_x^{(n)}, \tag{7.38}$$

$$\bar{\varepsilon}^{(n)}(x,t) = P[\bar{\sigma}^{(n)}(x,\cdot)](t). \tag{7.39}$$

Using the fact that

$$|\sigma^{(n)}(x,t) - \bar{\sigma}^{(n)}(x,t)|^2 \le \max_j |\sigma_j(t) - \sigma_{j-1}(t)|^2 \le \frac{C_9}{n},$$

we can pass to the limit and check that $\sigma, v$ are the desired solutions to (7.7)–(7.13). ■

# 8 A model problem motivated by porous media flow

Consider a bounded open domain $\Omega \subset \mathbb{R}^n$ representing a porous medium partially filled with liquid, and a time interval $[0,T]$. We assume that $\Omega$ has Lipschitzian boundary. We describe the flow of a fluid in terms of a scalar quantity $s(x,t) \in [0,1]$ (the saturation) and a vector $q(x,t)$ (the mass flux). The mass conservation principle reads

$$s_t + \operatorname{div} q = 0 \tag{8.1}$$

in $\Omega \times (0,T)$. For the mass flux, we assume for simplicity the linear Darcy law

$$q = -\kappa \nabla u, \tag{8.2}$$

where $u(x,t)$ is the pressure. In the constitutive relation between the pressure and the saturation, strong hysteresis effects are observed, see [9], and we assume that they can be described by the Preisach operator

$$s(x,t) = F[u(x,\cdot), \lambda(x,\cdot)](t), \tag{8.3}$$

generated by a function $\mu$ as in the previous section, with initial memory states $\lambda(x,\cdot) \in \Lambda$, which may vary from one point $x \in \Omega$ to another. By virtue of (3.19), for a fixed $\lambda$, the operator $F[\cdot,\lambda]$ is Lipschitz continuous from $L^p(\Omega; C[0,T])$ to $L^p(\Omega; C[0,T])$ for every $1 \le p \le \infty$.

We thus formally have the equation

$$F[u,\lambda]_t - \kappa \Delta u = 0, \tag{8.4}$$

which we couple with the boundary condition

$$q \cdot \nu = \alpha(u - p)\,, \tag{8.5}$$

on $\partial\Omega$, where $\nu$ is the unit outward normal vector, $p \in L^\infty(\partial\Omega \times (0,T))$ is a prescribed outer pressure, and $\alpha > 0$ is a constant. Hence, if the outer pressure $p$ is higher than the inner pressure $u$, the fluid flows in and vice versa. Finally, we prescribe an initial condition

$$u(x,0) = u_0(x)\,. \tag{8.6}$$

We make the following hypotheses.

**Hypothesis 8.1.** *The following conditions are assumed to hold.*

(i) *The generating function $\mu$ of the Preisach operator $F$ satisfies the conditions of Theorem 3.4;*

(ii) *We have $\lambda \in L^\infty(\Omega \times (0,\infty))$, $\lambda(x,\cdot) \in \Lambda$ for a.e. $x \in \Omega$, $\lambda(\cdot,r) = 0$ a.e. for all $r \ge \varrho$;*

(iii) *The function $p_t$ belongs to $L^2(\partial\Omega \times (0,T))$, and $|p(x,t)| \le \varrho$ a.e.;*

(iv) *The function $u^0$ belongs to $L^\infty(\Omega) \cap W^{1,2}(\Omega)$, $|u^0(x)| \le \varrho$ a.e., and the compatibility condition*

$$\kappa \int_\Omega \nabla u^0(x) \nabla\varphi(x)\,\mathrm{d}x + \alpha \int_{\partial\Omega} (u^0(x) - p(x,0))\varphi(x)\,\mathrm{d}S(x) = 0$$

*holds for every $\varphi \in W^{1,2}(\Omega)$.*

The difficulty of the problem consists in the fact that the ratio $F[u,\lambda]_t/u_t$ vanishes at turning points of $u$, so that an estimate for $F[u,\lambda]_t$ does not imply anything for $u_t$, which we need to get under control. This is why the compatibility condition is needed, and why we only construct the solution in the convexity domain of $F$, where additional estimates are available. The result reads as follows.

**Theorem 8.2.** *Let Hypothesis 8.1 hold. Then there exists a unique $u \in L^2(\Omega; C[0,T])$ satisfying the initial condition* (8.6) *a.e. in $\Omega$, $\nabla u, \nabla u_t \in L^2(\Omega \times (0,T))$, $|u(x,t)| \le \varrho$ a.e., and such that for all $\varphi \in W^{1,2}(\Omega)$ we have*

$$\int_\Omega \big(F[u,\lambda]_t(x,t)\varphi(x) + \kappa\nabla u \nabla\varphi\big)\,\mathrm{d}x + \alpha\int_{\partial\Omega}(u(x,t) - p(x,t))\varphi(x)\,\mathrm{d}S(x) = 0\,. \tag{8.7}$$

*Proof.* Instead of the operator $F$, we consider its convexification $F_\varrho$ as in Corollary 3.5. We first prove that if a solution of the problem

$$\int_\Omega \big(F_\varrho[u,\lambda]_t(x,t)\varphi(x) + \kappa\nabla u \nabla\varphi\big)\,\mathrm{d}x + \alpha\int_{\partial\Omega}(u(x,t) - p(x,t))\varphi(x)\,\mathrm{d}S(x) = 0 \tag{8.8}$$

for every test function $\varphi \in W^{1,2}(\Omega)$ with initial condition (8.6) exists, then it is necessarily unique and satisfies the bound $|u(x,t)| \le \varrho$ a.e., so that it is the desired solution of problem (8.7) as well.

Assume that there exist two functions $u_1, u_2$ with the required regularity that satisfy (8.8). We subtract the two equations and test by $\varphi = H_\varepsilon(u_1 - u_2)$, where $H_\varepsilon$ is an approximation of the Heaviside function defined as

$$H_\varepsilon(z) = \max\left\{0, \min\left\{\frac{z}{\varepsilon}, 1\right\}\right\}.$$

Since $H_\varepsilon$ is nondecreasing, we have

$$\int_\Omega \kappa\nabla(u_1 - u_2)\nabla H_\varepsilon(u_1 - u_2)\,\mathrm{d}x + \alpha\int_{\partial\Omega}(u_1 - u_2)H_\varepsilon(u_1 - u_2)\,\mathrm{d}S(x) \ge 0,$$

so that

$$\int_\Omega \frac{\partial}{\partial t}(F_\varrho[u_1,\lambda] - F_\varrho[u_2,\lambda])H_\varepsilon(u_1 - u_2)\,\mathrm{d}x \le 0.$$

Letting $\varepsilon$ tend to $0+$, we obtain e.g. by the Lebesgue dominated convergence theorem that

$$\int_\Omega \frac{\partial}{\partial t}(F_\varrho[u_1,\lambda] - F_\varrho[u_2,\lambda])H(u_1 - u_2)\,\mathrm{d}x \le 0.$$

By Proposition 3.6 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_0^\infty \left(g(r,\xi_r^1(x,t)) - g(r,\xi_r^2(x,t))\right)^+\,\mathrm{d}r \le 0 \quad \text{a.e.}$$

Hence, $\xi_r^1(x,t) \le \xi_r^2(x,t)$ a.e. for all $r > 0$ and, in particular, $u_1(x,t) \le u_2(x,t)$ a.e. Interchanging the roles of $u_1, u_2$, we obtain $u_1 = u_2$.

To prove the upper bound, we proceed in a similar way, testing the equation (8.8) by $H_\varepsilon(u(x,t) - \varrho)$. The argument is as above, with $u_1(x,t) = u(x,t)$, $u_2(x,t) \equiv \varrho$. We then have $\xi_r^2(x,t) \equiv (\varrho - r)^+$, and the inequality $\xi_r^1(x,t) \le \xi_r^2(x,t)$ a.e. for all $r > 0$ means in particular that $u(x,t) \le \varrho$. The inequality $u(x,t) \ge -\varrho$ is obtained by testing the equation by $H(-u(x,t) - \varrho)$.

This illustrates the importance of Hilpert's inequality. It implies at the same time both *uniqueness and the maximum principle*.

A solution to (8.8), (8.6) can be constructed by Galerkin approximations, see, e.g., [8]. We choose a basis in $W^{1,2}(\Omega)$ of eigenfunctions $\psi_k$ of the problem

$$\kappa\int_\Omega \nabla\psi_k\nabla\varphi\,\mathrm{d}x + \alpha\int_{\partial\Omega}\psi_k\varphi\,\mathrm{d}S(x) = \lambda_k\int_\Omega \psi_k\varphi\,\mathrm{d}x \tag{8.9}$$

for every $\varphi \in W^{1,2}(\Omega)$, $k = 0, 1, 2, \ldots$, and such that the system $\{\psi_k\}$ is orthonormal in $L^2(\Omega)$. Given $m \in \mathbb{N}$, we look for an approximate solution in the form

$$u^{(m)}(x,t) = \sum_{j=0}^m u_j(t)\psi_j(x), \tag{8.10}$$

satisfying the system

$$\frac{1}{m}\dot{u}_k(t) + \int_\Omega \big(F_\varrho[u^{(m)},\lambda]_t(x,t)\psi_k(x) + \kappa\nabla u^{(m)}\nabla\psi_k\big)\,\mathrm{d}x$$
$$+\alpha\int_{\partial\Omega}(u^{(m)}(x,t)-p(x,t))\psi_k(x)\,\mathrm{d}S(x) = 0 \tag{8.11}$$

for all $k = 0,1,\ldots,m$, with initial condition

$$u_k(0) = \int_\Omega u^0(x)\psi_k(x)\,\mathrm{d}x\,. \tag{8.12}$$

This is a system of $m+1$ first order ODEs with a hysteresis operator under the time derivative. Keeping still $m$ fixed, we approximate it by a second order system

$$\varepsilon\ddot{u}_k(t) + \frac{1}{m}\dot{u}_k(t) + \int_\Omega \big(F_\varrho[u^{(m)},\lambda]_t(x,t)\psi_k(x) + \kappa\nabla u^{(m)}\nabla\psi_k\big)\,\mathrm{d}x$$
$$+\alpha\int_{\partial\Omega}(u^{(m)}(x,t)-p(x,t))\psi_k(x)\,\mathrm{d}S(x) = 0 \tag{8.13}$$

for all $k = 0,1,\ldots,m$, with $\varepsilon > 0$, and with an additional initial condition

$$\dot{u}_k(0) = 0\,, \tag{8.14}$$

with the intention to pass to the limit as $\varepsilon \to 0+$. First, observe that (8.13) can be written as a first order system

$$\varepsilon\dot{u}_k(t) + \frac{1}{m}u_k(t) + \int_\Omega F_\varrho[u^{(m)},\lambda](x,t)\psi_k(x)\,\mathrm{d}x = v_k(t)\,, \tag{8.15}$$

$$\dot{v}_k(t) + \kappa\int_\Omega \nabla u^{(m)}\nabla\psi_k\,\mathrm{d}x + \alpha\int_{\partial\Omega}(u^{(m)}-p)\psi_k(x)\,\mathrm{d}S(x) = 0\,, \tag{8.16}$$

$k = 0,1,\ldots,m$. This is a system of ODEs with a Lipschitz continuous right hand side, hence it admits for every $\varepsilon > 0$ a unique solution.

In order to pass to the limit as $\varepsilon \to 0+$, let us evaluate first $\dot{v}_k(0)$. It follows from (8.12) that

$$\kappa\int_\Omega \nabla u^{(m)}(x,0)\nabla\psi_k\,\mathrm{d}x + \alpha\int_{\partial\Omega}(u^{(m)}(x,0)-p(x,0))\psi_k(x)\,\mathrm{d}S(x)$$
$$= \sum_{j=0}^m \int_\Omega u^0\psi_j\,\mathrm{d}x\left(\kappa\int_\Omega\nabla\psi_j\nabla\psi_k\,\mathrm{d}x + \alpha\int_{\partial\Omega}\psi_j\psi_k(x)\,\mathrm{d}S(x)\right) - \alpha\int_{\partial\Omega}p(x,0)\psi_k(x)\,\mathrm{d}S(x)\,.$$

We have

$$\kappa\int_\Omega\nabla\psi_j\nabla\psi_k\,\mathrm{d}x + \alpha\int_{\partial\Omega}\psi_j\psi_k(x)\,\mathrm{d}S(x) = \begin{cases} 0 & \text{for } j \neq k\,, \\ \lambda_k & \text{for } j = k\,. \end{cases}$$

Hence,

$$\begin{aligned}
&\kappa \int_\Omega \nabla u^{(m)}(x,0)\nabla\psi_k \,\mathrm{d}x + \alpha \int_{\partial\Omega} (u^{(m)}(x,0) - p(x,0))\psi_k(x)\,\mathrm{d}S(x) \\
&= \lambda_k \int_\Omega u^0 \psi_k \,\mathrm{d}x - \alpha \int_{\partial\Omega} p(x,0)\psi_k(x)\,\mathrm{d}S(x) \\
&= \kappa \int_\Omega \nabla u^0(x)\nabla\psi_k(x)\,\mathrm{d}x + \alpha \int_{\partial\Omega} (u^0(x) - p(x,0))\psi_k(x)\,\mathrm{d}S(x)\,.
\end{aligned}$$

The rightmost expression in the above identities is zero by virtue of the compatibility condition in Hypothesis 8.1 (iv). Hence, $\dot v_k(0) = 0$ for every $\varepsilon > 0$.

We now test (8.13) by $\dot u_k(t)$ and sum up over $k = 0, 1, \dots, m$. Using the fact that $F_\varrho[u^{(m)}, \lambda]_t u_t^{(m)} \ge 0$ a.e., we obtain the estimate

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=0}^m |\dot u_k(t)|^2\right) + \frac{1}{m}\sum_{k=0}^m |\dot u_k(t)|^2 + + \frac{\kappa}{2}\int_\Omega |\nabla u^{(m)}|^2\,\mathrm{d}x \\
&\qquad + \frac{\alpha}{2}\int_{\partial\Omega} |u^{(m)}(x,t) - p(x,t)|^2\,\mathrm{d}S(x) \le \alpha \int_{\partial\Omega} |p_t|\,|u^{(m)}|\,\mathrm{d}S(x), \qquad (8.17)
\end{aligned}$$

which yields that $u^{(m)}$ are bounded in $L^2(0,T; W^{1,2}(\Omega))$ independently of $\varepsilon$ and $m$, and $u_t^{(m)}$ are bounded in $L^2(\Omega \times (0,T))$ independently of $\varepsilon$, but the bound possibly depends on $m$. Keeping $m$ fixed, we thus can pass to the limit as $\varepsilon \to 0+$, and check that the limit function, still denoted by $u^{(m)}$, satisfies (8.11). Note that we have used the compact embedding of $W^{1,2}(\Omega \times (0,T))$ in $L^2(\Omega; C[0,T])$, and the continuity of $F$ in $L^2(\Omega; C[0,T])$. The operator $u^{(m)} \mapsto w^{(m)} = \frac{1}{m}u^{(m)} + F[u^{(m)}, \lambda]$ possesses the global convexity property, which yields the estimate

$$\int_0^t w^{(m)}(x,\tau)_{tt} u_t^{(m)}(x,\tau)\,\mathrm{d}\tau \ge \frac{1}{2}(w_t^{(m)}(x,t)u_t^{(m)}(x,t) - w_t^{(m)}(x,0)u_t^{(m)}(x,0))$$

a.e. In our case, we have in (8.15)–(8.16) for $\varepsilon = 0$ that $\dot v_k(0) = 0$, hence $w_t^{(m)}(x,0) = u_t^{(m)}(x,0) = 0$. Differentiating (8.11) with respect to $t$, testing by $\dot u_k(t)$, and summing up over $k = 0, 1, \dots, m$, we thus obtain the estimate

$$\int_\Omega |\nabla u_t^{(m)}|^2\,\mathrm{d}x + \int_{\partial\Omega} |u_t^{(m)}|^2\,\mathrm{d}S(x) \le C\,,$$

with a constant $C$ independent of $m$. This enables us, possibly after selecting a subsequence, to pass to the limit in (8.11) as $m \to \infty$, and check that the limit function $u$ is a solution to (8.8) using the compactness of $\{u^{(m)}\}$ in $L^2(\Omega; C[0,T])$. This completes the proof. ■

The compatibility condition in Hypothesis 8.1 (iv) seems necessary. Otherwise, we are not able to keep the initial time derivative under control due to the degeneracy of the problem.

# 9 Temperature dependence

The mechanical energy dissipated during a physical process does not disappear, but is transformed into heat. We show that this is compatible with the Preisach model introduced in Section 3. Still in 1D, for simplicity, we characterize the state by three state variables: stress $\sigma \in \mathbb{R}$, absolute temperature $\theta > 0$, and memory configuration $\lambda \in \Lambda$, cf. (3.3). At each memory level $r > 0$, we assume a constitutive relation for the partial strain $\varepsilon_r$ in the form

$$\varepsilon_r(t) = g(r, \lambda(r,t), \theta(t)), \tag{9.1}$$

where $g$ is a given function as in (3.18) with an additional dependence on $\theta$, and $\lambda(r,t)$ is the solution to the variational inequality

$$|\sigma(t) - \lambda(r,t)| \le r \qquad \forall t \in [0,T] \tag{9.2}$$

$$\lambda_t(r,t)(\sigma(t) - \lambda(r,t) - x) \ge 0 \qquad \forall |x| \le r \quad \text{a.e.} \tag{9.3}$$

$$\lambda(r,0) = \lambda_0(r) + P_r(\sigma(0) - \lambda_0(r)) \tag{9.4}$$

with a prescribed initial configuration $\lambda_0 \in \Lambda$, see (3.5)–(3.7). The total strain is then given by the integral

$$\varepsilon(t) = \frac{1}{E}\sigma + \int_0^\infty \varepsilon_r(t)\,\mathrm{d}r\,, \tag{9.5}$$

where $E$ is the elasticity modulus. Indeed, all quantities depend also on the space variable $x$. We assume here, however, that the material is homogeneous and $x$ just plays the role of a parameter. This is how the above formulas are to be understood.

The thermodynamic consistency of the model has to take into account energy exchange between mechanical and thermal energy. In classical thermodynamics, it is assumed that there exist state function $U$ (internal energy) and $S$ (entropy) such that

$$\dot{U} + \operatorname{div} q = \dot{\varepsilon}\sigma\,, \tag{9.6}$$

$$\dot{S} + \operatorname{div}\frac{q}{\theta} = p \ge 0\,, \tag{9.7}$$

where $q$ is the heat flux vector, and $p$ is the entropy production. Eq. (9.6) is the First principle of thermodynamics, which states that the mechanical work $\dot{\varepsilon}\sigma$ supplied to the system is partly used to energy increase $\dot{U}$, partly flows out by heat flow. This is in fact a generalized form of the energy conservation law (3.23). Eq. (9.7) is the Second principle in Clausius-Duhem form which says that in every energy exchange process, the entropy production is nonnegative.

It follows from (9.6)–(9.7) that

$$0 \le p\theta = \theta\dot{S} + \sigma\dot{\varepsilon} - \dot{U} - \frac{1}{\theta} q \cdot \nabla\theta\,. \tag{9.8}$$

The state function do not depend on $\nabla\theta$, hence the two terms on the right hand side of (9.8) must be nonnegative, that is, $q \cdot \nabla\theta \le 0$ (this is usually ensured by assuming the Fourier law $q = -\kappa\nabla\theta$ with a constant heat conductivity $\kappa > 0$), and

$$\theta\dot{S} + \sigma\dot{\varepsilon} - \dot{U} \ge 0\,. \tag{9.9}$$

Assume that at every memory level $r > 0$ the partial internal energy and partial entropy are functions of $(r, \sigma, \lambda(r), \theta)$, and that they satisfy the Second principle in the form

$$\theta \dot{S}_r + \sigma \dot{\varepsilon}_r - \dot{U}_r \geq 0\,. \tag{9.10}$$

We introduce the Gibbs energy $W_r = \theta S_r + \sigma \varepsilon_r - U_r$. The Second principle then reads

$$S_r \dot{\theta} + \varepsilon_r \dot{\sigma} - \dot{W}_r \leq 0\,. \tag{9.11}$$

Now, $W_r$ is a function of $(r, \sigma, \lambda(r), \theta)$ as well, and the chain rule yields

$$\dot{W}_r = \frac{\partial W_r}{\partial \sigma} \dot{\sigma} + \frac{\partial W_r}{\partial \lambda(r)} \dot{\lambda}(r) + \frac{\partial W_r}{\partial \theta} \dot{\theta}\,. \tag{9.12}$$

Both (9.11) and (9.12) must hold for every process, in particular for processes in which only $\dot{\sigma} \neq 0$ or only $\dot{\theta} \neq 0$. Hence, we must have

$$\varepsilon_r = \frac{\partial W_r}{\partial \sigma}\,, \quad S_r = \frac{\partial W_r}{\partial \theta}\,, \quad \frac{\partial W_r}{\partial \lambda(r)} \dot{\lambda}(r) \geq 0\,. \tag{9.13}$$

Since $\varepsilon_r$ does not explicitly depend on $\sigma$, we have

$$W_r = \sigma g(r, \lambda(r), \theta) - g_1(r, \lambda(r), \theta)\,, \tag{9.14}$$

where $g_1$ is an "integration constant". Then

$$\frac{\partial W_r}{\partial \lambda(r)} \dot{\lambda}(r) = \left( \sigma \frac{\partial g}{\partial \lambda(r)} - \frac{\partial g_1}{\partial \lambda(r)} \right) \dot{\lambda}(r)\,.$$

By virtue of (9.3), this term is nonnegative provided we choose

$$g_1(r, \lambda(r), \theta) = G(r, \lambda(r), \theta) - W_r^{cal}(r, \theta)\,, \tag{9.15}$$

with $G$ given by (3.22), and with another "integration constant" $W_r^{cal}(r, \theta)$, which is the purely caloric Gibbs energy. It can be determined by considering processes where no mechanics comes into play. The only quantity that can be measured is the specific heat $c_V(r, \theta) = \frac{\partial U_r^{cal}}{\partial \theta}$. We have

$$W_r^{cal}(r, \theta) = \theta S_r^{cal}(r, \theta) - U_r^{cal}(r, \theta) = \theta \frac{\partial W_r^{cal}}{\partial \theta}(r, \theta) - U_r^{cal}(r, \theta)\,,$$

hence

$$c_V(r, \theta) = \theta \frac{\partial^2 W_r^{cal}}{\partial \theta^2}(r, \theta)\,.$$

This is a differential equation for $W_r^{cal}$ which, for a constant specific heat $c_V(r)$, has thus the classical form

$$W_r^{cal}(r, \theta) = c_V(r) \theta \left( \log \left( \frac{\theta}{\theta_c} - 1 \right) \right) \tag{9.16}$$

with a constant reference temperature $\theta_c > 0$.

The full system of balance equations consists in combining the energy balance equation (9.6) with the momentum balance, say, (7.7)–(7.8). However, in this setting, the well-posedness of the system is still an open problem. Let us try to identify the difficulties.

We introduce the thermoplastic operator $\mathcal{P}$ and the internal energy operator $\mathcal{U}$ by the formulas

$$\mathcal{P}[\sigma,\theta] = \int_0^\infty g(r, \mathfrak{p}_r[\sigma,\lambda_0],\theta)\,\mathrm{d}r\,, \quad \mathcal{U}[\sigma,\theta] = \int_0^\infty (\theta g_\theta - \theta G_\theta + G)\,(r, \mathfrak{p}_r[\sigma,\lambda_0],\theta)\,\mathrm{d}r\,.$$

In 1D, with the linear Fourier law $q = -\kappa\theta_x$, the energy balance (9.6) reads

$$(C_V\theta + \mathcal{U}[\sigma,\theta])_t - \kappa\theta_{xx} = (\mathcal{P}[\sigma,\theta])_t\sigma\,, \tag{9.17}$$

where $C_V = \int_0^\infty c_V(r)\,\mathrm{d}r$. The equation is parabolic only if the coefficient in front of $\theta_t$ is positive. This means,

$$C_V + \int_0^\infty \theta(\sigma g_{\theta\theta} - G_{\theta\theta})(r,\lambda(r),\theta)\,\mathrm{d}r > 0 \tag{9.18}$$

for all values of the arguments. This is a considerable restriction of the model, and a more detailed discussion about this issue can be found in [5]. In fact, in order to obtain well-posedness of the system, further regularization is necessary, for example the viscous regularization with some viscosity coefficient $\nu > 0$

$$\varrho v_t = \sigma_x + \nu v_{xx}\,, \tag{9.19}$$

$$\varepsilon_t = v_x\,, \tag{9.20}$$

$$(C_V\theta + \mathcal{U}[\sigma,\theta])_t = \kappa\theta_{xx} + (\mathcal{P}[\sigma,\theta])_t\sigma + \nu v_x^2\,. \tag{9.21}$$

Note that the viscous dissipation $\nu v_x^2$ appears as a source term in the energy balance. For methods of solving such systems, we refer the reader again to [5].

# References

[1] M. Brokate, J. Sprekels: Hysteresis and phase transitions. Springer, New York, 1996.

[2] M.A. Krasnosel'skii, A.V. Pokrovskii: Systems with hysteresis. Nauka, Moscow, 1983, English edition Springer, New York, 1989.

[3] P. Krejčí: Hysteresis, convexity and dissipation in hyperbolic equations. Gakkotosho, Tokyo, 1996.

[4] P. Krejčí, Ph. Laurençot: Hysteresis filtering in the space of bounded measurable functions. Boll. Unione Mat. Ital. (8) **5**B (2002), 755–772.

[5] P. Krejčí, J. Sprekels: Temperature-dependent hysteresis in one-dimensional thermovisco-elastoplasticity. Appl. Math. **43** (1998), 173–205.

[6] P. Krejčí, I. Straškraba: A uniqueness criterion for the Riemann problem. Hiroshima Math. J. **27** (1997), 307–346.

[7] P. Krejčí: The Kurzweil integral and hysteresis. J. Physics: Conference Series **55** (2006), 144–154.

[8] J.-L. Lions: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Paris, 1969.

[9] J.P. O'Kane: The hysteretic linear reservoir – a new Preisach model. Physica B: Condensed matter **372** (2006), 388–392.

[10] A. Visintin: Differential models of hysteresis. Springer, Berlin, 1994.