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On the Katowice Problem

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Abstrakt: Práce pojednává o takzvaném Katovickém problému, tedy otázce zda je isomorfismus mezi $\mathcal{P}(\omega)/\operatorname{Fin}$ a $\mathcal{P}(\omega_1)/\operatorname{Fin}$ konzistentní s axiomy ZFC. Hlavním obsahem je vývoj nových forcingových technik pro důkazy konzistencí souvisejících s Katovickým problémem.

První kapitola obsahuje přehled známých výsledků, které se týkají této problematiky. Druhá kapitola je úvod do filter games, metody vyvinuté F. Galvinem a C. Laflammem. Je zde rovněž definována nová tower game a dokázáno, že první hráč nemá v této hře vyhrávající strategii, pokud příslušná hra generuje non-meager filtr. Tímto je zesílen (za předpokladu CH) klasický výsledek pro p-filter games. Tento výsledek je klíčový pro důkaz properness forcingů v následujících kapitolách.

Třetí kapitola obsahuje zjednodušenou presentaci výsledku S. Shelaha o konzistentní existenci pouze jediného p-bodu. Čtvrtá kapitola pojednává o strong-Q-posloupnostech v $\mathcal{P}(\omega)/\operatorname{Fin}$. Je podám přehled výsledků J. Stepranse z této oblasti a je vybudován ${}^{\omega}\omega$ bounding forcing přidávající strong-Q-posloupnost. To nám umožňuje dokázat konzistentní existenci countable like ideálu a takto zodpovědět otázku související s Katovickým problémem.

Poslední kapitola se zaměřuje na automorfismy $\mathcal{P}(\omega)/\operatorname{Fin}$. Je presentován důkaz K. P. Harta, že isomorfismus mezi $\mathcal{P}(\omega)/\operatorname{Fin}$ a $\mathcal{P}(\omega_1)/\operatorname{Fin}$ indukuje netriviální automorfismus na $\mathcal{P}(\omega)/\operatorname{Fin}$. Je zde rovněž představen forcing, který zamezuje existenci určitých netriviálních automorfismů. Tento forcing nám umožňuje vybudovat model ZFC, kde $\mathfrak{d} = \omega_1$ a každý automorfismus je triviální na nějaké množině z každého non-meager p-filtru. Za mnoha důkazy v této kapitole jsou myšlenky A. Dowa.

Klíčová slova: ω, Katovický problém, unikátní p-bod, netriviální automorfismus, ${}^{\omega}\omega$ bounding forcing, p-filter game, strong-Q-posloupnost

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Abstract: Main focus of this work is the so called Katowice problem, namely whether it is consistent with ZFC, that $\mathcal{P}(\omega)/\operatorname{Fin}$ is isomorphic with $\mathcal{P}(\omega_1)/\operatorname{Fin}$. The core of this work is development of new forcing notions and tools for establishing consistency results related to Katowice problem.

In the first chapter an overview of known results is given. In the second chapter we give an introduction to filter games (a method due to F. Galvin and C. Laflamme) and define a new tower game. We prove that player I has no winning strategy in this tower game if the involved tower generates a non-meager filter. This is a nontrivial strengthening (under CH) of the classical result for p-filter game. This result plays crucial role in the proof of properness of forcing notions from later chapters.

In the third chapter we present a simplification of S. Shelah's result, that existence of only one unique p-point is consistent with ZFC. The fourth chapter deals with strong-Q-sequences in $\mathcal{P}(\omega)/\operatorname{Fin}$. We review results of J. Steprans on this topic and introduce an ${}^{\omega}\omega$ bounding forcing, which creates a strong-Q-sequence. This enables us to prove consistency of existence of a countable like ideal in $\mathcal{P}(\omega)/\operatorname{Fin}$ and hence answering a weakening of Katowice problem question.

The last chapter focuses on automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$. We present a proof of K. P. Hart showing that isomorphism between $\mathcal{P}(\omega)/\operatorname{Fin}$ and $\mathcal{P}(\omega_1)/\operatorname{Fin}$ induces a nontrivial automorphism of $\mathcal{P}(\omega)/\operatorname{Fin}$. A new forcing method for reducing amount of nontrivial automorphisms is also introduced. We are able to use this method to build a model of ZFC where $\mathfrak{d} = \omega_1$ and each automorphism is trivial on some member of each non-meager p-filter. Most ideas behind this method are due to A. Dow.

Keywords: ω, Katowice problem, unique p-point, nontrivial automorphism, ${}^{\omega}\omega$ bounding forcing, p-filter game, strong-Q-sequence

CONTENTS

NOTATION AND CONVENTIONS

In this text we will work inside Zermelo-Fraenkel set theory with the axiom of choice, abbreviated ZFC. We will try to keep our approach and notation as standard as possible. The reader is expected to be familiar with usual set-theoretical language and methods. This includes the theory of proper forcing.

In some cases however, the notation and terminology used in this text goes beyond standard or is slightly less formal then expected in rigorous set-theoretical texts.

This chapter will review some basic notions essential for this work and establish the notation, which is less standard.

General Set Theory

For general set theory [Jec03, Kun80, BŠ01] are considered as standard reference. An excellent survey of set-theoretical methods is given is series of articles by A. Dow [Dow88a, Dow92, Dow95, Dow02].

The set theory we will work with is generally ZFC. In some cases, we will use statements like 'take countable elementary submodel M'. This will mean that we will resort to some fragment ZFC$^-$ of ZFC, which can miss powersets for some large sets.

For the cardinal hierarchy we will use the ω_α notation, the $\aleph$-notation will not be used. As usual, CH is abbreviation for the continuum hypothesis, i.e. $2^\omega = \omega_1$ and GCH, for generalized continuum hypothesis. In most cases when working with GCH, we will actually need just $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$.

Fin stands for the class of all finite sets. In practice, we will always work with Fin in context of powerset $\mathcal{P}(A)$ of same fixed set A. This enables to deal with the ideal (set) $\mathcal{P}(A) \cap \mathrm{Fin}$ instead. We will often use this ideal to work with classes of equivalence it generates.

For a set A, the equivalence class containing A is denoted $[A]$. It will always be clear from the context to which equivalence this refers to (usually $=^*$).

In our notation $A \subset B$ allows $A = B$. Generally relations with superscript $*$ mean modulo finite, e.g. $A \subset^* B$ is $|A \setminus B| < \omega$ and $A =^* B$ is $|A \Delta B| = |(A \setminus B) \cup (B \setminus A)| < \omega$. If $A \cap B =^* \emptyset$ we say that A and B are almost disjoint. For functions $f, g\colon \omega \to \omega$ the relation $f \leq^* g$ is defined as usually $|\{n \in \omega\colon g(n) < f(n)\}| < \omega$.

Functions

We will say that a function $f\colon \omega \to \omega$ is growing if it is nondecreasing and not eventually constant. A set of all functions from A to B is denoted ${}^A B$.

For a function $f\colon A \to B$, the associated function $\mathcal{P}(A) \to \mathcal{P}(B)$ is denoted by $f[\cdot]$, i.e.

$$f[\cdot]\colon X \mapsto \{f(x)\colon x \in X\}.$$

When the argument of this function is an equivalence class $[X]$, we use just $f[X]$ instead of the correct $f\big[[X]\big]$ in hope of creating less confusion than is being avoided. A notationally complicated situation arises when we are confronted with inverse function, e.g. what should be the range and domain of the inverse function. In general, we assume that the range of f^{-1} is subset of $\mathcal{P}(A)$ and we intentionally avoid establishing other conventions about inverse functions, since the intended meaning of our expressions will always be obvious from the context.

We use the convention that a function is a set of pairs. This allows us to express that f extends g by $g \subset f$ and if these two functions are compatible, we have a function $f \cup g$ which is their common extension.

For a sequence s of length α (for some ordinal α) we denote the sequence of length $\alpha+1$ obtained by appending x to the end of s as $s^\frown x$. The relation of end extension is denoted by $\sqsubset$. We will also say that set of ordinals A end extends B; $B \sqsubset A$ if $B \subset A$ and for each $\alpha \in A \setminus B$ and $\beta \in B$ is $\beta < \alpha$.

Trees

Let A be a set well ordered by $\lessdot$ (we are mostly interested in case where $(\lessdot, A)$ is isomorphic to $(\in, \omega)$). For set F (usually $F = 2$) we define ${}^{\lessdot A}F = \bigcup_{p \in A} {}^{\downarrow p}F$ where $\downarrow p = \{q \in A \colon q \lessdot p\}$. This set ordered with $\subset$ is a tree.

For a general tree T we call its node $t \in T$ splitting, if t has at least two immediate successors in T. For ordinal α we denote the αth level of T by $T^{[\alpha]} = \{t \in T \colon \downarrow t \text{ has order type } \alpha\}$. A level $T^{[\alpha]}$ is splitting if each element of $T^{[\alpha]}$ is a splitting node. The set of all (nonempty) splitting levels of T is denoted by $S(T)$. For $t \in T$ we denote $T[t]$ the subtree of T containing all nodes comparable with t, $T[t] = \{s \in T \colon s \leq t \text{ or } t \leq s\}$. A branch through T is a maximal set of pairwise comparable elements of T. For $t \in T$ we denote $[t]^T$ (or just $[t]$ in case T is clear from the context) the set of all branches through T containing t.

Cardinal characteristics of the continuum

We will explicitly need only few cardinal invariants. Dominating number $\mathfrak{d}$ and bounding number $\mathfrak{b}$ are defined in terms of cardinalities of sets in ordering $({}^\omega\omega, \leq^*)$.

$$\mathfrak{b} = \min\{|A| \colon (\forall f \in {}^\omega\omega)(\exists g \in A)(g \not\leq^* f)\}$$

$$\mathfrak{d} = \min\{|A| \colon (\forall f \in {}^\omega\omega)(\exists g \in A)(f \leq^* g)\}$$

The minimal cardinality of a character of a nonprincipal ultrafilter on ω is called ultrafilter number $\mathfrak{u}$. It is well known that $\omega_1 \leq \mathfrak{b} \leq \mathfrak{u}, \mathfrak{d} \leq 2^\omega$. No relation between $\mathfrak{d}$ and $\mathfrak{u}$ can be established in ZFC alone. A sequence

$$\mathcal{S} = \{f_\alpha \in {}^\omega\omega \colon \alpha \in \kappa\}$$

is called scale if $f_\alpha \leq^* f_\beta$ for $\alpha < \beta < \kappa$ and S is a dominating family (in $({}^\omega\omega, \leq^*)$). It follows that in this case $\kappa = \mathfrak{b} = \mathfrak{d}$ (in general, a scale exists if and only if $\mathfrak{b} = \mathfrak{d}$ and it's length is equal to this cardinal).

For more information about cardinal invariants we refer to [Bla10].

Filters and ideals

We will frequently deal with filters and ideals, mainly on ω. Unless said otherwise, all these filters and ideals are assumed to contain the Fréchet filter or ideal respectively. This enables us not to distinguish between these object as subsets of $\mathcal{P}(\omega)$ and $\mathcal{P}(\omega)/\operatorname{Fin}$. Filter $\mathcal{F}$ is generated by $\mathcal{A}$ if $\mathcal{A}$ together with the Fréchet is a subbase of $\mathcal{F}$. Analogously for ideals.

For filter $\mathcal{F}$ the dual ideal will be denoted $\mathcal{F}^*$ and the other way round, for ideal $\mathcal{I}$ the dual filter is denoted $\mathcal{I}^*$. For a set $\mathcal{J} \subset \mathcal{P}(\omega)$ the ideal perpendicular to $\mathcal{J}$ is denoted $\mathcal{J}^\perp = \{I \subset \omega \colon A \cap I =^* \emptyset \text{ for each } A \in \mathcal{J}\}$.

Various properties of filters and ideals will be considered in this text. We will follow a convention, that whenever there is some terminology for e.g. filters, the same terminology will be used for dual objects, in this case ideals. So we will speak e.g. about tall filters and rapid ideals (for concrete definitions see chapter 2).

For filters $\mathcal{F}, \mathcal{G} \subset \mathcal{P}(\omega)$ we have a filter on ω^2 defined by

$$\mathcal{F} \times \mathcal{G} = \left\{A \subset \omega^2 \colon \{x \in \omega \colon A_x \in \mathcal{F}\} \in \mathcal{G}\right\}$$

where $A_x = \{y \colon (x, y) \in A\}$. We will use this mainly for $\mathcal{G} = \{\omega\}$.

For filters $\mathcal{F}, \mathcal{G}$ on ω we say that $\mathcal{F}$ is in Rudin-Keisler below $\mathcal{G}$; $\mathcal{F} \leq_{RK} \mathcal{G}$ if there is a function $f \colon \omega \to \omega$ such that

$$\mathcal{F} = f_*(\mathcal{G}) = \{A \subset \omega \colon f^{-1}[A] \in \mathcal{G}\}.$$

If this function f is finite-to-one, then we say that $\mathcal{F}$ is Rudin-Blass bellow $\mathcal{G}$; $\mathcal{F} \leq_{RB} \mathcal{G}$. This relation commonly known is well studied in literature.

Forcing

Most of the material contained in this thesis is eventually focused on introducing various forcing notions and developing techniques for investigating these forcings. Hence a certain level of familiarity with the theory of forcing and especially proper forcing is expected. A good reference for forcing in general is [Kun80], for proper forcing one can use [Abr10, Gol93, She98a]. An unavoidable reference for forcings adding reals is [BJ95].

We will review some basic forcing results here. In our notation, stronger condition is smaller, i.e. q is stronger than p iff $q < p$. Groundmodel is usually denoted V, names for sets from the generic extension are generally labeled with dots - e.g. $\dot{\tau}$, and canonical names for sets from groundmodel are labeled with hats - e.g. $\hat{\omega}$. This convention sometimes abandoned and these object are not always distinguished by means of this notation. Expressions 'poset', 'forcing' and 'forcing notion' are used as synonyms. As usual, $H(\theta)$ denotes the set of all sets of hereditary cardinality $< \theta$.

Our approach to iteration of forcing notions is standard. When working with iterations of infinite length, we will use either finite support iteration or countable support iteration. While doing so, we will identify initial stages of the iteration with subposets of the resulting forcing via the canonical embeddings.

Definition. Let P be a poset. P is *κ-cc* if there is no antichain in P of size κ. An ω_1-cc poset is also called *ccc*.

P is *proper* if for each countable elementary submodel $M \prec H(\theta)$ (for some θ large enough) such that $P \in M$ and for each $p \in P \cap M$ there exist a (P, M) generic condition $q < p, q \in P$, i.e. a condition q, such that if $\dot{\tau} \in M$ is a P name for ordinal, then $q \Vdash \dot{\tau} \in M$.

The formulation 'θ large enough' in this definition can equivalently mean for all $\theta > 2^{|P|}$ or for some $\theta > 2^{|P|}$.

Definition. *Martin's Axiom* $\mathrm{MA}_\kappa(\mathscr{P})$ is the statement that for each poset P from the class $\mathscr{P}$ and each family $\mathcal{D} = \{D_\alpha \colon \alpha \in \kappa\}$ of dense subsets of P there exists a $\mathcal{D}$ generic filter on P, i.e. a filter meeting each set $D \in \mathcal{D}$. *MA* stands for MA_κ(cc-posets) for each $\kappa < 2^\omega$. *PFA* stands for MA_{ω_1}(proper posets).

From many consequences of these axioms let us mention just few of them.

Fact. *MA* $\Rightarrow \mathfrak{b} = \mathfrak{d} = \mathfrak{u} = 2^\omega = 2^\kappa$ *for each* $\kappa < 2^\omega$.

Fact. *PFA* $\Rightarrow$ *MA* + *OCA* + $2^\omega = \omega_2$.

Here OCA stands for Todorcevic's version of the Open Coloring Axiom. This axiom is in some literature also called Todorcevic's Axiom TA.

Definition. *Open Coloring Axiom* abbreviated OCA is the following statement.

Let X be an uncountable set of real numbers (with subspace topology) and let $X^2 = K_0 \cup K_1$ be a partition with K_0 open in the product topology. There either exists an uncountable $Y \subset X$ such that $Y^2 \subset K_0$ or $X = \bigcup_{n\in\omega} X_n$ and $X_n^2 \subset K_1$ for each $n \in \omega$.

For more details about PFA and related topics see [Tod89].

We will frequently work with some other properties of forcings and generic extensions.

Definition. Let P be a forcing notion. We say that P is *${}^\omega\omega$ bounding* if for each V generic filter G on P and each $f \in {}^\omega\omega \cap V[G]$ there exists a sequence of finite sets

$$\{H_n \in [\omega]^{<\omega} \colon n \in \omega\} \in V$$

such that $f \in \prod_{n\in\omega} H_n$.

Definition. Let P be a forcing notion. We say that P has *Sacks property* if there is a function $b \in {}^\omega\omega \cap V$ such that for each V generic filter G on P and each $f \in {}^\omega\omega \cap V[G]$ there exists a sequence of finite sets

$$\{H_n \in [\omega]^{b(n)} \colon n \in \omega\} \in V$$

such that $f \in \prod_{n\in\omega} H_n$.

In the definition of Sacks property, we can equivalently require that the same holds true for any growing function $b \in {}^\omega\omega \cap V$. We see that Sacks property is stronger than ${}^\omega\omega$ boundedness (the converse is not true).

The following theorems are well known and are used here only as a black box. The way they are stated here accord to the purpose of serving more as an outline or reminder than properly stated theorems. For detailed and cautiously correct formulations the reader is advised to see cited sources.

Theorem. *Finite support iteration of κ-cc forcing notions is κ-cc.*

Theorem. *Countable support iteration of proper and ${}^\omega\omega$ bounding forcing notions is proper and ${}^\omega\omega$ bounding.*

Theorem. *Countable support iteration of proper forcing notions with Sacks property is proper and has Sacks property.*

Theorem. *Countable support iteration of length ω_2 of proper forcing notions, each of size at most ω_1, is ω_2-cc.*

Theorem (Blass-Shelah [BS87])**.** *Countable support iteration of proper forcing notions preserving p-point $\mathcal{R}$ also preserves $\mathcal{R}$ (as a base of an ultrafilter).*

We will also need a generalization of these theorems using the notion ω_2-p.i.c. Relevant definitions and theorems are stated in chapter 5.

Specific models of ZFC in this thesis are obtained in a way characteristic for forcing constructions. Let's say that our goal is a model without p-points. The main difficulty and also our main focus is to define a forcing, which destroys single given p-point and is nice enough - (here proper ${}^\omega\omega$ bounding of size ω_1) preserves all we need to proceed in work (GCH) to the generic extension and not to introduce too many new p-points.

Once this is achieved, one can argue in this way; start in a model of GCH (and optionally e.g. $\diamondsuit_{\omega_2}$) and choose a suitable 'bookkeeping device'. Then iterate forcings which solve the issue one by one for each instance of the required statement (here kill given p-points). Continue in this iteration guided by the bookkeeping device (i.e. it tells us in each step with which instance to deal with at this stage - which p-point to kill) long enough (usually ω_2 steps) to encounter all instances from all intermediate models at some stage of the iteration (the bookkeeping device has to be chosen to ensure this - use e.g $\diamondsuit$).

In the end argue that each instance of our statement (here each p-point) from the final generic extension already appeared at earlier stage in some intermediate model (here, as a trace of that p-point on the intermediate model) and hence was dealt with.

This method is standard widely used (for more details see cited literature, a detailed proof is e.g. in [Woh08]). Hence when employing this method, we will usually articulate only key elements specific to the concrete construction, and we won't examine details of e.g. how to choose the bookkeeping device, …

CHAPTER 1

INTRODUCTION

The main object of our attention in the thesis is the Boolean algebra $\mathcal{P}(\omega)/\operatorname{Fin}$, the quotient of powerset of natural numbers modulo the ideal of it's finite elements, $[\omega]^{<\omega}$. Due to the Stone duality theorem, studying this algebra is equivalent to investigating properties of the topological space obtained as the remainder of Čech-Stone compactification $\omega^* = \beta\omega \setminus \omega$, i.e. the space of all nonprincipal ultrafilters on ω. This is of course true only if we assume the axiom of choice, in set theory without it there may for example be no ultrafilters at all while Boolean algebra $\mathcal{P}(\omega)/\operatorname{Fin}$ still exists, but we will not venture to follow this way of though (set theory without axiom of choice).

The structure in which we formulate question and prove theorems is usually the Boolean algebra $\mathcal{P}(\omega)/\operatorname{Fin}$, but there everything can be of course translated into the topological language of ω^*. See e.g. [BS89].

This algebra (or equivalently topological space) is of course a traditional object of interest in the history of set theory and topology. For an introduction into the problematic [vM84] is a good reference, a list of open problems is in a chapter by K. P. Hart and J. van Mill in Open problems in topology [HvM90].

There are several aspects of the behavior of $\mathcal{P}(\omega)/\operatorname{Fin}$. If we assume CH, then there is a very good understanding what this algebra actually is, inductive constructions of length ω_1 are a universal tool for solving most problems. In this case e.g. $\mathcal{P}(\omega)/\operatorname{Fin}$ is the unique universal algebra of size ω_1 [Par63]. Then there is some amount of results, which are provable in ZFC, and a whole heap of properties of $\mathcal{P}(\omega)/\operatorname{Fin}$, which are independent of ZFC. Because of this phenomenon $\mathcal{P}(\omega)/\operatorname{Fin}$ is called 'a monster having three heads' in [vM84]. With the later development of techniques like PFA and OCA, a 'fourth head' started to emerge [Far07].

It will turn out, that most results of this thesis belong to the 'second head' of independence results.

The first chapter (this one) introduces the main question we are interested in, the Katowice problem and reviews some related results. Unfortunately, this question still remains unsolved. Second chapter provides an introduction into the area of filter games. This chapter also contains a definition of a new game for towers and a characterization theorem for this game. This result is an essential tool for proving properness of forcings defined in chapter four. The author wouldn't be able to prove this result without what he learned from prof. Alan Dow.

The third chapter presents a nowadays classical method for building models with limited amount of p-ultrafilters due to S. Shelah. Forcings for no p-points, single selective ultrafilter and single p-point are presented. The reason for inclusion of this chapter is the simplified presentation of Shelah's original proof and also some common features, which this method shares with tools used in chapter four. Author of this text hopes, that this will make chapter four easier to understand.

Chapter four starts with a review of results of J. Steprans on strong-Q-sequences. The main result of this chapter is a new method for proving consistency of existence of strong-Q-sequence, which enables us to show this even together with $\mathfrak{d} = \omega_1$. This method is inspired by a forcing used in [JS91] and it was Michael Hrušák who observed the relevance of this work in scope of Katowice problem.

The last Chapter partially shifts the attention away from Katowice problem and studies automorphism of $\mathcal{P}(\omega)/\operatorname{Fin}$, namely their triviality and non-triviality. The main direction is to develop new approach to destroying non-trivial automorphism, which could be combined with forcings form previous chapters and which allows building

${}^{\omega}\omega$ bounding extensions of the groundmodel. These results are product of authors collaboration with Alan Dow, who is the author of most ideas involved. A section clarifying the relation between Katowice problem and existence of non-trivial automorphism of $\mathcal{P}(\omega)/\operatorname{Fin}$ is also included. The author of the proof in this section is Klaas Pieter Hart.

1.1 Katowice problem

There is a simple reason, why Boolean algebras $\mathcal{P}(\omega_1)$ and $\mathcal{P}(\omega)$ cannot be isomorphic, the different number of atoms (i.e. singletons) they contain. As it turns, the presence of atoms is the only obvious reason preventing existence of isomorphism between these algebras. If we remove them, the question whether $\mathcal{P}(\omega)/\operatorname{Fin} \cong \mathcal{P}(\omega_1)/\operatorname{Fin}$ suddenly becomes much more complicated.

There are many simple (and some not so simple) answers, why this consistently cannot be true. The simplest of them relies on comparing cardinalities; if $2^\omega \neq 2^{\omega_1}$ then there is even no bijection between these algebras. We will recall some more consistent answers in this chapter. But the general answer, if there some ZFC reason for non-isomorphism, is still missing.

Question 1 (Katowice problem)**.** *Is it consistent with ZFC that $\mathcal{P}(\omega)/\operatorname{Fin} \cong \mathcal{P}(\omega_1)/\operatorname{Fin}$?*

We will call *positive solution* the (consistent) existence of such isomorphism.

The name for this problem comes from the name of Polish city Katowice. It was originally discovered and discussed at the topological seminar of University of Silesia in Katowice in the 70s. The original equivalent formulation of this problem is the question if Boolean algebra $\mathcal{P}(\omega_1)/\operatorname{Fin}$ is homogeneous, i.e. whether it is isomorphic to all it's factor algebras.

Despite the simple statement, this problem is surprisingly hard to resolve. For a survey of historical development and obtained results (which are not numerous) see [Nyi07]. We will see that the core of this problem is the 'incompleteness' of these Boolean algebras.

Our approach to this problem in this work is to derive consequences of possible existence of isomorphism between $\mathcal{P}(\omega)/\operatorname{Fin} \cong \mathcal{P}(\omega_1)/\operatorname{Fin}$ and then test consistency of these consequences. The aim is to either prove some 'wild' consequences are not consistent with ZFC and hence refuting positive answer to Katowice problem or by building models of ZFC for those consequence discover a model witnessing the positive answer. Unfortunately, this goal is not fully achieved, we will 'only' establish consistency of existence of certain objects in $\mathcal{P}(\omega)/\operatorname{Fin}$.

Another twist to the Katowice problem would be not asking for general consistency, but asking 'Is it always possible to build a forcing extension of the groundmodel, where $\mathcal{P}(\omega)/\operatorname{Fin} \cong \mathcal{P}(\omega_1)/\operatorname{Fin}$? Do we need some additional axioms going beyond ZFC (e.g. large cardinals, ...)?' instead.

It is also worthwhile mentioning, that adding an additional assumption, that Luzin hypothesis $2^\omega = \omega_2$ holds, does not affect the problem. On the other hand, we do not know if positive solution implies Luzin hypothesis to hold.

Lemma 1.1.1. *If $\mathcal{P}(\omega)/\operatorname{Fin} \cong \mathcal{P}(\omega_1)/\operatorname{Fin}$ is consistent, then so is $\mathcal{P}(\omega)/\operatorname{Fin} \cong \mathcal{P}(\omega_1)/\operatorname{Fin} + 2^\omega = \omega_2$.*

Proof. Suppose we have a model of ZFC where $\mathcal{P}(\omega)/\operatorname{Fin} \cong \mathcal{P}(\omega_1)/\operatorname{Fin}$ and $2^\omega = \kappa > \omega_2$. The forcing for collapsing κ to ω_2 with conditions of size ω_1 is $< \omega_2$ closed (all descending chains of length $< \omega_2$ have lower bounds) and hence does not add any elements into $\mathcal{P}(\omega)/\operatorname{Fin}$ or $\mathcal{P}(\omega_1)/\operatorname{Fin}$. This means that the isomorphism from the model we started with still witnesses $\mathcal{P}(\omega)/\operatorname{Fin} \cong \mathcal{P}(\omega_1)/\operatorname{Fin}$. And since no new reals were added, we have $2^\omega = \omega_2$. $\square$

1.2 Algebras $\mathcal{P}(\kappa)/\operatorname{Fin}$

The natural generalization of Katowice problem would be asking the same question for different cardinal then ω and ω_1. However as was shown by Balcar and Frankiewicz [BF78], for different cardinals the question is much easier to resolve.

We will show this through a series of lemmas.

Lemma 1.2.1. *Let $\kappa < \lambda$ be cardinal numbers. If*

$$\mathcal{P}(\kappa)/\operatorname{Fin} \cong \mathcal{P}(\lambda)/\operatorname{Fin}$$

then
$$\mathcal{P}(\kappa)/\operatorname{Fin} \cong \mathcal{P}(\mu)/\operatorname{Fin} \cong \mathcal{P}(\lambda)/\operatorname{Fin}$$
for each $\kappa < \mu < \lambda$.

Proof. Suppose that $\kappa < \mu < \lambda$ is in the lexicographical ordering the smallest triple witnessing failure of the lemma and fix an isomorphism
$$f\colon \mathcal{P}(\lambda)/\operatorname{Fin} \to \mathcal{P}(\kappa)/\operatorname{Fin}.$$
If $\left|f[\mu]\right| = \kappa$ then f is a witness for
$$\mathcal{P}(\mu)/\operatorname{Fin} \cong \mathcal{P}(f[\mu])/\operatorname{Fin} \cong \mathcal{P}(\kappa)/\operatorname{Fin},$$
so $\left|f[\mu]\right| = \kappa_0 < \kappa$. Hence for $\kappa_0 < \kappa < \mu$ we have $\mathcal{P}(\kappa_0)/\operatorname{Fin} \not\cong \mathcal{P}(\kappa)/\operatorname{Fin} \not\cong \mathcal{P}(\mu)/\operatorname{Fin}$ and this contradicts the smallest choice of such triple. $\square$

Lemma 1.2.2. *Let* $\kappa < \lambda$ *be cardinal numbers such that*
$$\mathcal{P}(\lambda)/\operatorname{Fin} \cong \mathcal{P}(\lambda^+)/\operatorname{Fin}.$$
Then
$$\mathcal{P}(\kappa)/\operatorname{Fin} \cong \mathcal{P}(\kappa^+)/\operatorname{Fin}.$$

Proof. If $\mathcal{P}(\kappa)/\operatorname{Fin} \cong \mathcal{P}(\lambda)/\operatorname{Fin}$ use lemma 1.2.1.

We can suppose that for $\mu < \lambda$ is $\mathcal{P}(\mu)/\operatorname{Fin} \not\cong \mathcal{P}(\lambda)/\operatorname{Fin}$ (by taking minimal λ). Fix isomorphism
$$f\colon \mathcal{P}(\lambda)/\operatorname{Fin} \to \mathcal{P}(\lambda^+)/\operatorname{Fin}$$
and put $A = \lambda \setminus \bigcup_{\alpha<\lambda} f[\alpha]$. Each $f[\alpha]$ has cardinality less then λ hence $|A| = \lambda^+$. For each $\alpha < \lambda$ is
$$f^{-1}[A] \cap \alpha =^* \emptyset$$
hence $|f^{-1}[A]| = \omega$ and $\mathcal{P}(\omega)/\operatorname{Fin} \cong \mathcal{P}(\lambda^+)/\operatorname{Fin}$ and thus $\mathcal{P}(\kappa)/\operatorname{Fin} \cong \mathcal{P}(\kappa^+)/\operatorname{Fin}$. $\square$

Lemma 1.2.3. *Suppose that* $\mathcal{P}(\omega)/\operatorname{Fin} \cong \mathcal{P}(\kappa)/\operatorname{Fin}$ *for some regular uncountable cardinal* κ. *Then* $\mathfrak{b} = \mathfrak{d} = \kappa$.

Proof. Fix an isomorphism $f\colon \mathcal{P}(\kappa)/\operatorname{Fin} \to \mathcal{P}(\omega)/\operatorname{Fin}$ and a partition of κ into disjoint sets
$$\kappa = \bigcup\{B_n \in [\kappa]^\kappa \colon n \in \omega\}.$$
Fix $\{A_n \colon n \in \omega\}$ such that $\omega = \bigcup\{A_n \colon n \in \omega\}$, $[A_n] = f[B_n]$ and $A_n \cap A_m = \emptyset$ for $n \neq m \in \omega$.

Let $b\colon \omega \to \omega^2$ be a bijection such that $b[A_n] = \{n\} \times \omega$. For $\alpha < \kappa$ fix $D_\alpha \subset \omega$ such that $[D_\alpha] = f([\kappa \setminus \alpha])$. Put
$$f_\alpha(n) = \min\{i \in \omega \colon (n,i) \in b[D_\alpha]\}.$$

We will show that $\{f_\alpha \colon \alpha \in \kappa\}$ is a κ-scale. For $\alpha < \beta$ we have that $D_\beta \subset^* D_\alpha$ and $f_\alpha \leq^* f_\beta$ follows. Take arbitrary $f \in {}^\omega\omega$ and put $F = \{(n,i) \in \omega^2 \colon i < f(n)\}$. We have $b^{-1}[F] \cap A_n =^* \emptyset$ for each $n \in \omega$ hence $f^{-1}\left(b^{-1}[F]\right) \cap B_n =^* \emptyset$ for each $n \in \omega$ and $f^{-1}\left(b^{-1}[F]\right) \subset \alpha$ for some $\alpha \in \kappa$. Hence $F \cap b[D_\alpha] =^* \emptyset$ and $f \leq^* f_\alpha$. $\square$

And finally we can conclude.

Theorem 1.2.4 (Balcar [BF78], see also [Com77])**.** *Let* $\kappa < \lambda$ *be infinite cardinals such that*
$$\mathcal{P}(\kappa)/\operatorname{Fin} \cong \mathcal{P}(\lambda)/\operatorname{Fin}.$$
Then $\kappa = \omega$ *and* $\lambda = \omega_1 = \mathfrak{b} = \mathfrak{d}$.

Proof. Lemma 1.2.1 implies that $\mathcal{P}(\kappa)/\operatorname{Fin} \cong \mathcal{P}(\kappa^+)/\operatorname{Fin}$. If $\omega < \kappa$ then

$$\mathcal{P}(\omega)/\operatorname{Fin} \cong \mathcal{P}(\omega_1)/\operatorname{Fin} \cong \mathcal{P}(\omega_2)/\operatorname{Fin}$$

(lemma 1.2.2). But then lemma 1.2.3 would imply $\omega_1 = \mathfrak{b} = \omega_2$ hence $\kappa = \omega$. So $\omega_1 = \mathfrak{b} = \mathfrak{d} = \lambda$. □

The situation for algebras of form $\mathcal{P}(\kappa)/[\kappa]^{<\lambda}$ for other cardinals λ is somewhat different (and so is for other ideals then the Fréchet ideal). We will mention here just one basic result from this direction, for other results see e.g. [vD91, DH94].

Theorem 1.2.5. *If $\mathcal{P}(\kappa)/[\kappa]^{<\kappa} \cong \mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ then $cf(\kappa) = cf(\lambda)$.*

Proof. Suppose $cf(\kappa) < cf(\lambda)$ and decompose λ into disjoint sets of full size λ,

$$\lambda = \bigcup_{\alpha \in cf(\kappa)} A_\alpha.$$

For each $x \in [\lambda]^\lambda$ there is $\alpha \in cf(\kappa)$ such that $x \cap A_\alpha \in [\lambda]^\lambda$. On the other hand for each set S of size $cf(\kappa)$ of disjoint elements of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ we can find $x \in [\kappa]^\kappa$ such that $x \cap A < \kappa$ for each $A \in S$. So there is no set which could be the image of $\{A_\alpha \colon \alpha \in cf(\kappa)\}$ with a Boolean isomorphism. □

Completions of $\mathcal{P}(\kappa)/\operatorname{Fin}$

It is an important feature of the Katowice problem that it's essence is a question about the degree of incompleteness of involved Boolean algebras.

It is easy to see that from the forcing point of view all algebras $\mathcal{P}(\kappa)/\operatorname{Fin}$ are equivalent (the set of classes with countable representatives is dense in each such algebra).

For a nonzero element a of Boolean algebra $\mathbf{B}$ will $\mathbf{B}\restriction a$ denote the Boolean algebra with elements $\{b \in \mathbf{B} \colon \mathbf{0_B} \leq b \leq a\}$ and operations inherited from $\mathbf{B}$. The completion Boolean algebra $\mathbf{B}$ is denoted $\mathbf{c}(\mathbf{B})$.

Fact 1.2.6. *Let $\mathcal{A} = \{a_i \colon i \in \lambda\}$ be a maximal antichain in a complete Boolean $\mathbf{B}$. Then $f \colon \mathbf{B} \to \prod_{i\in\lambda} \mathbf{B}\restriction a_i$, $f \colon b \mapsto \langle b \wedge a_i \colon i \in \lambda\rangle$ is an isomorphism of (complete) Boolean algebras.*

Corollary 1.2.7. *If $\kappa^\omega = 2^\omega$ then $\mathbf{c}(\mathcal{P}(\omega)/\operatorname{Fin}) \cong \mathbf{c}(\mathcal{P}(\kappa)/\operatorname{Fin})$.*

This corollary can alternatively be seen through existence of base trees in these Boolean algebras [BPS80, Dow89, BDH].

Proof. There are maximal antichains $\mathcal{A} = \{a_i \colon i \in 2^\omega\}$, $B = \{b_i \colon i \in 2^\omega\}$ in algebras $\mathcal{P}(\omega)/\operatorname{Fin}$, $\mathcal{P}(\kappa)/\operatorname{Fin}$ consisting of countable sets. And

$$\mathbf{c}(\mathcal{P}(\omega)/\operatorname{Fin}) \cong \prod_{i\in 2^\omega} \mathbf{c}(\mathcal{P}(\omega)/\operatorname{Fin})\restriction a_i \cong \prod_{i\in 2^\omega} \mathbf{c}(\mathcal{P}(\kappa)/\operatorname{Fin})\restriction b_i \cong \mathbf{c}(\mathcal{P}(\kappa)/\operatorname{Fin}).$$

□

So Katowice problem can be vaguely stated as: Can $\mathcal{P}(\omega)/\operatorname{Fin}$ be incomplete in the precisely same way as $\mathcal{P}(\omega_1)/\operatorname{Fin}$ is? It is usually not problem to find a copy of any situation appearing in $\mathcal{P}(\omega)/\operatorname{Fin}$ into $\mathcal{P}(\omega_1)/\operatorname{Fin}$. The 'problem' may arise, when we want to copy 'incompleteness structures' the other way. The vague notion of incompleteness structure here stands for an infinite subset of Boolean algebra for which there either exists or does not exist an element of the Boolean algebra, with prescribed relation to this set. Examples of these are gaps (in the usual meaning) or so called strong-Q-sequences. It should be mentioned, that the structure of gaps in $\mathcal{P}(\omega_1)/\operatorname{Fin}$ is generally not extensively studied nor understood yet.

1.3 Some consequences of $\mathcal{P}(\omega_1)/\operatorname{Fin} \cong \mathcal{P}(\omega)/\operatorname{Fin}$

We will review consequences of positive answer for Katowice problem. All these consequences were previously known, though some of them were stated in a slightly different way.

Strong-Q-sequences in Boolean algebras

The concept of strong-Q-sequence were introduced by J. Steprans in [Ste85]. He defined them for general Boolean algebras as well as for the algebra $\mathcal{P}(\omega)/\operatorname{Fin}$.

Definition 1.3.1 (strong-Q-sequence)**.** Let $\mathcal{A}$ be a set of nonzero elements of a Boolean algebra $\mathbf{A}$. $\mathcal{A}$ is a *strong-Q-sequence* in $\mathbf{A}$ if the following holds true. For every $F\colon \mathcal{A} \to \mathbf{A}$ such that $F(a) \leq a$ there exists $c \in \mathbf{A}$ such that $c \wedge a = F(a)$ for each $a \in \mathcal{A}$. The element c is called *uniformization* of F.

Proposition 1.3.2. *Every strong-Q-sequence $\mathcal{A}$ in $\mathbf{A}$ is an antichain and if $\mathbf{A}$ is κ-complete than the converse is also true for antichains of size less than κ.*

Proof. If $a \wedge b \neq \mathbf{0_A}$ then any F such that $F\colon a \mapsto \mathbf{0_A}$ and $F\colon b \mapsto b$ has no uniformization.

If $\mathbf{A}$ is κ complete, $\mathcal{A}$ is an antichain in $\mathbf{A}$ of size less than κ and $F\colon \mathcal{A} \to \mathbf{A}$ is a function as in definition 1.3.1, then $\bigvee\{F(a)\colon a \in \mathcal{A}\}$ is a uniformization of F. □

The notion is stable with respect to subsets and restrictions.

Proposition 1.3.3. *Let $\mathcal{A}$ be a strong-Q-sequence in $\mathbf{A}$.*

1. *Every $\mathcal{B} \subset \mathcal{A}$ is a strong-Q-sequence in $\mathbf{A}$.*
2. *For every $b \in \mathbf{A}$ the set $\{a \wedge b\colon a \in \mathcal{A}, a \wedge b \neq \mathbf{0_A}\}$ is a strong-Q-sequence in $\mathbf{A}$.*

Proof. Easy. □

In agreement with established notation, for a subset $\mathcal{A}$ of Boolean algebra $\mathbf{A}$ we put

$$\mathcal{A}^{\perp} = \{x \in \mathbf{A} : x \wedge a = \mathbf{0_A} \text{ for each } a \in \mathcal{A}\}.$$

It is easy to see that $\mathcal{A}^{\perp}$ is an ideal in $\mathbf{A}$. The following theorem is due to J. Steprans.

Theorem 1.3.4 (Steprans)**.** *Let $\mathbf{A}$ and $\mathbf{B}$ be Boolean algebras. Suppose that $\mathcal{A} \subset \mathbf{A}$ and $\mathcal{B} \subset \mathbf{B}$ are strong-Q-sequences. Suppose furthermore that there is a bijection $\Psi : \mathcal{A} \to \mathcal{B}$ such that $\mathbf{A} \restriction a$ is isomorphic to $\mathbf{B} \restriction \Psi(a)$ for each $a \in \mathcal{A}$. Then $\mathbf{A}/\mathcal{A}^{\perp}$ is isomorphic to $\mathbf{B}/\mathcal{B}^{\perp}$.*

Proof. For each $a \in \mathcal{A}$ fix an isomorphism $\psi_a : \mathbf{A} \restriction a \to \mathbf{B} \restriction \Psi(a)$. For $x \in B$ and $a \in \mathcal{A}$ put $G_x(a) = \psi_a^{-1}(x \wedge \Psi(a))$. Define $\theta : \mathbf{B}/\mathcal{B}^{\perp} \to \mathbf{A}/\mathcal{A}^{\perp}$ by the rule $\theta(x) \wedge [a] = [G_x(a)]$ for each $a \in \mathcal{A}$. This obviously defines an isomorphism. □

We will be interested for strong-Q-sequences mainly in the algebra $\mathcal{P}(\omega)/\operatorname{Fin}$. Chapter 4 of this thesis is devoted to this topic.

Ideal of countable sets in $\mathcal{P}(\omega_1)/\operatorname{Fin}$

A distinguished object in the algebra $\mathcal{P}(\omega_1)/\operatorname{Fin}$ is the ideal of countable sets. This ideal will be denoted $\mathscr{C}$. It is not clear at all why there should be a subset of $\mathcal{P}(\omega)/\operatorname{Fin}$ mimicking properties of this ideal (in fact, this will be one of main results of this thesis).

Proposition 1.3.5. *Every p-ultrafilter on ω_1 intersects $\mathscr{C}$.*

For a definition of p-ultrafilter see 2.1.9.

Proof. Let $\mathcal{F}$ be a p-ultrafilter on ω_1. The principal case is clear so suppose opposite. The filter $\mathcal{F}$ can not be ω_1 complete (ω_1 is not a measurable cardinal); there is $D \in [\mathcal{F}]^{\omega}$ such that $\bigcap D = \emptyset$. There exists $p \in \mathcal{F}$ such that $p \subseteq^* d$ for each $d \in D$. This implies that $|p| = \omega$. □

The following fact was known to B. Balcar and P. Simon and was also independently observed by A. Dow.

Lemma 1.3.6. *Let $f\colon \mathcal{P}(\omega)/\operatorname{Fin} \to \mathcal{P}(\omega_1)/\operatorname{Fin}$ be a Boolean isomorphism and let $\mathscr{I} = f^{-1}[\mathscr{C}]$. Then $\mathscr{I}$ is a non-meager ideal.*

Proof. Suppose $\mathscr{I}$ were meager and let $\{I_n\colon n \in \omega\}$ be a sequence of disjoint finite subsets of ω such that for each $X \in [\omega]^\omega$ is $a_X = \bigcup_{n\in X} I_n \notin \mathscr{I}$ (i.e. $f[a_X]$ is uncountable).

Fix a copy $\{X_s\colon s \in {}^{<\omega}2\}$ of the binary tree $({}^{<\omega}2, \subset)$ in the ordering $([\omega]^\omega, \supset)$. Denote $A_s = f[a_{X_s}]$. There is some $\alpha \in \omega_1$ such that for all $s, t \in {}^{<\omega}2$ is $A_t \setminus A_s \subset \alpha$ if $s \subset t$ and $A_s \cap A_t \subset \alpha$ for s, t incompatible.

For each $\beta \geq \alpha$ there is at most one branch x through ${}^{<\omega}2$ such that $\beta \in A_s$ for all $s \in x$. Now use that $\omega_1 < 2^\omega$ to pick a branch y such that for each $\beta > \alpha$ there is $s \in y$ such that $\beta \notin A_s$. Take X an infinite pseudointersection of $\{X_s\colon s \in y\}$, i.e. $a_X \subset^* a_{X_s}$ for all $s \in y$. Hence $f[a_X]$ is uncountable and $f[a_X] \subset^* A_s$ for $s \in y$ so $\bigcap_{s\in y} A_s$ is also uncountable. This is a contradiction with the choice of y which ensured that $\bigcap_{s\in y} A_s \subset \alpha$. $\square$

We will call the ideal mimicking the behavior if $\mathscr{C}$ countable like.

Definition 1.3.7 (Countable like ideal)**.** We say that the ideal $\mathcal{I}$ in $\mathcal{P}(\omega)$ is *countable like* iff the following holds.

1. $\mathcal{I}$ is non-meager.
2. $\mathcal{I} \cap \mathcal{F} \neq \emptyset$ for each p-ultrafilter $\mathcal{F}$ in $\mathcal{P}(\omega)/\operatorname{Fin}$.
3. $\mathcal{I}$ is generated by an increasing tower $\mathcal{T} = \{T_\alpha\colon \alpha \in \omega_1\}$ in $\mathcal{P}(\omega)/\operatorname{Fin}$.
4. The set $\{T_{\alpha+1} \setminus T_\alpha\colon \alpha \in \omega_1\}$ is a strong-Q-sequence in $\mathcal{P}(\omega)/\operatorname{Fin}$.

Theorem 1.3.8. *Let $f\colon \mathcal{P}(\omega_1)/\operatorname{Fin} \to \mathcal{P}(\omega)/\operatorname{Fin}$ be a Boolean isomorphism and let $\mathscr{I} = f[\mathscr{C}]$. Then $\mathfrak{d} = \omega_1$ and $\mathscr{I}$ is countable like ideal.*

Proof. The fact that $\mathfrak{d} = \omega_1$ is theorem 1.2.4. Requirements 2. and 1. from the definition of countable like ideal are proved in lemma 1.3.6 and proposition 1.3.5 (p-ultrafilters correspond to p-ultrafilters in the isomorphism). To see that the other requirements are also fulfilled just notice that $\mathscr{C}$ is generated by $\{t_\alpha = [\alpha \cdot \omega]\colon \alpha \in \omega_1\}$ in $\mathcal{P}(\omega_1)/\operatorname{Fin}$ and $\{t_{\alpha+1} \setminus t_\alpha\colon \alpha \in \omega_1\}$ is a strong-Q-sequence in $\mathcal{P}(\omega_1)/\operatorname{Fin}$. $\square$

It will be shown in chapter 4 that there is model of ZFC where consequences of theorem 1.3.8 hold true. Thus this theorem alone is not strong enough to refute $\mathcal{P}(\omega_1)/\operatorname{Fin} \cong \mathcal{P}(\omega)/\operatorname{Fin}$.

For more combinatorial results (mainly under PFA) about $\mathcal{P}(\omega_1)/\operatorname{Fin}$ see [Dow88b] and for more results about $\mathscr{C}$ in $\mathcal{P}(\omega_1)$ see [Dow96].

CHAPTER 2

FILTER GAMES AND RELATIVES

In this chapter we review the technology of (ultra)filter games. This technology is nowadays classical, as its starting point is usually cited work of Galvin and Mackenzie [Gal80]. A good reference with systematic treatment is [Laf96, LL02]. Games for two filters appeared originally in [She82] and a systematically studied in [Eis01]. The concept of tower games in this thesis is original. All filters in this chapter are filters on ω.

The reason for our interest in these games is that they provide an essential tool for establishing properties of forcing notions, we will use. Namely, we will often encounter a forcing with an filter $\mathcal{F}$ as an parameter, conditions will approximate a characteristic function for generic real number with '$\mathcal{F}$ large amount of uncertainty' (for explicit definitions see chapters 3 and 4). For proving e.g. properness of such forcings we will need to be able to build a kind of fusion sequence of descending conditions and games are used to steer this fusion constructions.

2.1 About filters

We start by defining basic properties of filters (and ideals) we are interested in. All these notions and results are classical and well known.

Definition 2.1.1 (tall filter)**.** A filter $\mathcal{F}$ is a *tall* filter iff for each $A \in [\omega]^\omega$ there is some $B \in [A]^\omega$ such that $\omega \setminus B \in \mathcal{F}$.

Since a filter can be viewed as a subset of the Cantor space identified with $\mathcal{P}(\omega)$ via characteristic functions, we can talk about properties of filters defined in topological language.

Lemma 2.1.2 (Talagrand [Tal80])**.** *For a filter $\mathcal{F}$ in $\mathcal{P}(\omega)$ the following are equivalent:*

1. *$\mathcal{F}$ is non-meager subset of $\mathcal{P}(\omega)$.*
2. *$\mathcal{F}$ is unbounded, i.e. enumerating functions of sets in $\mathcal{F}$ are unbounded subset of $({}^\omega\omega, <)$.*
3. *For each decomposition of $\omega = \bigcup I_n$ into intervals there is a set $F \in \mathcal{F}$ such that $F \cap I_n = \emptyset$ for infinitely many intervals.*
4. *For each sequence of disjoint finite sets $\{a_n \colon a_n \in [\omega]^{<\omega}, n \in \omega\}$ there is a set $F \in \mathcal{F}$ such that $F \cap a_n = \emptyset$ for infinitely many n.*

Fact 2.1.3. *Every non-meager filter is tall.*

The concept of rapid filter was introduced in [Cho68]. Some authors also use terminology 'semi-Q(-point)' instead. This other terminology is usually in context of ultrafilters.

Definition 2.1.4 (rapid filter)**.** A filter $\mathcal{F}$ in $\mathcal{P}(\omega)$ is called *rapid* if enumerating functions of its subsets are dominating family in $({}^\omega\omega, <)$.

It follows from the definition that every rapid filter is non-meager.

Lemma 2.1.5 (Miller [Mil80])**.** *For a filter $\mathcal{F}$ on ω the following are equivalent:*

1. *$\mathcal{F}$ is rapid.*
2. *There exist an increasing sequence $\{a_i : i \in \omega\} \subset \omega$ and a function $f \in {}^\omega\omega$ such that for each subsequence $\{b_i : i \in \omega\}$ of $\{a_i : i \in \omega\}$ there exists some $F \in \mathcal{F}$ such that $|F \cap b_{i+1} \setminus b_i| < f(i)$ for each $i \in \omega$.*
3. *For each increasing sequence $\{a_i : i \in \omega\} \subset \omega$ and an for each growing function $f \in {}^\omega\omega$ there exists some $F \in \mathcal{F}$ such that $|F \cap a_{i+1} \setminus a_i| < f(i)$ for each $i \in \omega$.*
4. *There exist a function $f \in {}^\omega\omega$ such that for every sequence $\{t_i : t_i \in [\omega]^{<\omega}, i \in \omega\}$ there exists some $F \in \mathcal{F}$ such that $|F \cap t_i| < f(i)$ for each $i \in \omega$.*
5. *For each growing function $f \in {}^\omega\omega$ and each sequence $\{t_i : t_i \in [\omega]^{<\omega}, i \in \omega\}$ there exists some $F \in \mathcal{F}$ such that $|F \cap t_i| < f(i)$ for each $i \in \omega$.*

Note that the conditions 4 and 5 have no reference to the ordering of ω. This enables us to speak about rapid filters on a general countable set without declaring the respective ordering.

Proof. It is easy to see that $5 \Rightarrow 4 \Rightarrow 2$ and $5 \Rightarrow 3 \Rightarrow 2$.

To prove that $2 \Rightarrow 1$ take an infinite set $A \in [\omega]^\omega$ and we have to find $F \in \mathcal{F}$ such that the enumerating function e_A is dominated by e_F. Fix $\{b_i : i \in \omega\}$ such that $f(i+1) < |A \cap b_{i+1} \setminus b_i|$ for $i \in \omega$. According to 2 there exists $F' \in \mathcal{F}$ such that $|F' \cap b_{i+1} \setminus b_i| < f(i)$. Now $F = F' \setminus b_1$ is the desired set since $|F \cap b_{i+2} \setminus b_{i+1}| < f(i+1) < |A \cap b_{i+1} \setminus b_i|$ for $i \in \omega$.

Suppose that 1 holds and pick a growing function $f \in {}^\omega\omega$ and some sequence $\{t_i : i \in \omega\}$ of finite subsets of ω to prove 5. Define a function g such that for each $i \in \omega$ is $\max(t_i) < g(f(i))$. This is possible since f is growing. Because $\mathcal{F}$ is rapid there is some $F \in \mathcal{F}$ such that e_F dominates g. Now

$$|F \cap t_i| \leq |F \cap (\max(t_i) + 1)| = |e_F^{-1}[\max(t_i) + 1]| < |g^{-1}[\max(t_i) + 1]| < f(i).$$

□

The notion of rare filter was again introduces in [Cho68]. Among ultrafilters, object with this property are called q-points or q-ultrafilters almost exclusively. Since we will mainly deal with filters in general, we keep the original terminology.

Definition 2.1.6 (rare filter)**.** A filter $\mathcal{F}$ in $\mathcal{P}(\omega)$ is *rare* if for each sequence of disjoint finite sets

$$\{t_n : t_n \in [\omega]^{<\omega}, n \in \omega\}$$

there is a set $F \in \mathcal{F}$ such that $|F \cap t_n| \leq 1$ for each $n \in \omega$.

If $\mathcal{F}$ is a rare ultrafilter, it is called *q-ultrafilter* or *q-point*.

Fact 2.1.7. *Every rare filter is rapid.*

Lemma 2.1.8. *For a filter $\mathcal{F}$ on ω the following are equivalent.*

1. *$\mathcal{F}$ is rare.*
2. *There exist an increasing sequence $\{a_i : i \in \omega\} \subset \omega$ such that for each subsequence $\{b_i : i \in \omega\}$ of $\{a_i : i \in \omega\}$ there is $F \in \mathcal{F}$ such that $|F \cap [b_i, b_{i+1})| \leq 1$ for each $i \in \omega$.*

Proof. We need to show $2 \Rightarrow 1$. Let $\{t_n : t_n \in [\omega]^{<\omega}, n \in \omega\}$ consisting of disjoint sets be given. Choose $\{b_i : i \in \omega\}$ subsequence of $\{a_i : i \in \omega\}$ such that each t_n intersect only one or at most two (consecutive) intervals $[b_i, b_{i+1})$.

For $i \in 2$ find $F_i \in \mathcal{F}$ such that $|F_i \cap [b_{2n+i}, b_{2\cdot(n+1)+i})| \leq 1$ for each $n \in \omega$. For

$$F = (F_0 \cap F_1) \setminus b_1 \in \mathcal{F}$$

is $|F \cap t_n| \leq 1$ for each $n \in \omega$. □

We will make one exemption from the convention we established for this chapter, the definition of p-filter will be used for filters on other sets (also uncountable) than ω as well. Although we won't encounter any instance of this until chapter 5.

Definition 2.1.9 (p-filter)**.** Let $\mathcal{F}$ be a filter on some set. We say that $\mathcal{F}$ is a *p-filter* if for each $D \in [\mathcal{F}]^\omega$ there exists $p \in \mathcal{F}$ such that $p \subseteq^* d$ for each $d \in D$. The set p is called a *pseudointersection* of D. If $\mathcal{F}$ is both p-filter and an ultrafilter then $\mathcal{F}$ is a *p-ultrafilter* or *p-point*.

A rare p-ultrafilter is called *selective* ultrafilter or *Ramsey* ultrafilter.

2.2 Non-meager game and near coherence game

We will start with few simple games, which can be used for characterizing meagerness and similar properties of filters.

Definition 2.2.1 (non-meager game)**.** Let $\mathcal{F}$ be a filter in $\mathcal{P}(\omega)$. The following game is called *non-meager game $M_\mathcal{F}$*. In n-th move player I plays a finite set $A_n \in [\omega]^{<\omega}$ and player II responds with a finite set $B_n \in [\omega]^{<\omega}$ disjoint from A_n. After ω many moves player II wins if $\bigcup\{B_n : n \in \omega\} \in \mathcal{F}$ and player I wins otherwise.

The following observation will work in the same way for all games in this chapter.

Fact 2.2.2. *Player II has no winning strategy in the game $M_\mathcal{F}$ for any filter $\mathcal{F}$.*

Proof. The crucial observation is that if two games are played simultaneously (alternating moves between them), player I can force player II to pick sets B^0_n in the first game and B^1_n in the second game so that $\bigcup\{B^0_n : n \in \omega\}$ and $\bigcup\{B^1_n : n \in \omega\}$ are disjoint and so can not be both elements of $\mathcal{F}$. That means that that player II loses at least one of these two games and this wouldn't be possible if he had a winning strategy for $M_\mathcal{F}$. $\square$

As we will see, the situation may be very different if two games played simultaneously as in the previous proof are played with different filter each.

Existence of wining strategy for player I is not automatic. With Borel determinacy in hand we can argue that that if $\mathcal{F}$ is a Borel subset of $\mathcal{P}(\omega)$, then player I must have winning strategy since the game is determined. This is nice good agreement with the actual characterization.

Lemma 2.2.3. *Player I has winning strategy in the non-meager game $M_\mathcal{F}$ if and only if $\mathcal{F}$ is a meager filter.*

Proof. If $\mathcal{F}$ is a meager filter there is an interval partition $\{I_n : n \in \omega\}$ of ω witnessing this. Winning strategy for player I is in the n-th move to pick a $n_i \in \omega$ such that $\bigcup\{B_j : j \leq n\} \cap I_{n_i} = \emptyset$ and to play A_n such that $\bigcup\{I_{n_j} : i \leq n\} \subset A_n$.

Suppose $\mathcal{F}$ is a non-meager filter. We will show that player I has no winning strategy in the following modification of the non-meager game. The additional rule is that player II is in the n-th move allowed to play a non-empty set B_n only if $\bigcup\{B_i : i < n\} \subset n$. We will call this modified $M'_\mathcal{F}$. It is obvious that a winning strategy for player I in the game $M_\mathcal{F}$ is also winning in the modified game $M'_\mathcal{F}$.

Suppose $\mathcal{S}$ is a strategy for player I for the game $M'_\mathcal{F}$. For each n there are only finitely many sequences of moves of player II such that he is allowed to play a non-empty set in the n-th move. Denote this finite set $\mathcal{M}_n$. Let us choose inductively a sequence of integers $\{j_i : i \in \omega\}$. Start with $j_0 = 0$ and if j_i is defined pick $j_{i+1} > j_i$ such that $\mathcal{S}(m) \subset j_{i+1}$ for each $m \in \mathcal{M}_{j_i}$. Now use the non-meagerness of $\mathcal{F}$ to find an infinite set $I \subset \omega$ such that $\omega \setminus \bigcup\{[j_i, j_{i+1}) : i \in I\} \in \mathcal{F}$. Player II beats strategy $\mathcal{S}$ if he plays $B_{j_n} = [j_{n+1}, j_{\min(I \setminus (n+1))})$ if $n \in I$ and $B_n = \emptyset$ otherwise. $\square$

The game for characterizing rapidity has similar flavour as the non-meager game. It has one extra parameter, a function in ${}^\omega\omega$, but as will turn out, the concrete choice of function has little impact on the game.

Definition 2.2.4 (rapid game)**.** Let $\mathcal{F}$ be a filter in $\mathcal{P}(\omega)$ and f be an function in ${}^\omega\omega$. The following game is called *rapid game $R_{\mathcal{F},f}$*. In n-th move player I plays a finite set $A_n \in [\omega]^{<\omega}$ and player II responds with a finite set $B_n \in [\omega]^{f(n)}$ disjoint from A_n. After ω many moves player II wins if $\bigcup\{B_n : n \in \omega\} \in \mathcal{F}$ and player I wins otherwise.

It is possible to use the same argument as for non-meager game to see that player II never has a winning strategy for the rapid game $R_{\mathcal{F},f}$ for any $\mathcal{F}$ and f.

As suggested by name, existence of winning strategy for player I characterizes rapid filters.

Lemma 2.2.5. *The following conditions are equivalent for each filter $\mathcal{F}$ in $\mathcal{P}(\omega)$.*

1. *$\mathcal{F}$ is a rapid filter.*
2. *For each growing function $f \in {}^\omega\omega$ player I has no winning strategy for the rapid game $R_{\mathcal{F},f}$.*
3. *There is a function $f \in {}^\omega\omega$ such that player I has no winning strategy for the rapid game $R_{\mathcal{F},f}$.*

Proof. 1 $\Rightarrow$ 2: Pick a function f and again modify the game by allowing player II to play nonempty set B_n only if $\bigcup\{B_i : i < n\} \subset n$, denote this modified game $R'_{\mathcal{F},f}$. Again, if player I had winning strategy for $R_{\mathcal{F},f}$, then he would also have winning strategy for $R'_{\mathcal{F},f}$.

Suppose $\mathcal{S}$ is a strategy for player I for the game $R'_{\mathcal{F},f}$. For each n there are only finitely many sequences of moves of length $n-1$ of player II such that he is allowed to play a non-empty set in the n-th move. Denote this finite set $\mathcal{M}_n$. Let us choose inductively a sequence of integers $\{j_i : i \in \omega\}$. Start with $j_0 = 0$ and if j_i is defined pick $j_{i+1} > j_i$ such that $\mathcal{S}(m) \subset j_{i+1}$ for each $m \in \mathcal{M}_{j_i}$. Now use the non-meagerness of $\mathcal{F}$ to find an infinite set $I \subset \omega$ such that

$$\omega \setminus \bigcup\{[j_i, j_{i+1}) : i \in I\} \in \mathcal{F}.$$

Then use 5 of Lemma 2.1.5 for $\mathcal{F}$ and f to find $F \in \mathcal{F}$ such that $|B_{j_n}| < f(n)$ where

$$B_{j_n} = [j_{n+1}, j_{\min(I\setminus(n+1))}) \cap F$$

for each $n \in I$.

Player II beats strategy $\mathcal{S}$ if he plays B_{j_n} if $n \in I$ and $\emptyset$ otherwise.

2 $\Rightarrow$ 3 is clear. 3 $\Rightarrow$ 1: Suppose 3 holds for function f. We will show that clause 4 of Lemma 2.1.5 holds true for $g(i) = \sum_{j \le i} f(j)$. Let $\{t_i : t_i \in [\omega]^{<\omega}, i \in \omega\}$ be given and $\mathcal{S}$ be strategy for player I such that in the n-th move he always plays $A_n = \bigcup\{t_i : i < n\}$. This is not a winning strategy thus there is a sequence of moves $\{B_n : n \in \omega\}$ of player II which beats this strategy. So $F = \bigcup\{B_n : n \in \omega\} \in \mathcal{F}$ and $|F \cap t_i| < g(i)$. $\square$

Definition 2.2.6 (rare game)**.** The special case of rapid game $R_{\mathcal{F},\mathbf{1}}$, where $\mathbf{1} \in {}^\omega\omega$ is constantly equal 1 is called *rare game*.

The next lemma is proved in exactly the same way as lemma 2.2.5, so the proof is omitted.

Lemma 2.2.7. *Player I has no winning strategy in the rare game $R_{\mathcal{F},\mathbf{1}}$ if and only if $\mathcal{F}$ is a rare filter.*

We will now turn our attention back towards the proof of non-existence of winning strategy for player II. We observed that if we play two games simultaneously (alternating moves) and we require the second player to win both of them, player I has a winning. For this observation it was essential, that result of both games was evaluated with the same filter. The natural question one would ask is, how much similarity between filters in those two games is actually needed for this argument?

A notion relevant for this situation near coherence of filters introduced by A. Blass. It has close relation with the Rudin-Blass ordering.

Definition 2.2.8 (Blass [Bla86])**.** Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be filters. We say that $\mathcal{F}_0$ and $\mathcal{F}_1$ are near coherent if there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathcal{F}_0) \cup f(\mathcal{F}_1)$ has the finite intersection property.

For ultrafilters being non-coherent is the same as not having a common lower bound in the Rudin-Blass ordering $\le_{RB}$. For p-points this is equivalent to not having a common lower bound in the Rudin-Keisler ordering $\le_{RK}$.

Near coherence can be characterized in terms of partitions of ω into finite finite pieces. Especially the slightly technical item 3 will be very useful later.

Lemma 2.2.9 (Eisworth [Eis01])**.** *Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be two filters in $\mathcal{P}(\omega)$. The following are equivalent:*

1. *$\mathcal{F}_0$ and $\mathcal{F}_1$ are not nearly coherent.*

2. *For each partition $\{I_n : n \in \omega\}$ of ω into finite sets there exist two disjoint sets $A_0, A_1 \subset \omega$ such that $\bigcup\{I_n : n \in A_i\} \in \mathcal{F}_i$ for both $i \in 2$.*
3. *For each partition $\{I_n : n \in \omega\}$ of ω into finite sets there exist two disjoint sets $A_0, A_1 \subset \omega$ such that $\bigcup\{I_n : n \in A_i\} \in \mathcal{F}_i$ for both $i \in 2$. Moreover if $n \in A_i \Rightarrow n+1 \notin A_{1-i}$ for $i \in 2$.*

Proof. To see that 1 $\Leftrightarrow$ 2 note that the function taking $i \in I_n$ to n is not witnessing near coherence of $\mathcal{F}_0$ and $\mathcal{F}_1$ and for each finite-to-one function it is possible to define partition of ω into finite I_ns with the same same property.

3 $\Rightarrow$ 2 is obvious. 2 $\Rightarrow$ 3: Let $\{I_n : n \in \omega\}$ be a given partition. Because of 2 we can assume that

$$\bigcup\{I_{2n+i} : n \in \omega\} \in \mathcal{F}_i \text{ for } i \in 2.$$

(Take a coarser partition if necessary.) Use 2 to get $A, B \subset \omega$ such that

$$\bigcup\{I_{2n} \cup I_{2n+1} : n \in A\} \in \mathcal{F}_0,$$

$$\bigcup\{I_{2n} \cup I_{2n+1} : n \notin A\} \in \mathcal{F}_1,$$

$$\bigcup\{I_{2n-1} \cup I_{2n} : n \in B\} \in \mathcal{F}_0,$$

$$\bigcup\{I_{2n-1} \cup I_{2n} : n \notin B\} \in \mathcal{F}_1.$$

Put $A_0 = \{2n : n \in A \cap B\}$ and $A_1 = \{2n+1 : n \notin A, n+1 \notin B\}$. $\square$

Corollary 2.2.10. *If $\mathcal{F}_0$ and $\mathcal{F}_1$ are not nearly coherent then both these filters are non-meager.*

A game consisting of two different ultrafilter games played simultaneously appeared already in [She82]. A simple game characterizing near coherence was formulated by T. Eisworth.

Definition 2.2.11 (near coherence game [Eis01])**.** Let $\mathcal{F}_0, \mathcal{F}_1$ be filters in $\mathcal{P}(\omega)$. The following game is called *near coherence game $C_{\mathcal{F}_0,\mathcal{F}_1}$*. In n-th move player I plays a finite set $A_n \in [\omega]^{<\omega}$ and player II responds with a finite set $B_n \in [\omega]^{<\omega}$ disjoint from A_n. After ω many moves player II wins if $\bigcup\{B_{2n+i} : n \in \omega\} \in \mathcal{F}_i$ for both $i \in 2$ and player I wins otherwise.

The same argument as with previous games works here again (now with playing four games simultaneously).

Fact 2.2.12. *Player II has no winning strategy in the game $C_{\mathcal{F}_0,\mathcal{F}_1}$ for any couple of filters $\mathcal{F}_0, \mathcal{F}_1$.*

Lemma 2.2.13. *Player I has winning strategy in the near coherence game $C_{\mathcal{F}_0,\mathcal{F}_1}$ if and only if filters $\mathcal{F}_i$, $i \in 2$ are near coherent.*

Proof. If filters $\mathcal{F}_i$, $i \in 2$ are near-coherent, there is a partition $\{I_n : n \in \omega\}$ of ω for which there are no $A_0, A_1 \subset \omega$ which would fulfill condition 2 from Lemma 2.2.9. A winning strategy for player I is to play

$$A_n = \bigcup\{I_j : I_j \cap B_i \neq \emptyset \text{ for some } i < n\}.$$

Suppose $\mathcal{F}_i$, $i \in 2$ are not near coherent filters. We will again show that player I has no winning strategy in the modified game $C'_{\mathcal{F}_0,\mathcal{F}_1}$ (Again, player II is allowed to play in n-th move only if he played subsets of n so far.)

Take $\mathcal{S}$ a strategy for player I for the game $C'_{\mathcal{F}_0,\mathcal{F}_1}$. For each n there are only finitely many sequences of moves of player II such that he is allowed to play move n. Denote this finite set $\mathcal{M}_n$. Let us choose inductively a sequence of integers $\{j_i : i \in \omega\}$. Start with $j_0 = 0$ and if j_i is defined pick $j_{i+1} > j_i$ such that $\mathcal{S}(m) \subset j_{i+1}$ for each $m \in \mathcal{M}_{2j_i} \cup \mathcal{M}_{2j_i+1}$. Now use 3 from lemma 2.2.9 to find $A_0, A_1 \subset \omega$ such that $\bigcup\{[j_k, j_{k+1}) : k \in A_i\} \in \mathcal{F}_i$. For $i \in 2$ denote

$$A'_i = \{a \in A_i : \max((A_0 \cup A_1) \cap a \in A_{1-i})\}$$

and for $a \in A'_i$ define a^+ to be $\max(\{b \in A_i : [a, b) \cap A_{1-i} = \emptyset\})$.

Player II beats strategy $\mathcal{S}$ by playing $B_{2j_n+i} = [j_{n+1}, j^+_{n+1})$ if $n = a^+$ for some $a \in A'_{1-i}$ and $\emptyset$ otherwise. $\square$

Remark. We can change the definition of near coherence game by allowing player II to play only sets with size bounded by some function f. This combination of near coherence game and rapid game provides the expected characterization: player I has no winning strategy iff involved filters are not near coherent and rapid.

2.3 P-filter game and variations

The game for characterizing p-filters was invented by Galvin and Mackenzie in [Gal80] (originally for p-points).

Definition 2.3.1 (p-filter game)**.** Let $\mathcal{F}$ be a filter in $\mathcal{P}(\omega)$. The following game is called *p-filter game* $P_{\mathcal{F}}$. In n-th move player I plays a filter set $F_n \in \mathcal{F}$ and player II responds with its finite subset $B_n \in [F_n]^{<\omega}$. After ω many moves player II wins if $\bigcup\{B_n : n \in \omega\} \in \mathcal{F}$ and player I wins otherwise.

For winning strategy of player II we have again the same argument with two simultaneous games.

Fact 2.3.2. *The second player never has a winning strategy in the p-filter game.*

The p-filter game yields characterization of non-meager p-filters.

Lemma 2.3.3. *Filter $\mathcal{F}$ is non-meager p-filter in $\mathcal{P}(\omega)$ if and only if player I has no winning strategy in the p-filter game $P_{\mathcal{F}}$.*

Proof. Assume that there is no winning strategy for player I in the game $P_{\mathcal{F}}$. To prove that $\mathcal{F}$ is p-filter take any $\{F_i : i \in \omega\} \subset \mathcal{F}$ and let the first player play $\bigcap\{F_i : i \leq n\}$ in the n-th move of the game $P_{\mathcal{F}}$. There is a sequence $\{B_i : i \in \omega\}$ of moves of player II which beats this strategy, $\bigcup\{B_i : i \in \omega\} \in \mathcal{F}$ and $\bigcup\{B_i : i \in \omega\} \subseteq^* F_j$ for each $j \in \omega$.

If $\mathcal{F}$ is meager, player I can use his winning strategy for the non-meager game $M_{\mathcal{F}}$.

For the other implication assume that $\mathcal{F}$ is a non-meager p-filter. We will again show that player I has no winning strategy in the modified game $P'_{\mathcal{F}}$ (Again, player II is allowed to play in n-th move only if he played subsets of n so far.)

Let

$$\left\{A_s : s \in {}^{<\omega}\left[[\omega]^{<\omega}\right]\right\} \subset \mathcal{F}$$

be a strategy for player I in the game $P'_{\mathcal{F}}$ (A_s is the response to a sequence s of moves of player II. We have to introduce a sequence of moves for player II which beats this strategy. For $n \in \omega$ put

$$A_n = \bigcap\{A_s : s \text{ is a sequence of legal moves of player II of length } < n\} \in \mathcal{F}$$

and denote $A \in \mathcal{F}$ the pseudointersection of A_n's. Fix an increasing function $f \in {}^{\omega}\omega$ such that $A \subset A_n \cup f(n)$ for each $n \in \omega$. Denote $i_n = f^{(n)}(0)$ for $n \in \omega$. Player II will try to hit as much elements of A as possible. Note that if he is to play in move i_n, he can legally play any finite subset of $A \setminus i_{n+1}$.

The filter $\mathcal{F}$ is non-meager hence there is a set $F \in \mathcal{F}$ and an infinite increasing sequence $\{k_n : n \in \omega\} \subset \omega$ such that $F \cap [i_{k_n}, i_{k_n+1}) = \emptyset$ for each $n \in \omega$.

Let $B_i = A \cap F \cap [i_{k_n+1}, i_{k_{n+1}})$ for $i = i_{k_n}$ and $B_i = \emptyset$ if $i \notin \{i_{k_n} : n \in \omega\}$. The sequence $\{B_i : i \in \omega\}$ is a sequence of legal moves for player II and $\bigcup\{B_i : i \in \omega\} = (A \cap F) \setminus i_{k_0} \in \mathcal{F}$. □

We can combine rules of previously defined games to get characterizations of various kinds of filters.

Definition 2.3.4 (rapid p-filter game)**.** Let $\mathcal{F}$ be a filter on $\mathcal{P}(\omega)$ and f be an function in ${}^{\omega}\omega$. The following game is called *rapid p-filter game* $RP_{\mathcal{F},f}$. In n-th move player I plays a filter set $F_n \in \mathcal{F}$ and player II responds with its finite subset $B_n \in [F_n]^{f(n)}$. After ω many moves player II wins if $\bigcup\{B_n : n \in \omega\} \in \mathcal{F}$ and player I wins otherwise.

Definition 2.3.5 (near coherence p-filter game)**.** Let $\mathcal{F}_0, \mathcal{F}_1$ be filters in $\mathcal{P}(\omega)$. The following game is called *near coherence p-filter game* $CP_{\mathcal{F}_0,\mathcal{F}_1}$. In move $2n+i$ (for $n \in \omega, i \in 2$) player I plays a filter set $F_{2n+i} \in \mathcal{F}_i$ and player II responds with its finite subset $B_{2n+i} \in [F_{2n+i}]^{<\omega}$. After ω many moves player II wins if $\bigcup\{B_{2n+i} : n \in \omega\} \in \mathcal{F}_i$ for both $i \in 2$ and player I wins otherwise.

Following lemmas are proved just by combining techniques used in previous proofs.

Lemma 2.3.6. *The following conditions are equivalent for each filter $\mathcal{F}$ in $\mathcal{P}(\omega)$.*

1. *$\mathcal{F}$ is a rapid p-filter.*
2. *For each growing function $f \in {}^{\omega}\omega$ player I has no winning strategy for the rapid game $RP_{\mathcal{F},f}$.*

3. *There is a function $f \in {}^\omega\omega$ such that player I has no winning strategy for the rapid game $RP_{\mathcal{F},f}$.*

Lemma 2.3.7. *Player I has no winning strategy in the rare p-filter game $RP_{\mathcal{F},\mathbf{1}}$ if and only if $\mathcal{F}$ is a rare p-filter.*

Lemma 2.3.8. *Player I has no winning strategy in the near coherence p-filter game $CP_{\mathcal{F}_0,\mathcal{F}_1}$ if and only if filters $\mathcal{F}_i$, $i \in 2$ are non near coherent p-filters.*

Remark 2.3.9. *Generally if alternate moves in two filter games (for rapid game, p-filter game, ...) and player II has to win both games, player I has no winning strategy (in the composed game) iff he has no winning strategy in both separate games and the two involved filters are not near coherent.*

The situation is significantly different if the alternating is done in a different way; player I plays his moves in both games and only then player II chooses his responses in both games.

The following game is an example of such situation. This game is due to Shelah and is needed in proof of theorem 3.4.2.

Definition 2.3.10 (refining game for p-filter and rare p-filter)**.** Let $\{I_n \in [\omega]^{<\omega} : n \in \omega\}$ be sequence of disjoint intervals. Let $\mathcal{R}, \mathcal{F}$ be filters in $\mathcal{P}(\omega)$ such that $\bigcup_{n\in R} I_n \in \mathcal{F}$ iff $R \in \mathcal{R}$. (i.e. $\mathcal{R} \leq_{RB} \mathcal{F}$).

In n-th move player I plays a filter set $F_n \in \mathcal{F}$ and player II responds with an integer b_n and a finite set $B_n \subset I_{b_n} \cap F_n$. After ω many moves player II wins if $\bigcup\{B_n : n \in \omega\} \in \mathcal{F}$ and player I wins otherwise.

Note that if player II won, then $\{b_n : n \in \omega\} \in \mathcal{R}$. Also the sequence $\{b_n : n \in \omega\}$of moves of player II is not relevant. This integers are introduced just for easier notation and can be reconstructed from B_ns.

The following lemma is what will be needed in proof of 3.4.2.

Lemma 2.3.11. *Let $\mathcal{R}, \mathcal{F}$ be filters as in Definition 2.3.10. If $\mathcal{R}$ is rare and $\mathcal{F}$ is a p-filter then player I has no winning strategy in the refining game from 2.3.10.*

Proof. Pretend that player II plays only B_ns and follow the proof of lemma 2.3.3 with minor modifications.

Function f can be defined such that all it's values are end points of intervals I_n, $n \in \omega$.

Then use that $\mathcal{R}$ is rare to find a set $R \in \mathcal{R}$ such that for each $n \in \omega$ is

$$\{j \in R : I_j \cap [i_{k_n+1}, i_{k_{n+1}}) \neq \emptyset\} \subset \{j_n\}$$

for some $j_n \in \omega$.

Hence $b_i = j_i$, $B_i = A \cap F \cap [i_{k_n+1}, i_{k_{n+1}}) \cap I_{j_i}$ for $i = i_{k_n}$ and $b_i = 0$, $B_i = \emptyset$ if $i \notin \{i_{k_n} : n \in \omega\}$ is a sequence of legal moves of player II which beats this strategy. □

2.4 Tower games

We will investigate the situation for filters generated by decreasing towers in $\mathcal{P}(\omega)$. Let us recall that tower is a sequence $\{T_\alpha \subset \omega : \alpha \in \theta\}$ such that $T_\alpha \subset^* T_\beta$ for $\beta < \alpha < \theta$. Filter generated by $\mathcal{T}$ is denoted $\langle \mathcal{T} \rangle$.

Our motivation for this kind of games will become apparent in chapter 4. We will need to construct fusion sequences in some forcings, where the 'steering' provided by plain filter games is not good enough.

Let us start with a technical lemma which demonstrates, how the technique of countable elementary submodels can be used when dealing with towers.

Lemma 2.4.1. *Let $\mathcal{T}_i = \{T^i_\alpha : \alpha \in \kappa_i\}$ be a decreasing tower in $\mathcal{P}(\omega)$ generating filter $\mathcal{F}_i$ for $i \in 2$. Let*

$$\{t_n : t_n \in [\omega]^{<\omega}, i \in \omega\}$$

be a sequence of disjoint sets and let f be a growing function in ${}^\omega\omega$. Let θ be cardinal large enough and let M be a countable elementary submodel of $H(\theta)$ such that $\{t_n : n \in \omega\}, f, \mathcal{T} \in M \prec H(\theta)$. Denote $\sup M \cap \kappa_i = \varepsilon^i_M$. Then

1. *$\mathcal{F}_0$ is a non-meager filter $\Rightarrow$ there is an infinite $A \subset \omega$ such that $t_n \cap T^0_{\varepsilon^0_M} = \emptyset$ for each $n \in A$.*

2. *$\mathcal{F}_0$ is rapid filter $\Rightarrow$ there exists $n_0 \in \omega$ such that $|t_n \cap T^0_{\varepsilon^0_M}| < f(n)$ for each $n > n_0$.*

3. *$\mathcal{F}_0, \mathcal{F}_1$ are not near coherent and $\omega = \bigcup\{t_n : n \in \omega\} \Rightarrow$ there are $A_0, A_1 \subset \omega$ such that*

$$T^i_{\varepsilon^i_M} \subset^* \bigcup\{t_n : n \in A_i\}$$

for $i \in 2$ and if $n \in A_i \Rightarrow n, n+1 \notin A_{1-i}$.

Proof. We will show only 1, the rest is analogous. The assumption implies that there is some $\alpha \in M$ such that $T_\alpha \in M$ misses infinitely many t_is. Hence T_{ε_M} has the same property since $T_{\varepsilon_M} \subset^* T_\alpha$. □

Some p-filters are generated by decreasing towers in $\mathcal{P}(\omega)$. Moreover if we assume CH then each p-filter is of this kind, so investigating only such p-filters may not be a restriction at all. Also while doing forcing constructions, we can usually assume CH in the groundmodel.

Suppose that a tower $\mathcal{T}$ generates a (p-)filter $\mathcal{F}$. We can equivalently redefine the p-filter game $P_{\mathcal{F}}$.

Definition (p-filter game for towers)**.** Let $\mathcal{T} = \{T_\alpha : \alpha \in \kappa\}$ be a tower in $\mathcal{P}(\omega)$. In n-th move player I plays an ordinal $\alpha_n \in \kappa$ and a finite set $A_n \in [\omega]^{<\omega}$ and player II responds with a finite set $B_n \subset T_{\alpha_n} \setminus A_n$. After ω many moves player II wins if there exists $\gamma \in \kappa$ such that $T_\gamma \subset^* \bigcup\{B_n : n \in \omega\}$ and player I wins otherwise.

From the previous section we know following lemma.

Lemma. *The tower $\mathcal{T}$ generates a non-meager (p-)filter in $\mathcal{P}(\omega)$ if and only if player I has no winning strategy in the p-filter game for towers.*

Further modification (this time not equivalent) of the p-filter game produces what will be called the tower game. This is a stronger notion (harder game for player II) than just plain p-filter game. Its significant applications will be seen in further chapters.

Definition 2.4.2 (tower game)**.** Let $\mathcal{T} = \{T_\alpha : \alpha \in \kappa\}$ be a decreasing tower in $\mathcal{P}(\omega)$. The following game is called *tower game $TG_{\mathcal{T}}$*. In n-th move player I plays an ordinal $\alpha_n \in \kappa$ and a finite set $A_n \in [\omega]^{<\omega}$ and player II responds with an ordinal $\beta_n \in \kappa$ and a finite set $B_n \subset T_{\alpha_n} \setminus A_n$. After ω many moves player II wins if $\gamma = \sup(\beta_n : n \in \omega) \in \kappa$ and $T_\gamma \subset^* \bigcup\{B_n : n \in \omega\}$ and player I wins otherwise.

So the tower game requires that player II has not only to collect a set in the filter generated by $\mathcal{T}$, he is also supposed to correctly guess the level of $\mathcal{T}$ which witnesses this.

Fact 2.4.3. *Player II never has a winning strategy in the tower game.*

We show, that although the tower game seems to be significantly harder for the second player than the p-filter game, this is not the case. If there was no winning strategy for player I in the p-filter game, then he does not have winning strategy in the filter game.

Theorem 2.4.4. *The decreasing tower $\mathcal{T} = \{T_\alpha : \alpha \in \kappa\}$ generates a non-meager filter in $\mathcal{P}(\omega)$ if and only if player I has no winning strategy in the tower game $TG_{\mathcal{T}}$.*

Note that if cofinality of κ is countable then $\mathcal{T}$ generates a meager filter.

Proof. If the filter generated by $\mathcal{T}$ is meager, then player I can use his winning strategy for the p-filter game $P_{\langle\mathcal{T}\rangle}$ to play so that he forces $\bigcup\{B_n : n \in \omega\} \notin \langle\mathcal{T}\rangle$.

Suppose that $\mathcal{T}$ generates a non-meager filter. We will again show that player I has no winning strategy in the modified game $TG'_{\mathcal{T}}$ where player II is allowed to play in the n-th move a nonempty set B_n only if in previous moves $i < n$ he only played subsets of n i.e. $B_i \subset n$.

Let $\mathcal{S} = \{(\alpha_s, A_s) : \alpha \in \kappa, A \in [\omega]^{<\omega}\}$ be a strategy for player I in the game $TG'_{\mathcal{T}}$. Here (α_s, A_s) is the response to a sequence s of legal moves of player II. We have to introduce a sequence of moves for player II which beats this strategy.

Pick a large enough cardinal θ and an increasing sequence of countable elementary submodels $\{M_k \colon k \in \omega\}$ such that $\mathcal{T}, \mathcal{S} \in M_k \prec M_{k+1} \prec H(\theta)$, $M_k \in M_{k+1}$ for each $k \in \omega$ and put $M = \bigcup\{M_k \colon k \in \omega\} \prec H(\theta)$. For each elementary submodel N denote $\varepsilon_N = \sup(N \cap \kappa)$ and fix a sequence of ordinals

$$\{\beta'_k \colon k \in \omega, \beta'_k \in M_k, \sup\{\beta'_k \colon k \in \omega\} = \varepsilon_M\}.$$

We will inductively build a sequence of sequences of integers $J^k = \{j^k_i \colon k, i \in \omega\}$ such that for each k is J^k an increasing sequence and $J^k \in M_{k+1}$. Also for each k is J^{k+1} a subsequence of J^k.

Start defining J^0 by choosing j^0_0 such that $T_{\varepsilon_M} \setminus j^0_0 \subset T_{\varepsilon_{M_0}}$. Suppose j^0_i is defined. The set $\mathcal{M}^0_i$ of possible sequences of length j^0_i of legal moves of player II, such that he played always the ordinal β'_0 and he is allowed to play a nonempty set $B_{j^0_i}$ in the move j^0_i, is only finite. Choose a $j^0_{i+1} \in \omega$, $j^0_i < j^0_{i+1}$ such that $\bigcup\{A_s \colon s \in \mathcal{M}^0_i\} \subset j^0_{i+1}$ and $T_{\varepsilon_{M_0}} \subset T_{\alpha_s} \cup j^0_{i+1}$ for each $s \in \mathcal{M}^0_i$. Note that if player II is to play in move j^0_i and he has only played ordinal β'_0 and subsets of j^0_i so far, he can legally play any finite subset of $T_{\varepsilon_M} \setminus j^0_{i+1}$.

Now assume that J^{k-1} is already defined and construct J^k in the following way: Choose $j^k_0 \in J^{k-1}$ such that $T_{\varepsilon_M} \subset T_{\varepsilon_{M_k}} \cup j^k_0$. Suppose j^k_i is defined. The set $\mathcal{M}^k_i$ of possible sequences of length j^k_i of legal moves of player II, such that he played only ordinals $\beta_n \in \{\beta'_l \colon l \leq k\}$ and he is allowed to play a nonempty set $B_{j^k_i}$ in the move j^k_i, is only finite. Choose $j^k_{i+1} \in J^{k-1}$, $j^k_i < j^k_{i+1}$ such that $\bigcup\{A_s \colon s \in \mathcal{M}^k_i\} \subset j^k_{i+1}$ and $T_{\varepsilon_{M_k}} \subset T_{\alpha_s} \cup j^k_{i+1}$ for each $s \in \mathcal{M}^k_i$. Again, if player II is to play in move j^k_i and he has only played ordinals β'_l, $l \leq k$ and subsets of j^k_i so far, he can legally play any finite subset of $T_{\varepsilon_M} \setminus j^k_{i+1}$.

Use the non-meagerness of filter generated by $\mathcal{T}$ together with $J^k \in M$ for each $k \in \omega$ to see that for each k the set of integers j^k_i such that $[j^k_i, j^k_{i+1}) \cap T_{\varepsilon_M} = \emptyset$ is infinite.

Finally choose a sequence $d^k \in J^k, k \in \omega$ in the following way. Start with some $d^0 = j^0_{i_0} \in J^0$ such that $[j^0_i, j^0_{i+1}) \cap T_{\varepsilon_M} = \emptyset$. If $d^k \in J^k$ is defined, choose d^{k+1} to be some $j^{k+1}_{i_{k+1}}$ such that $d^k < d^{k+1}$ and

$$[j^{k+1}_{i_{k+1}}, j^{k+1}_{i_{k+1}+1}) \cap T_{\varepsilon_M} = \emptyset.$$

For notational reasons define $d^{-1} = 0$.

Player II beats strategy $\mathcal{S}$ by the following sequence of moves: in move n he plays

- $(\beta'_k, \emptyset)$ if $n \in (d^{k-1}, d^k)$ for $k \in \omega$
- $\big(\beta'_k, T_{\varepsilon_M} \cap [j^k_{i_k+1}, d^{k+1})\big)$ if $n = d^k$ for $k \in \omega$.

This is a legal sequence of moves since

$$T_{\varepsilon_M} \cap [j^k_{i_k+1}, d^{k+1}) \subset T_{\varepsilon_{M_k}},$$

$$\bigcup\{A_s \colon s \in \mathcal{M}^k_{i_k}\} \subset j^k_{i_k+1}$$

and

$$T_{\varepsilon_{M_k}} \setminus j^k_{i_k+1} \subset T_{\alpha_s} \text{ for } s \in \mathcal{M}^k_{i_k}.$$

Note that $\varepsilon_M = \sup\{\beta'_k \colon k \in \omega\}$ and $T_{\varepsilon_M} \cap [d^k, d^{k+1}) = T_{\varepsilon_M} \cap [j^k_{i_k+1}, d^{k+1})$ for each $k \in \omega$. □

In the same way we modified definition of p-filter game for towers, we can also modify other games. The resulting game again won't be more difficult for player II than the unmodified version.

Definition 2.4.5 (rapid tower game)**.** Let $\mathcal{T} = \{T_\alpha \colon \alpha \in \kappa\}$ be a decreasing tower in $\mathcal{P}(\omega)$ and f be an function in ${}^\omega\omega$. The following game is called *rapid tower game* $RT_{\mathcal{T},f}$. In n-th move player I plays an ordinal $\alpha_n \in \kappa$ and a finite set $A_n \in [\omega]^{<\omega}$ and player II responds with an ordinal $\beta_n \in \kappa$ and a finite set $B_n \subset T_{\alpha_n} \setminus A_n$, $|B_n| \leq f(n)$. After ω many moves player II wins if $\gamma = \sup(\beta_n \colon n \in \omega) \in \kappa$ and $T_\gamma \subset^* \bigcup\{B_n \colon n \in \omega\}$ and player I wins otherwise.

Theorem 2.4.6. *The following conditions are equivalent for each decreasing tower $\mathcal{T} = \{T_\alpha \colon \alpha \in \kappa\}$ in $\mathcal{P}(\omega)$.*

1. *$\mathcal{T}$ generates rapid filter.*
2. *For each growing function $f \in {}^\omega\omega$ player I has no winning strategy for the rapid tower game $RT_{\mathcal{T},f}$.*

3. *There is a function $f \in {}^{\omega}\omega$ such that player I has no winning strategy for the rapid tower game $RT_{\mathcal{T},f}$.*

Proof. Implication 2 $\Rightarrow$ 3 is obvious and 3 $\Rightarrow$ 1 follows from lemma 2.2.5.

To prove 1 $\Rightarrow$ 2 it is sufficient to combine ideas from proofs of theorem 2.4.4 and lemma 2.2.5.

Let $\mathcal{S} = \{(\alpha_s, A_s) : \alpha \in \kappa, A \in [\omega]^{<\omega}\}$ be a strategy for player I in the modified game $RT'_{\mathcal{T},f}$. Again, pick a large enough cardinal θ and an increasing sequence of countable elementary submodels $\{M_k : k \in \omega\}$ such that $\mathcal{T}, \mathcal{S}, f \in M_k \prec M_{k+1} \prec H(\theta)$, $M_k \in M_{k+1}$ for each $k \in \omega$ and put $M = \bigcup\{M_k : k \in \omega\} \prec H(\theta)$. For each elementary submodel N denote $\varepsilon_N = \sup(N \cap \kappa)$ and fix an increasing sequence of ordinals

$$\{\beta'_k : k \in \omega, \beta'_k \in M_k, \sup\{\beta'_k : k \in \omega\} = \varepsilon_M\}.$$

We will inductively build a sequence of sequences of integers $J^k = \{j^k_i : k, i \in \omega\}$ such that for each k is $J^k \in M_{k+1}$ an increasing sequence and J^{k+1} is a subsequence of J^k.

Start defining J^0 by picking some j^0_0 such that $T_{\varepsilon_M} \setminus j^0_0 \subset T_{\varepsilon_{M_1}} \setminus j^0_0 \subset T_{\varepsilon_{M_0}}$. Suppose j^0_i is defined. The set $\mathcal{M}^0_i$ of possible sequences of length j^0_i of legal moves of player II, such that he played always the ordinal β'_0 and he is still allowed to play a nonempty set $B_{j^0_i}$ in the move j^0_i, is only finite. Choose $j^0_{i+1} > j^0_i$ such that $\bigcup\{A_s : s \in \mathcal{M}^0_i\} \subset j^0_{i+1}$ and $T_{\varepsilon_{M_0}} \setminus j^0_{i+1} \subset T_{\alpha_s}$ for each $s \in \mathcal{M}^0_i$ (this is possible since $\alpha_s \in M_0$).

Note that if player II is to play in move j^0_i and he has only played ordinal β'_0 and subsets of j^0_i so far, he can legally play any subset of $T_{\varepsilon_{M_1}} \setminus j^0_{i+1}$ of size less than $f(j^0_i)$.

Now assume that J^{k-1} is already defined and construct J^k in the following way.

1) Case k is odd: Choose any increasing sequence

$$J^k = \{j^k_i = j^{k-1}_{l^k_i} \in J^{k-1} : i \in \omega\}$$

such that

$$[j^k_i, j^{k-1}_{l^k_i+1}) \cap T_{\varepsilon_{M_k}} = \emptyset.$$

This is possible since $\mathcal{T}$ generates a non-meager filter and $J^{k-1} \in M_k$.

2) Case k even: Choose $j^k_0 \in J^{k-1}$ such that

$$T_{\varepsilon_M} \setminus j^k_0 \subset T_{\varepsilon_{M_{k+1}}} \setminus j^k_0 \subset T_{\varepsilon_{M_k}}.$$

Suppose j^k_i is defined. The set $\mathcal{M}^k_i$ of possible sequences of length j^k_i of legal moves of player II, such that he played only ordinals $\beta_n \in \{\beta'_m : m \leq k\}$ and he is still allowed to play a nonempty set $B_{j^k_i}$ in the move j^k_i is again finite. Choose $j^k_{i+1} \in J^{k-1}, j^k_{i+1} > j^k_i$ such that $\bigcup\{A_s : s \in \mathcal{M}^k_i\} \subset j^k_{i+1}$ and $T_{\varepsilon_{M_k}} \setminus j^k_{i+1} \subset T_{\alpha_s}$ for each $s \in \mathcal{M}^k_i$. Again, if player II is to play in move j^k_i and he has only played ordinals $\beta'_m, m \leq k$ and subsets of j^k_i so far, he can legally play any subset of $T_{\varepsilon_{M_{k+1}}} \setminus j^k_{i+1}$ of size less than $f(j^k_i)$.

Finally choose an increasing sequence of integers $d^k \in J^{2k+1}$, $k \in \omega$ in the following way. Start with some $d^0 = j^1_{i_0} \in J^1$ such that for each $i \geq i_0$ is $\left|[j^1_i, j^1_{i+1}) \cap T_{\varepsilon_M}\right| < f(j^1_i)$. To see that this is possible note that both J^1 and f are elements of M, remember that $\mathcal{T}$ generates a rapid filter and use 5 from lemma 2.1.5.

If $d^{k-1} \in J^{2k-1}$ is defined, choose d^k to be some $j^{2k+1}_{i_k}$ such that $d^{k-1} < d^k$ and for each $i \geq i_k$ is

$$\left|[j^{2k+1}_i, j^{2k+1}_{i+1}) \cap T_{\varepsilon_M}\right| < f(j^{2k+1}_i).$$

Player II beats strategy $\mathcal{S}$ by the following sequence of moves: in move n he plays

- $(\beta'_0, \emptyset)$ if $n < d^0$
- $(\beta'_k, \emptyset)$ if $n \in [d^k, d^{k+1})$ and $n \notin J^{2k+1}$ for some $k \in \omega$
- $\left(\beta'_k, T_{\varepsilon_M} \cap [j^{2k+1}_i, j^{2k+1}_{i+1})\right)$ if $n \in [d^k, d^{k+1})$ and $n = j^{2k+1}_i \in J^{2k+1}$ for some $i, k \in \omega$.

This is a legal sequence of moves since

$$T_{\varepsilon_M} \cap [j^{2k+1}_i, j^{2k+1}_{i+1}) \subset T_{\varepsilon_{M_{2k+1}}},$$

$$\bigcup\{A_s : s \in \mathcal{M}^{2k+1}_i\} \subset j^k_{i_k+1},$$

$$T_{\varepsilon_{M_{2k+1}}} \setminus j^{2k+1}_{i+1} \subset T_{\alpha_s} \text{ for } s \in \mathcal{M}^{2k+1}_i$$

and

$$\left|T_{\varepsilon_M} \cap [j^{2k+1}_i, j^{2k+1}_{i+1})\right| < f(j^{2k+1}_i).$$

Note that $\varepsilon_M = \sup\{\beta'_k \colon k \in \omega\}$ and

$$T_{\varepsilon_M} \cap [d^k, d^{k+1}) = \bigcup \{T_{\varepsilon_M} \cap [j^{2k+1}_i, j^{2k+1}_{i+1}) \colon j^{2k+1}_i \in [d^k, d^{k+1})\}$$

for each $k \in \omega$. □

The same proof also yields a lemma for rare filters generated by towers.

Theorem 2.4.7. *The decreasing tower $\mathcal{T} = \{T_\alpha \colon \alpha \in \kappa\}$ generates a rare filter in $\mathcal{P}(\omega)$ if and only if player I has no winning strategy in the game $RG_{\mathcal{T},f}$.*

And we can redefine the near coherence game for p-filters as well.

Definition 2.4.8 (near coherence tower game)**.** Let $\mathcal{T}_0 = \{T^0_\alpha \colon \alpha \in \kappa_0\}, \mathcal{T}_1 = \{T^1_\alpha \colon \alpha \in \kappa_1\}$ be decreasing towers in $\mathcal{P}(\omega)$. The following game is called *near coherence tower game $CT_{\mathcal{T}_0,\mathcal{T}_1}$*. In move $2n+i$ (for $n \in \omega$, $i \in 2$) player I plays an ordinal $\alpha_{2n+i} \in \kappa_i$ and a finite set $A_{2n+i} \in [\omega]^{<\omega}$, player II responds with an ordinal $\beta_{2n+i} \in \kappa_i$ and a finite set $B_{2n+i} \subset T^i_{\alpha_{2n+i}} \setminus A_{2n+i}$. After ω many moves player II wins if $\gamma_i = \sup(\beta_{2n+i} \colon n \in \omega, i \in 2) \in \kappa_i$ and $T^i_{\gamma_i} \subset^* \bigcup\{B_{2n+i} \colon n \in \omega\}$ for both $i \in 2$ and player I wins otherwise.

The resulting game has again the same conditions for existence of winning strategy as the original one.

Theorem 2.4.9. *Player I has winning strategy in the near coherence tower game $CT_{\mathcal{T}_0,\mathcal{T}_1}$ if and only if $\mathcal{T}_0, \mathcal{T}_1$ do not generate near coherent filters.*

Proof. If $\langle\mathcal{T}_0\rangle$ and $\langle\mathcal{T}_1\rangle$ are near coherent, lemma 2.2.13 implies that player I has winning strategy.

Suppose that $\langle\mathcal{T}_0\rangle, \langle\mathcal{T}_1\rangle$ are not near coherent and let us prove that no strategy

$$\mathcal{S} = \{(\alpha_s, A_s) \colon \alpha \in \kappa_i, A \in [\omega]^{<\omega}\}$$

is winning for player II in the modified game $CT'_{\mathcal{T}_0,\mathcal{T}_1}$.

We will follow the proof of theorem 2.4.4 with some modifications. We choose M_0 such that both $\mathcal{T}_0, \mathcal{T}_1 \in M_0$ and ${\beta^i_n}' \in M_n$ such that $\sup\{{\beta^i_n}' \colon n \in \omega\} = \kappa_i$ for $i \in 2$. For some countable elementary submodel N we will denote $\sup N \cap \kappa_i = \varepsilon^i_N$.

Whenever we were defining some $j \in J^k$ satisfying some condition for the single tower $\mathcal{T}$, choose this j to fulfill analogous condition for both towers $\mathcal{T}_i$ and ε^i instead. Also the set $\mathcal{M}^k$ of possible moves of player II will contain all moves containing $\beta^{i\prime}_l$ for $i \in 2$ instead of just ${\beta_l}'$.

After J^k is defined for each $k \in \omega$, choose sets $A^k_i \subset \omega$ such that

$$T^i_{\varepsilon^i_M} \subset^* \bigcup\{[j^k_{n+1}, j^k_{n+2}) \colon n \in A^k_i\}$$

and $n \in A_i \Rightarrow n, n+1 \notin A_{1-i}$. (Use lemma 2.4.1.)

Then define increasing sequence $\{d^k \colon k \in \omega\}$, $d_k \in J^k$. Start with $d^0 = j^0_{n_0} \in J^0$, $n_0 \in A^0_0$ and

$$T^i_{\varepsilon^i_M} \setminus d^0 \subset \bigcup\{[j^0_{n+1}, j^0_{n+2}) \colon n \in A^0_i\}$$

for both $i \in 2$.

If $d^{k-1}$is defined for some $k \in \omega$ pick $d^k = j^k_{n_k} \in J^k$, $d^{k-1} < d^k$, $n_k \in A^k_0$ and

$$T^i_{\varepsilon^i_M} \setminus d^k \subset \bigcup\{[j^k_{n+1}, j^k_{n+2}) \colon n \in A^k_i\}$$

for both $i \in 2$.

Player II beats strategy $\mathcal{S}$ by the following sequence of moves: in move n he plays

- $({\beta^i_0}', \emptyset)$ if $n < d^0$, and $n = i \mod 2$

- $({\beta^i_k}', \emptyset)$ if $n \in [d^k, d^{k+1})$ for some $k \in \omega$, $n = i \mod 2$ and $n \neq j^k_l$ for each $l \in A^k_i$
- $\left({\beta^i_k}', T^i_{\varepsilon_M} \cap [j^k_{l+1}, j^k_{l+2})\right)$ if $n \in [d^k, d^{k+1})$ for some $k \in \omega$, $n = i \mod 2$ and $n = j^k_l$ for some $l \in A^k_i$

$\square$

It is possible to play simultaneously one modified tower game and one unmodified filter game.

Definition 2.4.10 (Mixed near coherence game)**.** Let $\mathcal{T} = \{T_\alpha \colon \alpha \in \kappa\}$ be a decreasing tower and $\mathcal{F}$ a filter in $\mathcal{P}(\omega)$. The following game is called *mixed near coherence game* $CM_{\mathcal{T},\mathcal{F}}$. In move $2n$ (for $n \in \omega$) player I plays an ordinal $\alpha_{2n} \in \kappa$ and a finite set $A_{2n} \in [\omega]^{<\omega}$, player II responds with an ordinal $\beta_{2n} \in \kappa$ and a finite set $B_{2n} \subset T_{\alpha_{2n}} \setminus A_{2n}$. In move $2n+1$ player I plays a set $F_{2n+1} \in \mathcal{F}$, player II responds with a finite set $B_{2n+1} \subset F_{2n}$. After ω many moves player II wins if $\gamma = \sup(\beta_{2n} \colon n \in \omega) \in \kappa$, $T_\gamma \subset^* \bigcup\{B_{2n} \colon n \in \omega\}$ and $\bigcup\{B_{2n+1} \colon n \in \omega\} \in \mathcal{F}$. and player I wins otherwise.

The result is again the expected one.

Theorem 2.4.11. *Player I has no winning strategy in the mixed near coherence game $CM_{\mathcal{T},\mathcal{F}}$ if and only if $\mathcal{F}$ is a p-filter and $\mathcal{T}$ does not generate filter near coherent with $\mathcal{F}$.*

Proof. Essentially the same proof as for theorem 2.4.9 works. The only necessary modification is that after choosing the sequence of elementary submodels $\{M_n \colon n \in \omega\}$ and M, we need to choose $F_N \in \mathcal{F}$ such that $F_N \subset^* F$ for each $F \in \mathcal{F} \cap N$ for each chosen elementary submodel N. Then the proof continues in the same way, we only need to use F_N in place of T_{ε_N}. $\square$

Remark 2.4.12. *It is possible to combine tower and mixed near coherent game with the game for rapidity or game of rareness. Combining arguments used in previous proofs it is not difficult to prove the expected result; player I has no winning strategy iff both (generated) filters are rapid and not near coherent.*

CHAPTER 3

DESTROYING AND PRESERVING P-POINTS

We will present here well known results of S. Shelah from [She82, She98a] concerning building models with limited amount of p-points (or no p-points at all). We will mostly follow the Shelah's original proofs, for a slightly different approach (but using same ideas) see [Wim82]. A nice presentation of the no p-points consistency is also in [Woh08].

There are multiple reasons for inclusion of this chapter. Mainly, existence of p-points has influence on the Katowice problem. As we saw in theorem 1.3.8, if there were an isomorphism between $\mathcal{P}(\omega_1)/\operatorname{Fin}$ and $\mathcal{P}(\omega)/\operatorname{Fin}$, all p-points on ω would need to intersect the image of the ideal of countable subsets of ω_1. In next chapter our goal will be to construct a countable like ideal, and for that some p-point killing is necessary.

Other reasons is the similarity of forcings (and arguments applied) for killing p-points to forcing notions we introduce in order to achieve different goal, namely forcing a strong-Q-sequence.

And the author of this text believes, that the slightly different presentation of these proofs, which is provided here, has the advantage of being simpler and more canonical than the original one. This applies mainly for section 3.4, where we use simpler than the one the original proof of Shelah.

3.1 Forcing with filters, Grigorieff and Sacks

Our tools for killing p-points will be two similar forcing notions. One of them is traditionally called Grigorieff's forcing and the other one we will call Sacks forcing. It should be mentioned that we use the name Sacks forcing for a different forcing notion, than the one usually called Sacks in the literature (i.e. forcing with perfect trees).

The main features (besides killing p-points) of these forcing notions are properness and being ${}^{\omega}\omega$ bounding (and some ultrafilter preservation for Sacks). It is possible to further refine these forcing methods to get even stronger preservation properties, for Grigorieff see [GS90] and for Sacks see [She92]. Using refined versions, it is even possible to get model with no nowhere dense ultrafilter. For more information about this topic see [Bau95, She98b, Bre99].

Definition 3.1.1 (Grigorieff's forcing [Gri71])**.** Let $\mathcal{F}$ be a filter on ω. Put

$$G(\mathcal{F}) = (\{g\colon I \to 2; I \in \mathcal{F}^*\}, \supset)\,.$$

The forcing notion $G(\mathcal{F})$ is called *Grigorieff's forcing*.

Grigorieff's forcing has size at most 2^ω (but can be countable if $\mathcal{F}$ is Fréchet filter). Depending on the choice $\mathcal{F}$, it is either iteration of σ-closed and a *ccc* forcing or it collapses ω_1. For proof see [Rep88].

Definition 3.1.2 (Sacks forcing with a filter)**.** Let $\mathcal{F}$ be a filter on ω. A condition p in the forcing $S(\mathcal{F})$ is a subtree of the binary tree ${}^{<\omega}2$ such that the set $S(p)$ of splitting levels of p is in the filter $\mathcal{F}$;

$$S(p) = \{n \in \omega \colon \forall s \in p\,(|s| = n) \Rightarrow \big(s^\frown 0, s^\frown 1 \in p\big)\} \in \mathcal{F}.$$

The ordering is inclusion, i.e. $q < p$ iff $q \subset p$. The forcing notion $S(\mathcal{F})$ is called *Sacks forcing with a filter*.

Sacks forcing has again size at most 2^ω. Note that if we add an additional requirement for conditions $p \in S(\mathcal{F})$ that for each $s, t \in p$ such that $|s| = |t|$ is $s^\frown 0 \Leftrightarrow t^\frown 0$ and $s^\frown 1 \Leftrightarrow t^\frown 1$, the resulting forcing is isomorphic to $G(\mathcal{F})$. For such p the set of all branches through p is precisely a set of all functions extending a condition in the Grigorieff's forcing $G(\mathcal{F})$. So the Grigorieff forcing can be regarded as an uniform version of the Sacks forcing. In some cases we will be rather dealing with subtrees of ${}^{\lessdot A}2$ for a general countable set A. In these cases there will be always declared some ordering $\lessdot$ on A in which A will be order isomorphic with $(\omega, \in)$.

Generic objects for both these forcing notions are characteristic functions of a subset of ω (union of conditions in the generic filter in case of Grigorieff forcing, intersection of all condition in the Sacks case).

If we replace the filter $\mathcal{F}$ with a base for this filter (in $\mathcal{P}(\omega)$) then we get a dense subset of the original forcing (and hence an equivalent forcing notion).

Lemma 3.1.3. *Let $\mathcal{F}$ be a non-meager p-filter on ω. Both $G(\mathcal{F})$ and $S(\mathcal{F})$ are proper ${}^\omega\omega$ bounding forcing notions. If $\mathcal{F}$ is moreover rapid then both this forcings have Sacks property.*

We will at first prove a helpful 'one step' lemma.

Lemma 3.1.4.

1. *Pick any $p \in G(\mathcal{F})$. Suppose $p \Vdash \dot{x} \in \hat{X}$ and fix a finite set $a \in [\omega \setminus \mathrm{Dom}(p)]^n$. Then there exists a condition $q \in G(\mathcal{F})$, $q < p$ and a finite set $Y \in [X]^{\leq 2^n}$ such that $q \Vdash \dot{x} \in \hat{Y}$ and $\mathrm{Dom}(q) \cap a = \emptyset$.*
2. *Pick any $p \in S(\mathcal{F})$. Suppose $p \Vdash \dot{x} \in \hat{X}$ and fix a finite set $a \in [S(p)]^n$. Then there exists a condition $q \in S(\mathcal{F})$, $q < p$ and a finite set $Y \in [X]^{\leq 2^n}$ such that $q \Vdash \dot{x} \in \hat{Y}$ and $a \subset S(q)$.*

Proof. Start with Grigorieff. Let $\{t_i \colon i \in 2^n\}$ be an enumeration of ${}^a2$ and denote $q_0 = p$. Now for $i \in 2^n$ repeat inductively the following procedure:

$\mathrm{Dom}(q_i) \cap a = \emptyset$ hence $q_i \cup t_i \in G(\mathcal{F})$. Find $q'_i \in G(\mathcal{F})$, $q'_i < q_i \cup t_i$ and $x_i \in X$ such that $q'_i \Vdash \dot{x} = \hat{x}_i$. Put $q_{i+1} = q'_i \restriction (\omega \setminus a)$.

Finally put $q = q_{2^n}$ and $Y = \bigcup\{x_i \colon i \in 2^n\}$.

The Sacks case is even easier. Fix $k \in \omega$ such that $a \subset k$ and a condition $p' < p$ such that $S(p') \cap k = a$. For each $l \in p'^{[k]}$ fix $q_l \in S(\mathcal{F}), x_l \in X$ such that $q_l < p'[l]$ and $q_l \Vdash \dot{x} = \hat{x}_i$. Put $q = \bigcup\{q_l \colon l \in p'^{[k]}\} \in S(\mathcal{F})$. Note that $|p'^{[k]}| = 2^n$ and $S(q) = a \cup \bigcap\{S(q_l) \colon l \in p'^{[k]}\} \in \mathcal{F}$. ☐

The lemma still holds true when X is not a set but a proper class. In this case we can find some set $X' \subset X$ such that $p \Vdash \dot{x} \in \hat{X}'$ and then use the lemma for X'. We will abuse this fact later.

Proof of 3.1.3. The same proof works for both Grigorieff and Sacks forcing. The Sacks case will be presented here.

At first we will prove that $S(\mathcal{F})$ is ${}^\omega\omega$ bounding. Fix any $g \in S(\mathcal{F})$ and $\dot{f}$ such that $g \Vdash \dot{f} \in {}^\omega\omega$.

Two players will play the p-filter game $P_\mathcal{F}$ and player I will follow this strategy: At first he denotes g as h_0 and puts $a_0 = \emptyset$. In the n-th move he has some condition $h_n \leq g$ and a set $a_n \in [\omega]^{<\omega}$ such that $a_n \subset S(h_n)$. Now he uses lemma 3.1.4 for $h_n \Vdash \dot{f}(n) \in \omega$ and the finite set a_n to get a condition $h_{n+1} < h_n$ and a finite set $Y_n \in [\omega]^{<\omega}$ such that $h_{n+1} \Vdash \dot{f}(n) \in Y_n$, $a_n \subset S(h_{n+1})$. The n-th move (for $n \in \omega$) of player I is $S(h_{n+1}) \in \mathcal{F}$. To this player II responds in n-th move with some set $b_n \in [S(h_{n+1})]^{<\omega}$. Player I denotes $a_{n+1} = a_n \cup b_n$ (so $a_{n+1} \subset S(h_{n+1})$) and continues with move $n+1$.

When the game is over, player I collected a sequence of conditions $\{h_n \colon n \in \omega\} \subset S(\mathcal{F})$, $h_{n+1} < h_n \leq g$ and a sequence of finite sets $\{Y_n \colon n \in \omega\}$ such that $h_{n+1} \Vdash \dot{f}(n) \in Y_n$. According to lemma 2.3.3 the described strategy is not winning for player I so we can assume that the actual course of this game was won by player II. Hence $h = \bigcap\{h_n \colon n \in \omega\} \in S(\mathcal{F})$ because $S(h) \supset \bigcup\{a_n \colon n \in \omega\} \in \mathcal{F}$. Now $h < h_n$ for each $n \in \omega$ thus $h \Vdash \dot{f} \in \prod\{Y_n \colon n \in \omega\}$ and we proved that $S(\mathcal{F})$ is ${}^\omega\omega$ bounding.

The proof of properness of $S(\mathcal{F})$ is similar to the proof of boundedness, we just have to be little more cautious in some details. Take any countable elementary submodel M of $H(\theta)$ (for sufficiently large θ) containing $S(\mathcal{F})$

and a condition $g \in S(\mathcal{F}) \cap M$. Enumerate $\{\dot\tau_n : n \in \omega\}$ all $S(\mathcal{F})$-names for ordinal numbers belonging to M. We need to find a condition in $S(\mathcal{F})$ which is stronger than g and forces $\dot\tau_n \in M$ for each $n \in \omega$.

Players again play the game $P_{\mathcal{F}}$ in $H(\theta)$ but the actual moves will take place in M (this is automatic for player II). Player I will follow this strategy: At first he denotes g as h_0 and puts $a_0 = \emptyset$. In the n-th move he has some condition $h_n \le g$, $h_n \in M$ and a set $a_n \in [\omega]^{<\omega}$ such that $a_n \subset S(h_n)$. Now he uses lemma 3.1.4 in M for $h_n \Vdash \dot\tau_n \in On$ and the finite set a_n to get a condition $h_{n+1} < h_n$, $h_{n+1} \in M$ and a finite set $Y_n \in [On]^{<\omega}$ such that $h_{n+1} \Vdash \dot\tau_n \in Y_n$ (in M), $a_n \subset S(h_{n+1})$. Note that $Y_n \subset M$. The n-th move of player I is $S(h_{n+1}) \in \mathcal{F}$. To this player II responds with some set $b_n \in [S(h_{n+1})]^{<\omega}$. Player I denotes $a_{n+1} = a_n \cup b_n$ (hence $a_{n+1} \subset S(h_{n+1})$) and continues with move $n+1$.

When the game is over, player I collected a sequence of conditions $\{h_n : n \in \omega\} \subset S(\mathcal{F}) \cap M$, $h_{n+1} < h_n \le g$ and a sequence $\{Y_n : n \in \omega\}$ of finite subsets of $On \cap M$ such that $h_{n+1} \Vdash \dot\tau_n \in Y_n$. According to lemma 2.3.3 the described strategy is not winning for player I (in $H(\theta)$) so we can assume that the actual course of this game was won by player II.

Hence $h = \bigcap\{h_n : n \in \omega\} \in S(\mathcal{F})$ because $S(h) \supset \bigcup\{a_n : n \in \omega\} \in \mathcal{F}$. Now $h < h_n$ for each $n \in \omega$ and $h \Vdash \dot\tau_n \in Y_n$ thus $h \Vdash \dot\tau_n \in M$ (since Y_n is finite).

Now assume moreover that $\mathcal{F}$ is rapid and we prove that $S(\mathcal{F})$ has Sacks property. (The proof for $G(\mathcal{F})$ is again analogous.)

Start by choosing a growing function $e \in {}^\omega\omega$. Then continue in the same way as if proving the ${}^\omega\omega$ bounding property only instead of playing p-filer game, the rapid p-filter game $RP_{\mathcal{F},e}$ is played. This ensures that $b_n < e(n)$ thus $a_n < n \cdot e(n)$ and lemma 3.1.4 produces sets Y_n such that $|Y_n| < 2^{n \cdot e(n)}$ for each $n \in \omega$.

In the end again $h \Vdash \dot f \in \prod\{Y_n : n \in \omega\}$ and the Sacks property is proved. $\square$

3.2 Killing p-points

We have already proved basic preservation properties of Grigorieff and Sacks forcing sufficient for to establish the basic result (about p-points). Now let us turn our attention the other aspect; what these forcings destroy.

In this section we will be working with forcings $G(\mathcal{F})$ and $S(\mathcal{F})$, where the filter $\mathcal{F}$ is not on ω but rather on ω^2 or on

$$\bigtriangledown = \{(i,j) \in \omega^2 : i < j\}.$$

On the later set we will use the inversed lexicographic ordering $(x,y) \lessdot (x',y')$ iff $y < y'$ or $y = y'$ and $x < x'$. With this ordering $\bigtriangledown$ is order isomorphic with ω so we can talk about forcing consisting of subtrees of ${}^{\lessdot\bigtriangledown}2$.

To simplify notation we will denote $\bigtriangledown^y = \{(x,y) \in \bigtriangledown : x \in \omega\}$. Let h be a subtree of ${}^{\lessdot\bigtriangledown}2$. Then for $n \in \omega$ we write

$${}^{[n]}h = \left\{\eta \in h : \ \mathrm{Dom}(\eta) = \bigcup\{\bigtriangledown^y : y \in n\}\right\}.$$

Definition 3.2.1 (filter $\mathcal{F} \times \omega$)**.** Let $\mathcal{F}$ be a filter on ω. Denote

$$\mathcal{F} \times \omega = \left\{A \subset \bigtriangledown : \ \{y : (x,y) \in A\} \in \mathcal{F} \text{ for each } x \in \omega\right\}.$$

In words, $\mathcal{F} \times \omega$ consists of subsets of $\bigtriangledown$ with each vertical section in the filter $\mathcal{F}$. It is easy to see that $\mathcal{F} \times \omega$ is a filter.

If $\mathcal{F}$ is a p-filter, the situation is a bit easier to deal with.

Claim. *Let $\mathcal{F}$ be a p-filter. The filter $\mathcal{F} \times \omega$ has base consisting of sets $\bigtriangledown(F, b)$ for $F \in \mathcal{F}$ and $b \in {}^\omega\omega, b(x) > x$ where*

$$\bigtriangledown(F, b) = \{(x,y) \in \omega \times F : b(x) < y\}.$$

In case of p-filter, it is easy to show that the product filter $\mathcal{F} \times \omega$ inherits some properties of filter $\mathcal{F}$.

Lemma 3.2.2. *Let $\mathcal{F}$ be a non-meager p-filter on ω. Then $\mathcal{F} \times \omega$ is a non-meager p-filter on $\bigtriangledown$.*

Proof. Let's start with proving that $\mathcal{F} \times \omega$ is a p-filter. Take any sequence $\{\bigtriangledown(F_n, b_n) \colon n \in \omega\}$ of sets from base of $\mathcal{F} \times \omega$. Since $\mathcal{F}$ is a p-filter, there exists $F \in \mathcal{F}$, $F \subseteq^* F_n$ for each $n \in \omega$ and fix increasing $b \in {}^\omega\omega$ which eventually dominates b_n for each $n \in \omega$. We get that $\bigtriangledown(F, b) \subseteq^* \bigtriangledown(F_n, b_n)$ for each $n \in \omega$.

According to lemma 2.1.2, for the proof of non-meagerness it is enough to show the following: Pick any increasing function $g \in {}^\omega\omega$, $g(0) = 0$ and denote $I_n = [g(n), g(n+1))$, $J_n = \bigcup\{\bigtriangledown^j \colon j \in I_n\}$. Then there exists $A \in \mathcal{F} \times \omega$ which has empty intersection with infinitely many J_n's.

The assumption that $\mathcal{F}$ is non-meager gives us $F \in \mathcal{F}$ which misses infinitely many I_n's and $\bigtriangledown(F, id)$ is the desired set in $\mathcal{F} \times \omega$ (whenever $F \cap I_n = \emptyset$ then $\bigtriangledown(F, id) \cap J_n = \emptyset$). $\square$

Lemma 3.2.3. *Let $\mathcal{F}$ be a rapid p-filter on ω. Then $\mathcal{F} \times \omega$ is a rapid p-filter on $\bigtriangledown$.*

Proof. Suppose that condition (2) in lemma 2.1.5 holds for $\mathcal{F}$ and a function f'. We will check condition (2) for $a_i = \bigtriangledown^i$ and $f(i) = (i+1) \cdot f'$.

We need to show the following: For any increasing function $g \in {}^\omega\omega$, $g(0) = 0$ denote $I_n = [g(n), g(n+1))$, $J_n = \bigcup\{\bigtriangledown^j \colon j \in I_n\}$. There exists $A \in \mathcal{F} \times \omega$ such that $|J_n \cap A| < (n+1) \cdot f'(n) = f(n)$ for each $n \in \omega$.

Our assumption give us $F \in \mathcal{F}$ such that $|I_n \cap F| < f'(n)$ for each $n \in \omega$. Now $A = \bigtriangledown(F, g)$ is the desired set in $\mathcal{F} \times \omega$ (since $|J_n \cap A| < (n+1) \cdot |I_n \cap F| < (n+1) \cdot f'(n)$). $\square$

The following theorem is the crucial point of this method. It shows that forcing with $\mathcal{F} \times \omega$ prevents $\mathcal{F}$ from being a subset of a p-filter in the extension.

Theorem 3.2.4. *Suppose that $\mathcal{F}$ is a non-meager p-filter on ω and G is a $P(\mathcal{F} \times \omega)$ generic filter over V, where P is either Grigorieff or Sacks forcing with a filter. Let $N = V[G][G_1]$ be an ${}^\omega\omega$ bounding generic extension of $V[G]$. Then*

$$N \models \text{ there is no p-ultrafilter extending } \mathcal{F}.$$

Proof. Assume towards a contradiction that there is a p-ultrafilter $\tilde{\mathcal{F}}$ in N extending $\mathcal{F}$. Denote $\{x_i \colon i \in \omega\}$ the sequence of reals introduced by G corresponding to the restriction of G to columns $\{i\} \times (i, \omega)$.

For $i \in \omega$ put $c(i) = 0$ iff $\{n \in \omega \colon x_i(n) = 0\} \in \tilde{\mathcal{F}}$ and $c(i) = 1$ otherwise. Suppose that c is not eventually constant and fix a function $f' \in {}^\omega\omega \cap N$ such that for each $k \in \omega$ there exists some $j \in \big(k, f'(k)\big)$ for which $c(k) = c(j)$. Since N is a ${}^\omega\omega$ bounding extension of V we can fix an increasing function $f \in {}^\omega\omega \cap V$ dominating f'.

For $n \in \omega$ denote $i(n) = f^{(n)}(0)$, $I_n = \big[i(n), i(n+1)\big)$ and $J_n = I_n \setminus \big\{i(n)\big\}$.

For each $n \in \omega$ there exists $j \in J_n$ such that $c\big(i(n)\big) = c(j)$. If c is constant on $\omega \setminus k$ we can achieve the same effect by putting $f(n) = n + k + 2$.

Note that

$$A_n = \{m \in \omega \colon \exists j \in J_n, x_{i(n)}(m) = x_j(m)\} \in \tilde{\mathcal{F}} \cap V[G].$$

$\tilde{\mathcal{F}}$ is a p-filter so there exists some set $A \in \tilde{\mathcal{F}} \cap N$ and a function $g' \in {}^\omega\omega \cap N$ such that $A \subset A_n \cup g'(n)$ for each $n \in \omega$. Using ${}^\omega\omega$ boundedness deduce that there is an increasing function $g \in {}^\omega\omega \cap V$ such that $A \subseteq A_n \cup g(n)$.

Put $h \in {}^\omega\omega \cap V$ a function defined by $h(m) = g(n)$ iff $m \in I_n$ and $h(m) = 0$ bellow $i(0)$.

There is a condition $q \star \dot{q}_1 \in G \star \dot{G}_1$ which forces that everything we constructed so far is done correctly. We can suppose (taking stronger condition if necessary) that $D(q) = \bigtriangledown \setminus \bigtriangledown(F, b)$ (Grigorieff case) or $S(q) = \bigtriangledown(F, b)$ (Sacks case) for some $F \in \mathcal{F}$ and b a nondecreasing function which dominates h and is constant on each I_n for all $n \in \omega$. Denote $o(n)$ the value of b on I_n.

Extend condition q into q' by defining

Grigorieff case:

- $q'(y, z) = 1$ iff $(y, z) \in \big\{i(n)\big\} \times \Big(F \cap \big(o(n), o(n+1)\big]\Big)$ for some $n \in \omega$
- $q'(y, z) = 0$ iff $(y, z) \in J_n \times \Big(F \cap \big(o(n), o(n+1)\big]\Big)$ for some $n \in \omega$
- $q'(y, z) = q(y, z)$ iff $(y, z) \in \mathrm{Dom}(q)$.

Sacks case: We require that q' contains precisely those $s \in q$ for which

- $s(y, z) = 1$ if $(y, z) \in \mathrm{Dom}(s)$ and $(y, z) \in \big\{i(n)\big\} \times \Big(F \cap \big(o(n), o(n+1)\big]\Big)$ for some $n \in \omega$

- $s(y,z) = 0$ if $(y,z) \in \mathrm{Dom}(s)$ and $(y,z) \in J_n \times \Big(F \cap \big(o(n), o(n+1)\big]\Big)$ for some $n \in \omega$.

Hence $q' \in P(\mathcal{F} \times \omega)$, $q' \star \dot{q}_1 < q \star \dot{q}_1$ and for each $z \in F \setminus (o(0)+1)$ we have $q' \star \dot{q}_1 \vdash z \notin \dot{A}$. Thus $q' \star \dot{q}_1$ forces $|F \cap \dot{A}| < \omega$ and this is contradiction with the choice of $q \star \dot{q}_1$ which knew that $F \cap \dot{A} \in \tilde{\mathcal{F}}$. □

Now we have all ingredients to build a model with no p-points.

Theorem 3.2.5 (Shelah)**.** *It is consistent with ZFC that there are no p-points.*

Proof. This is a typical example of countable support iteration of length ω_2 of proper forcings of size $2^\omega = \omega_1$. Start in a model where GCH holds. Then do countable support iteration of forcings $G(\mathcal{F}_\alpha \times \omega)$ and use a bookkeeping device to make sure that each p-point from each intermediate model appeared as some (subset of) $\mathcal{F}_\alpha$ at some stage. We are using that in each intermediate model $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$ and so there are always only ω_2 many p-points. Also note, that for each p-point $\mathcal{U}$ from the resulting model there is some intermediate model V_α such that $\mathcal{U} \cap V_\alpha \in V_\alpha$ and $\mathcal{U} \cap V_\alpha$ is a p-point. □

The reference for this result is [She98a], for a detailed proof see [Woh08]. A slightly different approach is developed in [Wim82].

3.3 Preserving selective ultrafilter I

So far, there wasn't any significant difference between destroying p-points with Sacks and Grigorieff forcing. We will see that the additional complexity of Sacks forcing is rewarded by achieving some control over which ultrafilters on ω are not destroyed.

Let us at first review some general preservation results.

Lemma 3.3.1. *Let P be a proper forcing notion and let $\mathscr{S}$ be a p-filter. Then $\mathscr{S}$ is a base of a p-filter in the generic extension by P.*

Proof. We need to prove that each set $S \in [\mathscr{S}]^\omega$ in the extension has some pseudointersection in $\mathscr{S}$. Since P is proper, there is some set $S' \in [\mathscr{S}]^\omega \cap V$ such that $S \subset S'$. And any pseudointersection of S' can serve as pseudointersection of S. □

Now we will see that ${}^\omega\omega$ bounding extensions preserve many properties of filters. The following fact is a simple consequence of characterization of non-meager and definition of rapid filter from section 2.1.

Fact 3.3.2.

1. *Let N be an ${}^\omega\omega$ bounding extension of V and let $\mathscr{S}$ be a non-meager filter in V. Then $\mathscr{S}$ is a base of non-meager filter in N.*
2. *Let N be an ${}^\omega\omega$ bounding extension of V and let $\mathscr{S}$ be a rapid filter in V. Then $\mathscr{S}$ is a base of rapid filter in N.*

Rare filters are also preserved with in ${}^\omega\omega$ bounding extensions. The key observation for proving that is this lemma.

Lemma 3.3.3. *Let N be an ${}^\omega\omega$ bounding extension of V. For each interval partition $\{I_n : n \in \omega\} \in N$ of ω there exist an interval partition $\{J_n : n \in \omega\} \in V$ such that for each $n \in \omega$ there are at most two $m_0, m_1 \in \omega$ such that $I_n \cap J_{m_i} \neq \emptyset$ for $i \in 2$.*

Proof. Suppose that all I_ns are nonempty. There is an increasing function $f \in V$ such that $|I_n| < f(n)$. Put

$$J_n = \Big[f^{(n)}(0), f^{(n+1)}(0)\Big)$$

for $n \in \omega$. If $I_n \cap J_m \neq \emptyset$ and $I_n \cap J_{m+1} \neq \emptyset$ then

$$|J_{m+1}| = f(\min(I_n \cap J_{m+1})) \geq f(n) > |I_n|$$

and $I_n \cap J_{m+2} = \emptyset$. □

And using argument somewhat similar to proof of lemma 2.2.9, we can prove preservation lemma for rare filters.

Lemma 3.3.4. *Let N be an ${}^{\omega}\omega$ bounding extension of V and let $\mathscr{S}$ be a rare filter in V. Then $\mathscr{S}$ is a base of rare filter in N.*

Proof. Let $\{I_n : n \in \omega\} \in N$ be an interval partition of ω. Use lemma 3.3.3 to find $\{J_n : n \in \omega\} \in V$ such that each I_n intersects at most two J_m. For $i \in 2$ find $S_i \in \mathscr{S}$ such that

$$|S_i \cap (J_{2n+i} \cup J_{2n+1+i})| \leq 1$$

for each $n \in \omega$. For

$$S = (S_0 \cap S_1) \setminus \min J_1 \in \mathscr{S}$$

is $|S \cap I_n| \leq 1$ for each $n \in \omega$. □

And for a pair of filters, being not near coherent is preserved as well.

Lemma 3.3.5. *Let N be an ${}^{\omega}\omega$ bounding extension of V and let $\mathscr{S}_0, \mathscr{S}_1$ be not near coherent filters in V. Then $\mathscr{S}_0, \mathscr{S}_1$ generate not near coherent filters in N.*

Proof. Let $\{I_n : n \in \omega\} \in N$ be any interval partition of ω. Use lemma 3.3.3 to find $\{J_n : n \in \omega\} \in V$ such that each I_n intersect at most two J_m. Now use 3. of lemma 2.2.9 to find $A_0, A_1 \subset \omega$ such that $\bigcup\{J_n : n \in A_i\} \in \mathscr{S}_i$ for $i \in 2$ and $n \in A_i \Rightarrow n + 1 \notin A_{1-i}$. Put

$$B_i = \{n \in \omega : \exists k \in A_i, I_n \cap J_k \neq \emptyset\}$$

for $i \in 2$. We have that B_0 and B_1 are disjoint and

$$\bigcup\{I_n : n \in B_i\} \supset \bigcup\{J_k : k \in A_i\} \in \mathscr{S}_i$$

for both $i \in 2$. □

And here comes the promised preservation theorem for ultrafilters. It shows that if we force with Sacks forcing, then while some ultrafilters are destroyed, other survive.

Theorem 3.3.6. *Let $\mathscr{R}$ be a selective ultrafilter and $\mathcal{F}$ be p-filter not near coherent with $\mathscr{R}$ (and hence non-meager). The Sacks forcing $S(\mathcal{F} \times \omega)$ preserves $\mathscr{R}$ as a base of a selective ultrafilter.*

Proof. It is sufficient to prove that for a given $S(\mathcal{F} \times \omega)$ name $\dot{A}$ for a subset of ω there is a dense set of conditions deciding that there is some $R \in \mathscr{R}$ such that $R \subset \dot{A}$ or $R \cap \dot{A} = \emptyset$.

Fix a condition $p \in S(\mathcal{F} \times \omega)$. We can suppose that there is no $q < p$ such $q \Vdash \dot{A} \notin \langle \mathscr{R} \rangle$ i.e. for each $q < p$ is

$$R_q = \{s \in \omega : \exists q' < q : q' \Vdash n \in \dot{A}\} \in \mathscr{R}.$$

Two players will play the near coherence game for p-filter in even moves and rare p-filter in odd moves. Player I will follow this strategy: At first he denotes p as h_0 and puts $a_0 = \emptyset$. We can suppose that $S(h_0) = \bigtriangledown(F_0, f_0)$ for some $F_0 \in \mathcal{F}$ and $f_0 \in {}^{\omega}\omega$.

Let n be even. In the n-th move player I has some condition $h_n \leq p \in S(\mathcal{F} \times \omega)$, $S(h_n) = \bigtriangledown(F_n, f_n)$ for $F_n \in \mathcal{F}$ and $f_n \in {}^{\omega}\omega$, and a set $a_n \in [F_n]^{<\omega}$. Now fix $k_n \in \omega$ such that $a_n \subset k_n$ and $k_n > f_n(n)$. The n-th move of player I is $F_n \setminus k_n \in \mathcal{F}$. To this player II responds with some set $b_n \in [F_n \setminus k_n]^{<\omega}$. Player I denotes $a_{n+1} = a_n \cup b_n$ (so $a_{n+1} \subset F_n$ and for $l \in b_n$ is $f_n(n) < l$), $h_{n+1} = h_n$ and continues with the odd move $n + 1$.

Now n is odd. Player I has condition h_n, $S(h_n) = \bigtriangledown(F_n, f_n)$ for $F_n = F_{n-1} \in \mathcal{F}$ and $f_n = f_{n-1} \in {}^{\omega}\omega$, and a set $a_n \in [F_n]^{<\omega}$.

Fix $k_n \in \omega$ such that $a_n \subset k_n$. Put

$$R(n) = \bigcap\{R_q : q = h_n[\eta], \eta \in {}^{[k_n]}h_n\} \in \mathscr{R}.$$

The n-th move of player I is $R(n)$.

To this player II responds with an integer $r_n \in R(n)$. For each condition $q = h_n[\eta], \eta \in {}^{k_n}h_n$ is $r_n \in R_q$ so there is a stronger condition $q' < q$ such that $q' \Vdash r_n \in \dot{A}$. Put

$$h'_{n+1} = \bigcup\{q' : q = h_n[\eta], \eta \in {}^{k_n}h_n\}.$$

We can take stronger condition $h_{n+1} \in S(\mathcal{F} \times \omega)$ such that $S(h_{n+1}) = \triangledown(F_{n+1}, f_{n+1})$ such that $a_n \subset F_{n+1}$ and $f_n \restriction n = f_{n+1} \restriction n$. Note that $h_{n+1} \Vdash r_n \in \dot{A}$. Put $a_{n+1} = a_n$ and continue with the next (even) move $n+1$.

When the game is over, player I collected a sequence of conditions $\{h_n : n \in \omega\} \subset S(\mathcal{F} \times \omega)$, $h_{n+1} < h_n \leq p$ and a sequence $\{r_n : n \in \omega\}$ such that $h_{n+1} \Vdash r_n \in \dot{A}$. According to remark 2.3.9 the described strategy is not winning for player I so we can assume that the actual course of this game was won by player II. Hence $h = \bigcap\{h_n : n \in \omega\} \in S(\mathcal{F})$ because $S(h) \supset \triangledown(F, f)$ Where $F = \bigcup\{a_n : n \in \omega\} \in \mathcal{F}$ and $f(n) = f_{n+1}(n)$. Now $h < h_n$ for each $n \in \omega$ thus $h \Vdash R = \{r_n : n \in \omega\} \subset \dot{A}$ and $R \in \mathscr{R}$. We proved that $h \Vdash \dot{A} \in \langle \mathscr{R} \rangle$. □

Now it is possible modify proof of theorem 3.2.5 to build a model with e.g. only single (up to isomorphism) selective ultrafilter on ω. It is achieved by picking a selective ultrafilter $\mathscr{R}$ in the groundmodel and iterating Sacks forcings for destroying all p-points not near coherent with $\mathscr{R}$ (this includes all selective ultrafilters non-isomorphic with $\mathscr{R}$), while omitting forcings for destroying the coherent ones. Theorem 3.3.6 ensures, that $\mathscr{R}$ is preserved at isolated steps of the iteration while theorem of Blass and Shelah (cited in the preliminary chapter, page 9) provides preservation in limit stages of iteration.

3.4 Preserving selective ultrafilter II

We will prove a counterpart of theorem 3.3.6 for selective ultrafilters, which are in Rudin-Blass ordering strictly bellow $\mathcal{F}$.

We will need a Ramsey like lemma for finite trees. Let $\{A_l : l < n\}$ be a finite sequence of finite sets. Suppose branches of the tree $T = \bigcup_{k \leq n}\{\prod_{l<k} A_l\}$ are divided into two sets, $[\emptyset]^T = X_0 \cup X_1$. For $i \in 2$ we say that $u \subset n$ is *i-good* if there exists some S, a nonempty initial subtree of T, such that $[\emptyset]^S \subset X_i$ and for each $l \in u$ and $s \in S^{[l]}$ is $s^\frown a \in S$ for each $a \in A_l$ (i.e. nodes of S in levels from u have full splitting). If a u is i-good for some $i \in 2$, we say that it is *good.*

Lemma 3.4.1. *Let $\{A_l : l < n\}$, $T = \bigcup_{k \leq n}\{\prod_{l<k} A_l\}$, $[\emptyset]^T = X_0 \cup X_1$ be as above. At least one of the following holds true.*

1. *$n = u_0 \cup u_1 \cup u_2$ and all u_j are good for $j \in 3$.*
2. *$n = u_3 \cup \{x\}$ and u_3 is good.*

Proof. Note that $\emptyset$ is good and any subset of a good subset is also good. We show that if u is not good, then $n \setminus u$ is good.

Claim. *If u is not i-good, then $n \setminus u$ is $(1-i)$-good.*

Consider a game of length n for two players I and II. In move l, player I plays some $a \in A_l$ iff $l \notin u$ (while player II waits) and player II plays $a \in A_l$ iff $l \in u$ (and player I waits). After n moves player I wins iff the sequence of moves played belongs to X_i and player II wins otherwise. This game is finite and thus determined. A winning strategy for player I is a subtree of T witnessing that u is i-good and winning strategy for player II demonstrates that $n \setminus u$ is $(1-i)$-good. □

Case 1; good sets are not an ideal on n. Thus there are u_0, u_1; two disjoint good sets such that $u_0 \cup u_1$ is not good. We put $u_2 = n \setminus (u_0 \cup u_1)$ and we are done.

Case 2; good sets form an ideal. If n is good, then put $u_3 = n - 1$. Otherwise good sets are a proper maximal ideal. This ideal has to be generated by a good set u_3 of size $n - 1$. □

Now we can proceed to proving the preservation theorem. Note, that the assumption that $\mathscr{R}$ is strictly bellow the p-point $\mathcal{F}$ is used only to prevent existence of the most obvious counterexample.

Theorem 3.4.2. *Let $\mathscr{R}$ be a selective ultrafilter and $\mathcal{F}$ a p-ultrafilter such that $\mathscr{R} \lneq_{RK} \mathcal{F}$ (i.e. $\mathscr{R}$ is strictly below $\mathcal{F}$). The forcing $S(\mathcal{F} \times \omega)$ preserves $\mathscr{R}$ as a base of a selective ultrafilter.*

Proof. Fix an increasing sequence of integers $\{i(n)\colon n\in\omega\}$, $I_n=\big[i(n),i(n+1)\big)$ such that for each $R\in\mathscr{R}$ is $\bigcup\{I_n\colon n\in R\}\in\mathcal{F}$. Observe that if $F\subset\omega$ and $|F\cap I_n|\le 1$ for each $n\in\omega$ then $F\notin\mathcal{F}$ (otherwise $\mathcal{F}\le_{RK}\mathscr{R}$).

Given a condition $p\in S(\mathcal{F}\times\omega)$ and name $\dot{A}$ for a subset of ω we need to find a stronger condition deciding if $\dot{A}\in\langle\mathscr{R}\rangle$.

A condition $q\in S(\mathcal{F}\times\omega), S(q)=\triangledown(F,f)$ is called *positive* if there are sets $R_q\in\mathscr{R}$ and $F_q\in\mathcal{F}$, $F_q\subset F$ such that for each $n\in R_q$ there exist $\eta_q^n\in{}^{[i(n)]}q$ and a condition $s_q^n<q[\eta_q^n]$ such that $s_q^n\Vdash n\in\dot{A}$ and $S(s_q^n)\supset\triangledown^j\cap S(q)$ for each $j\in F_q\cap I_n$.

Condition is *negative* if the same is true, only $s_q^n\Vdash n\notin\dot{A}$.

Claim. *Each condition in $S(\mathcal{F}\times\omega)$ is positive or negative (or both).*

Take any condition $q\in S(\mathcal{F}\times\omega)$, $S(q)=\triangledown(F,f)$. We may suppose that for each $\nu\in q$ if $\mathrm{Dom}(\nu)\notin S(q)$ then $|\{\nu^\frown 0,\nu^\frown 1\}\cap q|=1$ i.e. ν is not a splitting node of q. For each $n\in\omega$ fix an $\eta^n\in{}^{[i(n)]}q$. For each e for which $\eta^n\cup e\subset\nu_e$ for some $\nu_e\in{}^{[i(n+1)]}q$ fix a condition $q_e<q[\nu_e]$, which decides $n\in\dot{A}$. Put

$$X^+=\left\{e\in\prod_{j\in I_n\cap F}{}^{\triangledown^j\cap S(q)}2\colon q_e\Vdash n\in\dot{A}\right\}$$

and

$$X^-=\left\{e\in\prod_{j\in I_n\cap F}{}^{\triangledown^j\cap S(q)}2\colon q_e\Vdash n\notin\dot{A}\right\}.$$

The set $X^+\cup X^-$ can be viewed as set of all branches through a finite tree T from lemma 3.4.1, hence we can define either sets ${}^{\square(0)}u_0^n, {}^{\square(1)}u_1^n, {}^{\square(2)}u_2^n$ such that

$${}^{\square(0)}u_0^n\cup{}^{\square(1)}u_1^n\cup{}^{\square(2)}u_2^n=I_n\cap F$$

or ${}^{\square(3)}u_3^n\subset I_n\cap F$, $|{}^{\square(3)}u_3^n|=|I_n\cap F|-1$ (where $\square(k)$ stands for $+$ or $-$ and depends on n) and such that for $k\in 4$, for which ${}^{\square(k)}u_k^n$ is defined, there exists S subtree of T such that $[\emptyset]^S\subset X^{\square(k)}$ and $\triangledown^j\cap S(q)\subset S(S)$ for each $j\in{}^{\square(k)}u_k^n$.

Now we use that $\mathcal{F}$ is an ultrafilter which is not $\le_{RK}$ below $\mathscr{R}$ to see that there is $k\in 4,\square\in\{+,-\}$ and $R_q\in\mathscr{R}$, such that ${}^{\square}u_k^n$ is defined for each $n\in R_q$ and

$$F_q=\bigcup\left\{{}^{\square}u_k^n\colon n\in R_q\right\}\in\mathcal{F}.$$

If $\square$ is $+$ then q is positive, if $\square$ is $-$ then negative. To see that R_q,F_q and η^n for $n\in R_q$ work, put

$$s_q^n=\bigcup\left\{q_e\colon e\in\prod_{j\in{}^{\square}u_k^n}{}^{\triangledown^j\cap S(q)}2\right\}$$

$\square$

Now it is enough to show that if the set of positive conditions is dense below some $p'<p$, we can find a stronger condition $h<p'$ and set $R\in\mathscr{R}$ such that $h\Vdash R\subset\dot{A}$. If there is a dense set of negative conditions, the same proof produces h and $R\in\mathscr{R}$ such that $h\Vdash R\cap\dot{A}=\emptyset$. So from now on we will work only with positive conditions.

The refining game for a p-filter $\mathcal{F}$ and a selective ultrafilter $\mathscr{R}$ will be played. Player I denotes $h_0=p'$ and $n_0=0$. We may suppose that $S(h_0)=\triangledown(F_0,f_0)$ for some $F_0\in\mathcal{F}$, $f_0\in{}^{\omega}\omega$.

In move m player I has a condition h_m, $S(h_m)=\triangledown(F_m,f_m)$ and an integer $n_m\in\omega$. All conditions $h_m[\nu]$ for $\nu\in{}^{[i_{n_m+1}]}h_m$ are positive (substitute with stronger condition if necessary), so there are sets $R_{h_m[\nu]},F_{h_m[\nu]}$ witnessing this. Player I chooses integer $n'_m>n_m$ such that $i(n'_m)>f_m(m)$ and plays

$$F'_m=\bigcap\left\{F_{h_m[\nu]}\colon\nu\in{}^{[i_{n_m+1}]}h_m\right\}\setminus i(n'_m)\in\mathcal{F}.$$

Note that $F'_m\subset F_m$. Player II answers with some $n_{m+1}\in\omega$ and $b_m\in[F'_m\cap I_{n_{m+1}}]^{<\omega}$. For each $\nu\in{}^{[i(n_m+1)]}h_m$ there is some $\eta_{h_m[\nu]}^{n_{m+1}}$ and $s_{h_m[\nu]}^{n_{m+1}}<h_m[\eta_{h_m[\nu]}^{n_{m+1}}]$ such that $s_{h_m[\nu]}^{n_{m+1}}\Vdash n_{m+1}\in\dot{A}$.

Hence

$$h'_{m+1} = \bigcup \left\{ s^{n_{m+1}}_{h_m[\nu]} \colon \nu \in {}^{[i(n_m+1)]}h_m \right\} \Vdash n_{m+1} \in \dot{A}.$$

Also note that $S(h'_{m+1}) \cap \triangledown^j = S(h_m) \cap \triangledown^j$ for each $j < i(n_m+1)$ since the union is taken over all $\nu \in {}^{[i(n_m+1)]}h_m$ and for all $j \in b_m$ since this was true for all $s^{n_{m+1}}_{h_m[\nu]}$.

Now fix h_{m+1} to be some condition stronger then h'_{m+1} such that $S(h_{m+1}) = \triangledown(F_{m+1}, f_{m+1})$ for some $F_{m+1} \in \mathcal{F}$ and $f_{m+1} \in {}^{\omega}\omega$ such that

$$F_{m+1} \cap i(n_{m+1}+1) = F_n \cap i(n_m+1) \cup b_m$$

and

$$f_m \restriction (m+1) = f_{m+1} \restriction (m+1)$$

and player I can proceed to next move.

This cannot be a winning strategy for player I (see lemma 2.3.11) so we can suppose that the game was won by player II. Hence $R = \{n_{m+1} \colon m \in \omega\} \in \mathscr{R}$ and $h = \bigcap\{h_m \colon m \in \omega\} \in S(\mathcal{F} \times \omega)$ since $S(h) = \triangledown(\bigcup_{m \in \omega} b_m, f)$ where $f(m) = f_m(m)$.

And $h \Vdash n \in \dot{A}$ for each $n \in R$ so $h \Vdash R \subset \dot{A}$. The negative case works in precisely the same way. □

Now we have all tools to prove the following.

Theorem 3.4.3 (Shelah)**.** *Suppose GCH and let $\mathcal{S}$ be a set containing only selective ultrafilters. There exist a forcing extension $V[G]$ such that each p-point in $V[G]$ is a permutation of a selective ultrafilter generated by some $\mathscr{R} \in \mathcal{S}$ and each $\mathscr{R} \in \mathcal{S}$ is a base of selective ultrafilter in $V[G]$.*

Proof. This is proved in exactly the same way as theorem 3.2.5. We only need to utilizing the forcing $S(\mathcal{F}_\alpha \times \omega)$ instead of the Grigorieff variant (so that we preserve all selective ultrafilters not isomorphic to F_α) and we avoid forcing with any $S(\mathcal{F}_\alpha \times \omega)$ when $\mathcal{F}_\alpha$ is isomorphic to some $\mathscr{R} \in \mathcal{S}$. See [She98a]. □

CHAPTER 4

STRONG-Q-SEQUENCES

We will turn our attention toward a topic directly connected with Katowice problem, namely existence of strong-Q-sequences (as defined in 1.3.1) in the Boolean algebra $\mathcal{P}(\omega)/\operatorname{Fin}$. We will present a result of J. Steprans establishing a consistency of existence of a strong-Q-sequence in $\mathcal{P}(\omega)/\operatorname{Fin}$ and a we introduce a new method of creating strong-Q-sequences. This method which enables us to build models with a countable like ideal and $\mathfrak{d} = \omega_1$.

4.1 Strong-Q-sequences in $\mathcal{P}(\omega)/\operatorname{Fin}$

The following definition is a reformulation of the notion strong-Q-sequence mentioned in chapter 1 for the Boolean algebra $\mathcal{P}(\omega)/\operatorname{Fin}$. From now on, the term strong-Q-sequence will refer to this definition unless stated otherwise.

Other authors use also in some context the term uniformizable AD system. The name strong-Q-sequence is used because in $\mathcal{P}(\omega)/\operatorname{Fin}$ it is a strengthening of the notion Q-set (See fact 4.1.2).

Definition 4.1.1. Let

$$\mathcal{A} = \{A_\alpha \colon A_\alpha \in [\omega]^\omega, \alpha \in \kappa\}.$$

$\mathcal{A}$ is a *strong-Q-sequence* (of size κ) iff for each $F = \{f_\alpha \colon A_\alpha \to 2\}$ there exists $f_F \colon \omega \to 2$, such that $f_F \restriction A_\alpha =^* f_\alpha$. The family of all such F for $\mathcal{A}$ will be denoted $\mathscr{F}_\mathcal{A}$ and the function f_F is called *uniformization* of F.

A subset $\mathcal{A}$ of the Cantor space 2^ω is a Q-set, if all its subsets are F_σ (or equivalently G_δ) in $\mathcal{A}$ with the subspace topology. Since there are only 2^ω F_σ subsets, existence of a Q-set $\mathcal{A}$ implies $2^{|\mathcal{A}|} = 2^\omega$. The name of strong-Q-set has origin in the following fact.

Fact 4.1.2. *Every strong-Q-sequence is a Q-set in the Cantor space (identified with $\mathcal{P}(\omega)$).*

Proof. Let X be a subset of a strong-Q-sequence $\mathcal{A} = \{A_\alpha \colon \alpha \in \kappa\}$. Let f_α be constantly 1 if $A_\alpha \in X$ and constantly 0 otherwise. Find a uniformization f_F and put $E = f_F^{-1}\{1\}$. Now $\{A \colon A \subseteq^* E\}$ is a F_σ set containing X and disjoint with $\mathcal{A} \setminus X$. □

Note that for proving the previous fact we only needed existence of uniformizations for systems $F \in \mathscr{F}_\mathcal{A}$ containing constant functions.

Corollary 4.1.3. *If there is a strong-Q-sequence of size κ, then $2^\omega = 2^\kappa$.*

We already know that every strong-Q-sequence forms an AD system. In the context of $\mathcal{P}(\omega)/\operatorname{Fin}$ it cannot be a MAD system.

Proposition 4.1.4. *If $\mathcal{A} = \{A_\alpha \colon \alpha \in \kappa\}$ is a strong-Q-sequence then $\mathcal{A}$ is not a maximal AD system.*

Proof. Define $F \in \mathscr{F}_\mathcal{A}$ to consist of $f_\alpha \colon A_\alpha \to 2$ constantly 1 if $\alpha < \omega$ and constantly 0 otherwise. Take f some uniformization of F. Construct inductively an infinite set

$$D = \{n_i \in A_i \setminus \bigcup_{j<i} A_j \colon i \in \omega, f(n_i) = 1\}.$$

Note that $D \cap A_\alpha =^* \emptyset$ for all $\alpha \in \kappa$. □

We recall here definition of Luzin gap and few well known facts about this objects.

Definition 4.1.5 (Luzin gap)**.** An AD system $\mathcal{L} = \{L_\alpha \colon \alpha \in \omega_1\} \subset [\omega]^\omega$ is a *Luzin gap* if for each $\alpha \in \omega_1$ and each $n \in \omega$ is $|\{\beta < \alpha \colon L_\alpha \cap L_\beta \subset n\}| < \omega$.

Theorem 4.1.6. *A Luzin gap exists in each model of ZFC.*

Fact 4.1.7. *Let $\mathcal{L} = \{L_\alpha \colon \alpha \in \omega_1\}$ be a Luzin gap and $A, B \in [\omega_1]^{\omega_1}$ be disjoint. There is no $X \subset \omega$ such that $L_\alpha \subset^* X$ if $\alpha \in A$ and $L_\alpha \cap X =^* \emptyset$ if $\alpha \in B$.*

Example 4.1.8. There are AD systems in $\mathcal{P}(\omega)$ of size ω_1 which are not strong-Q-sequences.

One such AD system is the Luzin gap $\mathcal{L}$, with no uniformization even for some $F \in \mathscr{F}_\mathcal{L}$ consisting of constant functions.

Other example of such AD system is built from nodes of the complete binary tree $({}^{<\omega}2, \subset)$. Pick B_α, $\alpha \in \omega_1$ distinct maximal branches through ${}^{<\omega}2$. Now $\mathcal{A} = \{B_\alpha \colon \alpha \in \omega_1\}$ forms an AD system which is not a strong-Q-sequence.

Proof. To show this put $f_\alpha(s) = i$ iff $s^\frown i \in B_\alpha$ for $\alpha \in \omega_1$, $s \in B_\alpha$ and $i \in 2$. For contradiction, assume that there is a uniformization $f_F \colon {}^{<\omega}2 \to 2$. For each $\alpha \in \omega_1$ there is some $s_\alpha \in B_\alpha$ such that $f_\alpha(s) = f_F(s)$ for each $s \in B_\alpha$, $s_\alpha \subseteq s$. For some $\alpha \neq \beta$ we have $s_\alpha = s_\beta$ and this implies $B_\alpha = B_\beta$, contradiction. □

Note that on the other hand $MA_{\omega_1}(\sigma\text{-centered})$ implies that this AD system is a Q-set.

Proof. Take any $F = \{f_\alpha \colon \alpha \in \omega_1\} \in \mathscr{F}_\mathcal{A}$ containing only constant functions. Consider partial order P consisting of finite approximations of the desired uniformization; $p = (g_p, \mathcal{A}_p) \in P$ iff there is $n_p \in \omega$ such that $g_p \colon {}^{\leq n}2 \to 2$, $\mathcal{A}_p \in [\omega_1]^{<\omega}$ and for each $\alpha \in \mathcal{A}_p$ and $s \in B_\alpha \cap {}^n2$ is $g_p(s) = f_\alpha(s)$; $(g_p, \mathcal{A}_p) < (g_q, \mathcal{A}_q)$ iff $g_q \subset g_p$ and for each $\alpha \in \mathcal{A}_q$ and $s \in B_\alpha \cap \mathrm{Dom}(p) \setminus \mathrm{Dom}(q)$ is $g_p(s) = f_\alpha(s)$.

This poset P is σ-centered, set of conditions sharing the same g_p is centered. It is also easy to see that $D_\alpha = \{p \in P \colon \alpha \in \mathcal{A}_p\}$ and $H_n = \{p \in P \colon n_p \geq n\}$ are dense sets in P for all $\alpha \in \omega_1$ and $n \in \omega$. If MA holds, there is a filter on P intersecting all these sets and union of first parts of elements of this filter is a uniformization of F. □

Note that both these examples are absolute in the sense, that they can never become strong-Q-sequence in any larger model of ZFC unless ω_1 from the groundmodel is collapsed (i.e. the AD system becomes countable). This means that if we want to force an AD system $\mathcal{A}$ to be a strong-Q-sequence in some extension, we have to be careful with the choice of $\mathcal{A}$. Two possible ways how to choose $\mathcal{A}$ will be presented in this chapter.

Every countable AD system is obviously a strong-Q-sequence. The consistency of existence of an uncountable strong-Q-sequences was proved by J. Steprans in [Ste85] and by S. Shelah [She82]. Their proofs are very similar. The approach is to start with adding an AD system generic on finite conditions (originally due to S. Hechler [Hec72]) and than iterate ccc forcing notions adding uniformizations, which yield a ccc forcing extension where this AD system is a strong-Q-sequence. This proof will be presented in next section of this chapter.

In [Ste85] Steprans also showed, that it follows from MA that there are no uncountable strong-Q-sequences. This will follow from two lemmas, which in combination show, that under MA every AD system locally looks like the AD system from example 4.1.8.

At first we reduce any AD system to an AD system of branches on finitely branching tree.

Lemma 4.1.9 (Steprans)**.** *Assume $MA_{\omega_1}(\sigma\text{-centered})$. Let $\mathcal{A} = \{A_\alpha \colon \alpha \in \omega_1\}$ be an AD family on ω. There exists $T \subset \omega$ and $\leq_T$ such that $(T, \leq_T)$ is a finitely branching tree of height ω with no leaves and there are uncountably many $A \in \mathcal{A}$ such that $A \cap T$ is a branch through T.*

Proof. Consider a partial order P consisting of finite approximations of T. A triple $p = (T_p, \leq_p, \mathcal{A}_p)$ is an element P iff $(T_p, \leq_p)$ is a finite tree, $\mathcal{A}_p \in [\mathcal{A}]^{<\omega}$, $T_p \subset \bigcup \mathcal{A}_p$ and for each $A \in \mathcal{A}_p$ is $A \cap T_p$ a branch through $(T_p, \leq_p)$. For $p, q \in P$ is $p \leq q$ iff $T_q \subset T_p$, $\leq_p$ end-extends $\leq_q$ and $\mathcal{A}_q \subset \mathcal{A}_p$. Define

$$\mathscr{B}(p) = \{A \in \mathcal{A} \colon A \cap T_p \text{ is a branch through } (T_p, \leq_p)\}.$$

Put

$$\mathcal{A}' = \mathcal{A} \setminus \bigcup \{\mathscr{B}(p) \colon p \in P, \mathscr{B}(p) < \omega_1\}$$

(so $|\mathcal{A}'| = \omega_1$) and

$$P' = \{p \in P \colon \emptyset \neq \mathcal{A}_p \subset \mathcal{A}'\}.$$

The poset P' is σ-centered (conditions sharing the same $(T_p, \leq_p)$ are centered). Denote

$$D_\beta = \{p \in P' \colon \exists \alpha > \beta, A_\alpha \in \mathcal{A}_p\}$$

and $H_n = \{p \in P' \colon \text{ all branches of } (T_p, \leq_p) \text{ are longer than } n\}$. Note that all these sets are dense in P' and use MA to find a filter G intersecting all of them. Now

$$\left(\bigcup \{T_p \colon p \in G\}, \bigcup \{\leq_p \colon p \in G\}\right)$$

is the desired tree. □

And then we reduce such AD system to branches of binary tree.

Lemma 4.1.10 (Steprans)**.** *Assume $MA_{\omega_1}(\sigma\text{-linked})$. Let $(T, \leq_T)$ be a finitely branching tree of height ω with no leaves and $\mathcal{A} = \{A_\alpha \colon \alpha \in \omega_1\}$ be a set of branches through T. There exists S, a binary initial subtree of T with no leaves and $|\{A \in \mathcal{A} \colon |A \cap S| = \omega\}| = \omega_1$.*

Proof. For each $t \in T$ put $\mathcal{A}[t] = \{A \in \mathcal{A} \colon t \in A\}$, $T' = \{t \in T \colon |\mathcal{A}[t]| = \omega_1\}$ and

$$\mathcal{A}' = \{A \in \mathcal{A} \colon |A \cap T'| = \omega\}.$$

It is easy to see that T' is a nonempty initial subtree of T with no leaves and $|\mathcal{A}'| = \omega_1$.

Define p to be an element of poset P iff $p = (T_p, \mathcal{A}_p)$ where T_p is a finite binary initial subtree of T', $\mathcal{A}_p \in [\mathcal{A}']^{<\omega}$ and for each $A \in \mathcal{A}_p$ is $A \cap T_p$ a branch through T_p and $A \cap T_p \neq B \cap T_p$ for $A \neq B$. For $p, q \in P$ is $p \leq q$ iff T_q is an initial subtree of T_p and $\mathcal{A}_q \subset \mathcal{A}_p$. The poset P is σ-linked (conditions sharing the same T_p are linked).

Denote

$$D_\beta = \{p \in P \colon \exists \alpha > \beta, A_\alpha \in \mathcal{A}_p\}$$

and

$$H_n = \{p \in P \colon \text{ all branches of } T_p \text{ are longer than } n\}.$$

Note that all these sets are dense in P and use MA to find a filter G intersecting all of them. Now $S = \bigcup \{T_p \colon p \in G\}$ is the desired tree. □

Putting 4.1.9 and 4.1.10 together provides the result.

Theorem 4.1.11 (Steprans)**.** *$MA_{\omega_1}(\sigma\text{-linked})$ implies that there is no strong-Q-sequence of size ω_1.*

Proof. Suppose $\mathcal{A}$ were a strong-Q-sequence. There is $\mathcal{B} \in [\mathcal{A}]^{\omega_1}$ and $T \in [\omega]^\omega$ such that $\mathcal{B} \restriction T$ is isomorphic to a subset of branches of a binary tree. This system should remain a strong-Q-sequence (proposition 1.3.3) but according to example 4.1.8 this is not possible. □

4.2 Adding strong-Q-sequence with ccc forcing

We will present a proof from [Ste85] establishing the consistency of existence of an uncountable strong-Q-sequence. We already know that if we plan to force an AD system to become a strong-Q-set (by adding uniformizations), we should choose the AD carefully.

Hence the forcing construction starts with adding generically an AD system.

Definition 4.2.1 (forcing adding AD set [Hec72, Hec74])**.** Let κ be an infinite cardinal. A function p is a condition in the forcing A_κ if there is some $\Gamma_p \in [\kappa]^{<\omega}$ and $n_p \in \omega$ such that $p\colon \Gamma_p \times n_p \to 2$.

A condition q is stronger then p iff $p \subset q$ and for each $k \in n_q \setminus n_p$ is $|\{\alpha \in \Gamma_p\colon q(\alpha, k)\} = 1| \leq 1$.

To see the following fact just note that $\{p \in A\colon (\exists k > n) p(\alpha, k) = 1\}$ and $\{p \in A\colon \alpha, \beta \in \Gamma_p\}$ are dense sets for all $n \in \omega$ and $\alpha, \beta \in \kappa$.

Fact 4.2.2. *Let G be a generic filter on A_κ. Put $A_\alpha = \{k\colon (\exists p \in G)\, p(\alpha, k) = 1\}$. The set $\mathcal{A} = \{A_\alpha\colon \alpha \in \kappa\}$ is an AD system of infinite sets in $V[G]$.*

After fixing the AD system, we will add all uniformizations necessary for this AD system to be a strong-Q-sequence.

Definition 4.2.3. Let $\mathcal{A} = \{A_\alpha\colon \alpha \in \kappa\}$ be an AD system on ω and $F \in \mathscr{F}_{\mathcal{A}}$. The forcing $K(\mathcal{A}, F)$ consists of partial functions $g\colon \operatorname{Dom}(g) \to \omega$, $\operatorname{Dom}(g) \in [\kappa]^{<\omega}$ such that if $\alpha, \beta \in \operatorname{Dom}(g)$ and

$$n \in \big(A_\alpha \setminus g(\alpha)\big) \cap \big(A_\beta \setminus g(\beta)\big) \text{ then } f_\alpha(n) = f_\beta(n).$$

Condition g is stronger then h iff $h \subset g$.

Indeed, this forcing adds uniformizations.

Fact 4.2.4. *Let G be a generic filter on $K(\mathcal{A}, f)$. Then*

$$f = \bigcup \{f_\alpha \restriction (A_\alpha \setminus g(\alpha)) : g \in G, \alpha \in \operatorname{Dom}(g)\}$$

is a uniformization of F.

Proof. It is clear that f is a partial function from ω to 2. We only need to show that for each $\alpha \in \kappa$ the set

$$D_\alpha = \{g \in K(\mathcal{A}, F)\colon \alpha \in \operatorname{Dom}(g)\}$$

is dense. Take any $p \in K(\mathcal{A}, F)$, $\alpha \notin \operatorname{Dom}(p)$. There exists $n \in \omega$ such that $(A_\alpha \setminus n) \cap A_\beta = \emptyset$ for each $\beta \in \operatorname{Dom}(p)$. Now $g = p \cup \{(\alpha, n)\}$ is a condition in $K(\mathcal{A}, F) \cap D_\alpha$ bellow p. ☐

Let us introduce the whole iteration. We will work with iterated forcing $A_\kappa \star (P_\gamma, Q_\gamma)_{\gamma \in \lambda}$ which is a finite support iteration of length λ, $A_\kappa \star P_\gamma \Vdash Q_\gamma$ is $K(\hat{\mathcal{A}}, \dot{F}_\gamma)$ where $\mathcal{A}$ is the name for the AD set generically added by A_κ and P_γ forces that $\dot{F}_\gamma \in \mathscr{F}_{\hat{\mathcal{A}}}$.

A condition $(p, \dot{q}) \in A_\kappa \star (P_\gamma, Q_\gamma)_{\gamma \in \lambda}$ is *simple* if $p \Vdash \operatorname{Dom}(\dot{q}) = \hat{D}_q$ for some $D_q \in [\lambda]^{<\omega} \cap V$ and for each $\gamma \in D_q$ there is some $h_q(\gamma) \in V$ such that $(p, \dot{q} \restriction \gamma) \Vdash \dot{q(\gamma)} = \hat{h_q(\gamma)}$.

Claim. *The set of simple conditions is dense in $A_\kappa \star (P_\gamma, Q_\gamma)_{\gamma \in \lambda}$.*

Proof. The first part is easy. Suppose for contradiction, that the opposite is true. For a condition (p, q), which is not simple, define $\gamma_{(p,q)}$ to be the the maximal $\beta \in D_q$ preventing simplicity of (p, q). There is (p, q) with no simple condition below with minimal $\gamma_{(p,q)}$. Find condition $(r, s) \in A_\kappa \star (P_\gamma, Q_\gamma)_\lambda$, $(r, s) < (p, q)$ such that $(r, s \restriction \gamma_{(p,q)}$ forces $q(\gamma_{(p,q)}) = h_q(\hat{\gamma_{(p,q)}})$. and $q(\alpha) = s(\alpha)$ for $\alpha \geq \gamma_{(p,q)}$. Now $\gamma_{(r,s)} < \gamma_{(p,q)}$ contradicting minimality of $\gamma_{(p,q)}$. ☐

A simple condition $(p, \dot{q})$ is *nice* if for each $\gamma \in D_q$ is $(p, \dot{q} \restriction \gamma) \Vdash \operatorname{Dom}(\dot{q(\gamma)}) = \hat{\Gamma p}$.

Claim. *The set of nice simple conditions is dense in $A_\kappa \star (P_\gamma, Q_\gamma)_{\gamma\in\lambda}$.* $\square$

The crucial argument is, that this iterated forcing is ccc and hence the AD system added in the first step remains uncountable in the final generic extension.

Lemma 4.2.5. *The forcing $A_\kappa \star (P_\gamma, Q_\gamma)_{\gamma\in\lambda}$ is ccc.*

Proof. Let $\{(p_\alpha, q_\alpha)\colon \alpha \in \omega_1\}$ be a set of nice simple conditions. Using the Δ-system lemma and the pigeon hole principle we can thin out this set so that we can assume

1. $n_{p_\alpha} = n$ for all $\alpha \in \omega_1$.
2. $\{\Gamma_{p_\alpha}\colon \alpha \in \omega_1\}$ is a Δ-system with core Γ. Denote ${}^\alpha\varphi^\beta$ the unique order preserving isomorphism mapping Γ_{p_α} onto Γ_{p_β}.
3. For each $\alpha, \beta \in \omega_1$ is $p_\beta \circ {}^\alpha\varphi^\beta = p_\alpha$.
4. $\{D_{q_\alpha}\colon \alpha \in \omega_1\}$ is a Δ-system with core $D = \{d_i\colon i \in |D|, d_i < d_j \text{ for } i < j\}$.
5. For each $\alpha, \beta \in \omega_1$ and $\gamma \in D_{q_\alpha}$ is $h_{q_\beta}(\gamma) \circ {}^\alpha\varphi^\beta = h_{q_\alpha}(\gamma)$.

For $i \in \omega$ put $b(i) = 2^{1+i\cdot n\cdot|\Gamma|}$. We will show that among conditions $\{(p_\alpha, q_\alpha)\colon \alpha \in b(|D|)\}$ at least two are compatible. Fix (the unique) increasing enumeration

$$\bigcup\{D_{q_\alpha}\colon \alpha \in b(|D|)\} = \{\beta_k\colon k \in K\}$$

for some $K \in \omega$.

We will define a sequence of conditions

$$\{(r_k, s_k) \in A_\kappa \star (P_\gamma, Q_\gamma)_{\gamma\in\beta_k+1}\colon k \in K\}$$

such that

$$(r_{k+1}, s_{k+1} \restriction \beta_k + 1) \leq (r_k, s_k)$$

and a sequence of sets

$$\{\Omega_i \subset b(|D|)\colon i \in |D| + 1, |\Omega_i| = b(|D| - i)\}.$$

Start with $\Omega_0 = b(|D|)$ and let r_{-1} be the empty condition. If $\beta_k \in D_{q_\alpha} \setminus D$ for some $\alpha \in d_{|D|}$ choose (r_k, s_k) such that

$$(r_k, s_k \restriction \beta_k) \Vdash s_k(\beta_k) \leq q_\alpha(\beta_k).$$

If $\beta_k = d_i$ then choose $(r_k, s'_k) \in A_\kappa \star (P_\gamma, Q_\gamma)_{\gamma\in\beta_k}$ such that there is some function $g_k \in V$ such that for each $j \in n$ and $\sigma \in \Gamma_{q_\alpha}$ for each $\alpha \in \Omega_{k-1}$ is

$$(r_k, s'_k) \Vdash \dot{f}_\sigma(j) = \hat{g}_k(\sigma, j)$$

for $\dot{f}_\sigma \in \dot{F}_{\beta_k}$.

There exists

$$\Omega_{i+1} \in [\Omega_i]^{b(|D|-(i+1))}$$

such that for each $\alpha, \beta \in \Omega_{i+1}$ and $\sigma \in D_{q_\alpha}$ is

$$g_k({}^\alpha\varphi^\beta(\sigma), j) = g_k(\sigma, j)$$

for each $j \in n$ (pigeon hole principle). Hence (r_k, s'_k) forces that

$$s_k(\beta_k) = \bigcup\{h_{q_\alpha}(\beta_k)\colon \alpha \in \Omega_{i+1}\} \in Q_{\beta_k}$$

and we define $s_k = s'_k{}^\frown s(\beta_k)$.

Once (r_{K-1}, s_{K-1}) and $\Omega_{|D|} = \{\alpha, \beta\}$ are defined, we have that (r_{K-1}, s_{K-1}) is below both (p_α, q_α) and (p_β, q_β). $\square$

Theorem 4.2.6 (Steprans)**.** *For each cardinal κ it is consistent with ZFC that there exist a strong-Q-sequence of cardinality κ.*

Proof. Use the forcing we were considering in this section. At first add an AD system of required size κ and then use a bookkeeping device to add all uniformization with iteration of length 2^κ. The fact that the forcing is ccc ensures, that once the iteration is done, the AD system has still size κ. $\square$

4.3 Guided Grigorieff and guided Sacks forcing

Forcing constructed in previous chapter was defined to add uniformizations while being ccc. One disadvantage of the construction is that it provides no control over $\mathfrak{d}$ in the resulting model. Keeping in mind that we want to approximate a model where $\mathcal{P}(\omega)/\operatorname{Fin} \cong \mathcal{P}(\omega_1)/\operatorname{Fin}$, we should beside creating a strong-Q-sequence also aim for $\mathfrak{d} = \omega_1$. Forcing notions defined in this section are designed to add uniformizations while being proper and ${}^\omega\omega$ bounding and hence $\mathfrak{d}$ is not increased.

Similar forcing appeared implicitly in [JS91]. It was M. Hrušák who observed that method used there is relevant in the context of strong-Q-sequences.

The following forcing notions are somewhat similar to the Grigorieff and Sacks forcings. In fact, they are subposets of these forcings as defined in chapter 3.

Definition 4.3.1 (Guided Grigorieff forcing)**.** Let $\mathcal{T} = \{T_\alpha \colon T_\alpha \in [\omega]^\omega, \alpha \in \omega_1\}$ be a strictly increasing tower, i.e. $\mathcal{A} = \{A_\alpha = T_{\alpha+1} \setminus T_\alpha, \alpha \in \omega_1\}$ is an AD system consisting of infinite sets.

For $F = \{f_\alpha \colon A_\alpha \to 2\} \in \mathscr{F}_\mathcal{A}$ conditions in the forcing $G(\mathcal{T}, F)$ are partial functions $g \colon \operatorname{Dom}(g) \to 2$ for which there is $d(g) \in \omega_1$ such that $\operatorname{Dom}(g) =^* T_{d(g)}$. Moreover we require that for each $\alpha < d(g)$ is $g \restriction A_\alpha =^* f_\alpha$. The ordering is reversed inclusion; $g \leq h$ iff $h \subset g$.

This forcing notion $G(\mathcal{T}, F)$ we call *guided Grigorieff* forcing.

This forcing has size at most 2^ω. We will show that for right choice of $\mathcal{T}$ this forcing is proper and ${}^\omega\omega$ bounding (and it can also have Sacks property).

Proposition 4.3.2. *The set $S_\alpha = \{g \in G(\mathcal{T}, F) \colon \alpha \leq d(g)\}$ is dense in $G(\mathcal{T}, F)$ for each $\alpha \in \omega_1$.*

Proof. Take $p \in G(\mathcal{T}, F)$ such that $d(p) < \alpha$. The interval $[d(p), \alpha)$ is countable so there are pairwise disjoint sets A'_β for $\beta \in [d(p), \alpha)$, $A'_\beta \cap \operatorname{Dom}(p) = \emptyset$ and $A'_\beta =^* A_\beta$ for each $\beta \in [d(p), \alpha)$. Now any function g extending $p \cup \bigcup\{f_\beta \restriction A'_\beta \colon \beta \in [d(p), \alpha)\}$ with $\operatorname{Dom}(g) =^* T_\alpha$ is a condition below p and belongs S_α. □

Note that this proposition would be not possible to prove if we replaced ω_1 with some larger cardinal κ.

This shows that the forcing $G(\mathcal{T}, F)$ adds a generic function $f_F = \bigcup\{g \colon g \in G\}$ (where G is the generic filter) and $A_\beta \subset^* \operatorname{Dom}(f_F)$ for each $\beta \in \omega_1$. The function f_F is obviously an uniformization of F.

Next we introduce guided Sacks forcing. The relation between guided Grigorieff and guided Sacks is similar to relation between Grigorieff and Sacks forcing form chapter 3.

Definition 4.3.3 (Guided Sacks forcing)**.** Let $\mathcal{T} = \{T_\alpha \colon T_\alpha \in [\omega]^\omega, \alpha \in \omega_1\}$ be a strictly increasing tower, i.e. $\mathcal{A} = \{A_\alpha = T_{\alpha+1} \setminus T_\alpha, \alpha \in \omega_1\}$ is an AD system consisting of infinite sets and take $F = \{f_\alpha \colon A_\alpha \to 2\} \in \mathscr{F}_\mathcal{A}$.

A condition p in the forcing $S(\mathcal{T}, F)$ is a subtree of the binary tree ${}^{<\omega}2$ such that the set of splitting levels of p is $S(p) =^* \omega \setminus T_{d(p)}$ for some $d(p) \in \omega_1$. Moreover for each $\alpha < d(p)$ there exists $n^p_\alpha \in \omega$ such that for each $s \in p$ and $k > n^p_\alpha$, if $k \in \operatorname{Dom}(s) \cap A_\alpha$ then $s(k) = f_\alpha(k)$.

The ordering is inclusion, $g \leq p$ iff $g \subset p$.

This forcing notion $S(\mathcal{T}, F)$ we call *guided Sacks* forcing.

Sacks forcing has again size at most 2^ω and is a subposet of Sacks forcing with filter as defined in chapter 3. And again, guided Grigorieff can be regarded as a subposet consisting of uniform trees in the guided Sacks forcing.

Proposition 4.3.4. *The set $S_\alpha = \{p \in S(\mathcal{T}, F) \colon \alpha \leq d(p)\}$ is dense in $S(\mathcal{T}, F)$ for each $\alpha \in \omega_1$.*

Proof. Analogous to proof of proposition 4.3.2. □

Hence this forcing again adds a uniformization of F, namely the intersection of all conditions in generic filter.

Next lemma shows that the guided Sacks forcing poses the 'gluing' property characteristic for Sacks forcing.

Lemma 4.3.5. *Let $\{p_i \colon i < k\}$ be a finite set of conditions in $S(\mathcal{T}, F)$. There is a set of conditions $\{p'_i \colon i < k\}$ such that $p'_i < p_i$ for each $i < k$ and $\bigcup_{i<k} p'_i \in S(\mathcal{T}, F)$.*

Proof. Use proposition 4.3.4 to find $\alpha \in \omega_1$ and $p'_i < p_i$ such that $d(p'_i) = \alpha$ for all $i < k$. To see that $q = \bigcup_{i<k} p'_i \in S(\mathcal{T}, F)$ note that for n^q_α one can take $\max\{n^{p'_i}_\alpha : i < k\}$. □

Next we will show that guided versions of Grigorieff and Sacks forcings retain some 'nice' properties of their original unguided versions. A key element in fusion like arguments for these forcings is a lemma analogous to lemma 3.1.4.

Lemma 4.3.6.

1. *Pick any $p \in G(\mathcal{T}, F)$. Suppose $p \Vdash \dot{x} \in \hat{X}$ and fix a finite set $a \in [\omega \setminus \mathrm{Dom}(p)]^n$. Then there exists a condition $q \in G(\mathcal{T}, F)$, $q < p$ and a finite set $Y \in [X]^{\leq 2^n}$ such that $q \Vdash \dot{x} \in \hat{Y}$ and $\mathrm{Dom}(q) \cap a = \emptyset$.*

2. *Pick any $p \in S(\mathcal{T}, F)$. Suppose $p \Vdash \dot{x} \in \hat{X}$ and fix a finite set $a \in [S(p)]^n$. Then there exists a condition $q \in S(\mathcal{T}, F)$, $q < p$ and a finite set $Y \in [X]^{\leq 2^n}$ such that $q \Vdash \dot{x} \in \hat{Y}$ and $a \subset S(q)$.*

Proof. The proof of lemma 3.1.4 works with almost no modifications. (Just for guided Sacks use for 'gluing' lemma 4.3.5.) □

And the following lemma shows that our new forcings are nice indeed. Proof of this lemma needs the tower game defined if chapter 2. In fact, this proof is the main reason why the author of this text defined games with towers.

Lemma 4.3.7. *Let $\mathcal{T} = \{T_\alpha \colon \alpha \in \omega_1\}$ be an increasing tower generating non-meager (p-)ideal $\langle \mathcal{T} \rangle$ on ω and take any $F \in \mathscr{F}_{\mathcal{A}}$ (where $\mathcal{A} = \{T_{\alpha+1} \setminus T_\alpha \colon \alpha \in \omega_1\}$). Both $G(\mathcal{T}, F)$ and $S(\mathcal{T}, F)$ are proper ${}^\omega\omega$ bounding forcing notions. If $\langle \mathcal{T} \rangle$ is moreover rapid then both this forcings have Sacks property.*

Proof. This proof is similar to proof of lemma 3.1.3. The difficulty given by increased complexity of involved forcing notions is solved by using the tower game $TG_{\mathcal{T}^*}$ (see 2.4.2) instead of p-filter game. Here $\mathcal{T}^*$ is decreasing tower dual to $\mathcal{T}$; $\mathcal{T}^* = \{T^*_\alpha = \omega \setminus T_\alpha \colon \alpha \in \omega_1\}$.

Only the proof of guided Sacks being proper is presented here. All the other proofs are essentially the same (for proving Sacks property use the rapid tower game $RT_{\mathcal{T}^*, f}$ for some $f \in {}^\omega\omega$). For more details see proof of 3.1.3.

Take any countable elementary submodel M of $H(\theta)$ (for sufficiently large θ) containing $S(\mathcal{T}, F)$ and a condition $g \in S(\mathcal{T}, F) \cap M$. Enumerate $\{\dot{\tau}_n \colon n \in \omega\}$ all $S(\mathcal{T}, F)$-names for ordinal numbers belonging to M. We need to find a condition in $S(\mathcal{T}, F)$ which is stronger than g and forces $\dot{\tau}_n \in M$ for each $n \in \omega$.

Two players play the game $TG_{\mathcal{T}^*}$ in $H(\theta)$ but the actual moves will take place in M. Player I will follow this strategy: At first he denotes g as h_0 and puts $a_0 = \emptyset$ and $\alpha_0 = 0$. In the n-th move he has some condition $h_n \leq g$, $h_n \in M$, a set $a_n \in [S(h_n)]^{<\omega}$ and an ordinal $\alpha_n \in \omega_1 \cap M$ such that $a_n \subset S(h_n)$ and $\alpha_0 \leq d(h_n)$. Now he uses lemma 4.3.6 in M for $h_n \Vdash \dot{\tau}_n \in On$ and the finite set a_n to get a condition $h'_{n+1} < h_n$, $h'_{n+1} \in M$ and a finite set $Y_n \in [On]^{<\omega}$ such that $h'_{n+1} \Vdash \dot{\tau}_n \in Y_n$ (in M), $a_n \subset S(h'_{n+1})$. Note that $Y_n \subset M$.

Player I fixes a finite set $A_n \in [\omega]^{<\omega}$ such that

$$S(h'_{n+1}) \supset T^*_{d(h'_{n+1})} \setminus A_n$$

and his n-th move is $\big(d(h'_{n+1}), A_n\big)$. To this player II responds with

$$(\alpha_{n+1}, b_n) \in (\omega_1 \cap M) \times [S(h'_{n+1})]^{<\omega}.$$

Player I denotes $a_{n+1} = a_n \cup b_n$ (hence $a_{n+1} \subset S(h'_{n+1})$) and chooses some $h_{n+1} < h'_{n+1}$, $h_{n+1} \in M$, $\alpha_{n+1} \leq d(h_{n+1})$ and $a_{n+1} \subset S(h_{n+1})$. Now he can continue to move $n+1$.

When the game is over, player I collected a sequence of conditions $\{h_n \colon n \in \omega\} \subset S(\mathcal{T}, F) \cap M$, $h_{n+1} < h_n \leq g$ and a sequence $\{Y_n \colon n \in \omega\}$ of finite subsets of $On \cap M$ such that $h_{n+1} \Vdash \dot{\tau}_n \in Y_n$. According to theorem 2.4.4 the described strategy is not winning for player I (in $H(\theta)$) so we can assume that the actual course of this game was won by player II.

We will check that there exists $q \subset h = \bigcap\{h_n \colon n \in \omega\}$, such that $q \in S(\mathcal{T}, F)$. Put $\gamma = \sup\{\alpha_n \colon n \in \omega\}$. We know that $S(h) \supset \bigcup\{a_n \colon n \in \omega\} \supset^* T^*_\gamma$. Let q be any subtree of h such that $S(q) =^* \omega \setminus T_\gamma$.

To show that $q \in S(\mathcal{T}, F)$ we only need to check that for each $\alpha < \gamma$ there exists $n^q_\alpha \in \omega$ such that for each $s \in q$ and $k > n^p_\alpha$, if $k \in \mathrm{Dom}(s) \cap A_\alpha$ then $s(k) = f_\alpha(k)$. (See definition 4.3.3.) But there is some $n \in \omega$ such that $\alpha_n > \alpha$ and hence $d(h_n) > \alpha$. This shows that we can put $n^q_\alpha = n^{h_n}_\alpha$ (since $q \subset h \subset h_n$).

Now $q < h_n$ for each $n \in \omega$ and $q \Vdash \dot{\tau}_n \in Y_n$ thus $q \Vdash \dot{\tau}_n \in M$ (since Y_n is finite) and properness is proved. □

4.4 Preserving selective ultrafilter III

In section 3.3 we showed that Sacks forcing preserves selective ultrafilters not near coherent with the filter used as parameter. The same is true for the guided Sacks forcing as well.

The second preservation theorem (for selective ultrafilters strictly $\leq_{RK}$ bellow) is not of much interest in this context. The reason is, that it required the assumption, that the filter $\mathcal{F}$ used as parameter is also an ultrafilter. This would translate to $\mathcal{T}$ generates an maximal ideal. However, $\mathcal{T}$ would stop generating maximal ideal after the forcing with $S(\mathcal{T}, F)$ and since we would like to continue adding uniformizations, the preservation theorem would be of no use then.

Theorem 4.4.1. *Let $\mathcal{T} = \{T_\alpha\colon \alpha \in \omega_1\}$ be an increasing tower generating non-meager (p-)ideal $\langle\mathcal{T}\rangle$ on ω and take any $F \in \mathscr{F}_{\mathcal{A}}$ (where $\mathcal{A} = \{T_{\alpha+1} \setminus T_\alpha\colon \alpha \in \omega_1\}$). Let $\mathscr{R}$ be a selective ultrafilter which is not near coherent with the filter dual to ideal generated by $\mathcal{T}$.*

The forcing $S(\mathcal{T}, F)$ preserves $\mathscr{R}$ as a base of a selective ultrafilter.

Proof. This is only reformulation of proof of 3.3.6 in a similar fashion as in proof of 4.3.7. The mixed game from definition 2.4.10 for the tower $\mathcal{T}^*$ dual to $\mathcal{T}$ and a rare p-filter $\mathscr{R}$ is used.

It is sufficient to prove that for a given $S(\mathcal{T}, F)$ name $\dot{A}$ for a subset of ω there is a dense set of conditions deciding that there is some $R \in \mathscr{R}$ such that $R \subset \dot{A}$ or $R \cap \dot{A} = \emptyset$.

Fix a condition $p \in S(\mathcal{T}, F)$. We can suppose that there is no $q < p$ such $q \Vdash \dot{A} \notin \langle\mathscr{R}\rangle$ i.e. for each $q < p$ is $R_q = \{s \in \omega\colon \exists q' < q\colon q' \Vdash n \in \dot{A}\} \in \mathscr{R}$.

Two players will play the mixed game for tower $\mathcal{T}^* = \{T^*_\alpha = \omega \setminus T_\alpha\colon \alpha \in \omega_1\}$ in even moves and p-filter $\mathscr{R}$ with the Q property in odd moves. Player I will follow this strategy: At first he denotes p as h_0 and puts $a_0 = \emptyset$ and $\alpha_0 = 0$.

Let n be even. In the n-th move player I has some condition $h_n \leq p \in S(\mathcal{T}, F)$, $\alpha_n \geq d(h_n)$ and a set $a_n \in [S(h_n)]^{<\omega}$. Player I chooses a finite set $A_n \in [\omega]^{<\omega}$ such that $S(h_n) \supset T^*_{d(h_n)} \setminus A_n$ and his n-th move is $(d(h_n), A_n)$. To this player II responds with $(\alpha_{n+1}, b_n) \in \omega_1 \times [S(h_n)]^{<\omega}$. Player I denotes $a_{n+1} = a_n \cup b_n$ (so $a_{n+1} \subset S(h_n)$) and chooses some $h_{n+1} \in S(\mathcal{T}, F)$, $h_{n+1} < h_n$, $a_{n+1} \subset S(h_{n+1})$ and $d(h_{n+1}) > \alpha_{n+1}$ and continues with the odd move $n + 1$.

Now n is odd. Player I has condition h_n, $a_n \subset [S(h_n)]^{<\omega}$ and $\alpha_n \in \omega_1$.

Fix $k_n \in \omega$ such that $a_n \subset k_n$. Put

$$R(n) = \bigcap\{R_q\colon q = h_n[\eta], \eta \in {}^{[k_n]}h_n\} \in \mathscr{R}.$$

The n-th move of player I is $R(n)$.

To this player II responds with an integer $r_n \in R(n)$. For each condition $q = h_n[\eta]$, $\eta \in {}^{k_n}h_n$ is $r_n \in R_q$ so there is a stronger condition $q' < q$ such that $q' \Vdash r_n \in \dot{A}$. Put h_{n+1} to be a condition created by gluing conditions q' (lemma 4.3.5 and note that $h_{n+1} < h_n$, $a_n \subset S(h_{n+1})$ and $h_{n+1} \Vdash r_n \in \dot{A}$. Put $a_{n+1} = a_n$, $\alpha_{n+1} = \alpha_n$ and continue with the next (even) move $n + 1$.

When the game is over, player I collected a sequence of conditions $\{h_n\colon n \in \omega\} \subset S(\mathcal{T}, F)$, $h_{n+1} < h_n \leq p$ and a sequence $\{r_n\colon n \in \omega\}$ such that $h_{n+1} \Vdash r_n \in \dot{A}$. According to remark 2.4.12 the described strategy is not winning for player I so we can assume that the actual course of this game was won by player II. Denote $\gamma = \sup\{\alpha_n\colon n \in \omega\}$. We can use the same arguments as in proof of 4.3.7 that there is a condition $q \in S(\mathcal{T}, F)$, $q \leq h_n$ for each $n \in \omega$ with $d(q) = \gamma$. Thus $q \Vdash R = \{r_n\colon n \in \omega\} \subset \dot{A}$ and $R \in \mathscr{R}$. We proved that $q \Vdash \dot{A} \in \langle\mathscr{R}\rangle$. □

4.5 A Countable like ideal and small $\mathfrak{d}$

Now we have all tools for iterating forcings adding uniformizations and hence creating a strong-Q-sequence. In fact are ready to prove our main result, the consistency of existence of a countable like ideal. (See definition 1.3.7.)

Theorem 4.5.1. *It is consistent with ZFC that $\mathfrak{d} = \omega_1$ and there is countable like ideal $\mathcal{I}$ on ω.*

Proof. This is again a construction of forcing with countable support iteration similar to the one used in 3.2.5. Start in a model of GCH and pick $\mathcal{T} = \{T_\alpha \colon \alpha \in \omega_1\}$ some increasing tower in $\mathcal{P}(\omega)$ which generates non-meager ideal and denote $\mathcal{A}$ the AD system $\{T_{\alpha+1} \setminus T_\alpha \colon \alpha \in \omega_1\}$.

Now use countable support iteration of forcings $G(\mathcal{T}, F_\alpha)$ adding uniformization of F_α and $G(\mathcal{F}_\alpha \times \omega)$ and use some bookkeeping device to control that each p-filter in each intermediate model appears as some $\mathcal{F}_\alpha$ at some stage and that F_α ranges over $\mathscr{F}_\mathcal{A}$ in all intermediate models. We are using (besides arguments mentioned in proof of 3.2.5) that $\mathscr{F}_\mathcal{A}$ has always size ω_2 and that all involved forcings are ${}^\omega\omega$ bounding so $\mathcal{T}$ generates non-meager ideals in all intermediate models.

The whole iteration is proper and ${}^\omega\omega$ bounding so in the resulting model $\mathfrak{d} = \omega_1$, $\mathcal{T}$ generates a non-meager p-ideal and $\mathcal{A}$ is a strong-Q-sequence. In this model there are no p-points hence $\langle \mathcal{T} \rangle$ is a countable like ideal. $\square$

In the previous proof we could have also used guided Sacks forcing instead of the guided Grigorieff for adding uniformizations and forcings $S(\mathcal{F} \times \omega)$ for killing p-points. This, together with theorem 4.4.1 and lemma 3.3.5, enables us pick in the groundmodel any set of selective ultrafilters, such that all their isomorphic copies intersect $\langle \mathcal{T} \rangle$, and construct the iteration so that all these ultrafilters are preserved in the generic extension.

CHAPTER 5

AUTOMORPHISMS OF $\mathcal{P}(\omega)/\operatorname{Fin}$

5.1 Trivial and non-trivial automorphisms

The topic of the last chapter of this thesis is investigation of automorphisms of the Boolean algebra $\mathcal{P}(\omega)/\operatorname{Fin}$. Our attention will be limited to variants of property of such automorphisms called triviality.

We will leave out questions concerning the general structure of the automorphism group, for more information about this structure we refer to [vD90, Fuc92, Far00, Ste03]

Definition 5.1.1. Let A be a set. A partial 1-1 function $f\colon A \to A$ is an *almost permutation* (of A) iff

$$\operatorname{Dom}(f) =^* A =^* \operatorname{Rng}(f).$$

Each almost permutation $p\colon A \to B$ induces in the natural way a Boolean isomorphism

$$\varphi\colon \mathcal{P}(A)/\operatorname{Fin} \to \mathcal{P}(B)/\operatorname{Fin},$$

namely $\varphi[A] = [f[A]]$ for $A \in \mathcal{P}(A)$. So given any almost permutation f of ω we have an automorphism φ of the Boolean algebra $\mathcal{P}(\omega)/\operatorname{Fin}$. Automorphisms obtain in such way are called trivial.

Definition 5.1.2. Let $\varphi\colon \mathcal{P}(\omega)/\operatorname{Fin} \to \mathcal{P}(\omega)/\operatorname{Fin}$ be a Boolean automorphism of $\mathcal{P}(\omega)/\operatorname{Fin}$. We will denote the ideal of sets on which φ is trivial as $\operatorname{Triv}(\varphi)$. This set contains precisely those subsets A of ω such that $\varphi\restriction\big(\mathcal{P}(A)/\operatorname{Fin}\big)$ is induced by some one-to-one function $f\colon A' \to \omega$ (and $A' =^* A$).

Let $\mathcal{S}$ be a subset of $\mathcal{P}(\omega)$. The automorphism φ is *trivial on* $\mathcal{S}$ if $\mathcal{S} \cap \operatorname{Triv}(\varphi) \neq \emptyset$. We call φ *trivial* if it is trivial on $\{\omega\}$ and *somewhere trivial* if it is trivial on $[\omega]^\omega$. An automorphism is *nowhere trivial* if it is not somewhere trivial.

It is not immediately clear whether there need to exists non-trivial automorphisms at all. It was shown by [Rud56] that under CH this is the case indeed. In fact, he showed that under CH there are 2^{2^ω} many automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$ hence most of them must be non-trivial. Later Steprans proved that the cardinality of the group of automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$ can be any regular cardinal between 2^ω and 2^{2^ω}, see [Ste03].

The following result is well known and was proved independently by van Douwen and Baumgartner.

Proposition 5.1.3. *If there is a p-point of character ω_1 then there is a non-trivial automorphism of $\mathcal{P}(\omega)/\operatorname{Fin}$.*

Proof. Fix an $\subset^*$ increasing sequence of sets $\{A_\alpha\colon \alpha \in \omega_1\}$ which generates a maximal non-principle ideal $\mathcal{I}$ in $\mathcal{P}(\omega)$. Then define inductively functions f_α, f_α is a permutation of A_α with no fixed points and $f_\alpha \subset^* f_\beta$ for $\alpha < \beta$.

Define mapping $F\colon \mathcal{P}(\omega)/\operatorname{Fin} \to \mathcal{P}(\omega)/\operatorname{Fin}$ by

$$F[A] = \begin{cases} [f_\alpha[A]] & \text{for } A \in \mathcal{I} \text{ and } A \subset^* A_\alpha \text{ for some } \alpha \in \omega_1, \\ [\omega \setminus f_\alpha[\omega \setminus A]] & \text{for } A \notin \mathcal{I} \text{ and } \omega \setminus A \subset^* A_\alpha \text{ for some } \alpha \in \omega_1. \end{cases}$$

It is easy to see that F is an automorphism of $\mathcal{P}(\omega)/\operatorname{Fin}$. Assume there were an almost permutation f of ω inducing F. We can find three disjoint infinite sets X_i, $i \in 3$ such that $X_0 \cup X_1 \cup X_2 =^* \omega$ and $f[X_i] \cap X_i = \emptyset$ (since f can have only finitely many fixed points). Pick $i \in 3$ such that $X_i \notin \mathcal{I}$ and since $\mathcal{I}$ is maximal, X_i is in the dual filter and also $f[X_i]$ is in dual filter. This is a contradiction with $f[X_i] \cap X_i = \emptyset$. □

On the other hand, an important result in this field is that it is consistent with ZFC that all automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$ are trivial. This was first demonstrated by Shelah in [She82] using forcing method called oracle-cc forcing (see also [Jus92]). Afterwards it was shown in [SS88] that PFA implies that all automorphisms are trivial and this method was further refined by Velickovic in [Vel86, Vel93], where he proved that OCA + MA is strong enough for proving that all isomorphisms are trivial. He also showed that starting from a model of PFA, it is possible to construct a model with a non-trivial automorphism and where MA holds.

An important notion is lifting of an morphism of $\mathcal{P}(\omega)/\operatorname{Fin}$.

Definition 5.1.4. Let $\varphi\colon \mathcal{P}(\omega)/\operatorname{Fin} \to \mathcal{P}(\omega)/\operatorname{Fin}$ be a mapping. A function $\Phi\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is called *lifting* of φ if $[\Phi(A)] = \varphi[A]$ for each $A \in \mathcal{P}(\omega)$.

One of main tools proved and used in [Vel93] is the following theorem. We will reprove this result in a slightly more general form as proposition 5.3.6.

Theorem 5.1.5 (Velickovic)**.** *Let Φ be a lifting of an automorphism φ. If there exist Borel functions $F_n\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ for $n \in \omega$ such that for each $A \in \mathcal{P}(\omega)$ there is $n \in \omega$ for which $\Phi(A) =^* F_n(A)$, then φ is trivial.*

On the other hand, it was shown in [SS89] that in the model obtained by adding ω_2 Cohen reals to a model of CH, there is a non-trivial automorphism and its non-triviality is in certain sense absolute. This example shows, that if we need to force all automorphisms to be trivial, the only reasonable way is to prevent the non-trivial ones to be extendible in the generic extension, i.e. non-triviality generally can not be 'cured' by adding missing almost permutations.

It was also shown in [SS94] that it is consistent with MA that all automorphism are somewhere trivial while there can exist a non-trivial automorphism in the same model. And later in [SS02] was shown that MA is consistent with the existence of nowhere trivial automorphisms and that in model obtained from CH by iterating Silver forcing every automorphism is somewhere trivial (and $\mathfrak{d} = \omega_1$). For a short review of results and open question concerning the group of automorphisms see chapter of J. Steprans in [HvM90].

It is also worth mentioning that investigation of existence of non-trivial automorphisms has significant impact on the related topic of inner and outer automorphisms of Calkin algebras [Far11].

This chapter is written with several goals in mind. First of them is better understanding relation between Katowice problem and non-trivial automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$. This is partially achieved by inclusion of a recent result of K. P. Hart in section 5.4. Another point of interest is how can be forcing methods of previous chapters integrated with methods for controlling non-trivial automorphisms. This can be formulated as a search for proper ${}^\omega\omega$ bounding forcing notions destroying non-trivial automorphisms. And one more question is 'Is is it possible to build a model of ZFC where all automorphism are trivial and $\mathfrak{d} = \omega_1$?'

We will show that forcing with Grigorieff forcing $G(\mathcal{F})$ prevents any automorphism φ with $\operatorname{Triv}(\varphi) \cap \mathcal{F} = \emptyset$ to be extended to an automorphism in the forcing extension. To gain 'absolute' non-extendability, we will need to use method called gap freezing.

Using this technique, we are able e.g. to extend already mentioned result of Shelah and Steprans to:

Theorem 5.1.6. *It is consistent with ZFC that $\mathfrak{d} = \omega_1$ and every automorphism of $\mathcal{P}(\omega)/\operatorname{Fin}$ is trivial on each non-meager p-filter.*

This chapter arose from a collaboration of the author of this text with Alan Dow. Many ideas behind proofs presented here are due to Alan Dow.

5.2 Abraham-Todorcevic gap freezing

The purpose of this section is to provide a minimalistic exposition of certain version of a gap freezing forcing. This forcing is needed as tool in the next section, where a forcing iteration using this forcing for freezing ω_1, ω_1 gaps is constructed.

This section presents some known facts and results about gaps and gap freezing methods. It contains mainly results from [AT97]. The treatment of this topic here is rather minimal and the only purpose it serves is an attempt to make this text more self contained. For systematic treatment of the problematic of gaps in $\mathcal{P}(\omega)/\operatorname{Fin}$ see e.g. [Sch93, Far96, Yor03].

At certain place a forcing property called properness isomorphism condition will be mentioned. It's purpose is again to enable some iterated forcing constructions to work. These properties and related theorems are in this text used as black box. For exposition into these topics the reader can use [She98a, Abr10, Bur98].

Through this whole section GCH is assumed to hold.

Definition 5.2.1. A sequence $\{(a_\alpha, b_\alpha) \colon \alpha \in \omega_1\}$ is a *pregap* if $a_\alpha, b_\alpha \subset \omega$ and for each $\alpha \leq \beta < \omega_1$ is $a_\alpha \subset^* a_\beta \subset b_\beta \subset^* b_\beta$.

This pregap is a *gap* if there is no $x \subset \omega$ such that $a_\alpha \subset^* x \subset^* b_\alpha$ for all $\alpha \in \omega_1$.

Definition 5.2.2. A pregap $\{(a_\alpha, b_\alpha) \colon \alpha \in \omega_1\}$ is *Hausdorff gap* if

$$|\{\beta < \alpha \colon a_\beta \setminus b_\alpha \subset n\}| < \omega$$

for each $\alpha \in \omega_1$ and $n \in \omega$.

Fact 5.2.3. *Each pregap containing a Hausdorff gap (as a subsequence) is gap. Moreover this remains true in any larger model of ZFC in which $\omega_1 = {\omega_1}^{\mathbf{V}}$.*

For proof of the following theorem see e.g. [Yor03].

Theorem 5.2.4 (Kunen)**.** *For each gap $\mathcal{A}$ there exists a ccc forcing notion $K_\mathcal{A}$ such that in the generic extension $\mathcal{A}$ contains a Hausdorff gap.*

The main theorem we need for making non-trivial automorphisms absolutely inextendible is the following.

Theorem 5.2.5 (Abraham-Todorcevic)**.** *(GCH) Let $\mathcal{A} = \{(a_\alpha, b_\alpha) \colon \alpha \in \omega_1\}$ be a gap. There is a proper ω_2-p.i.c. forcing notion of size ω_2 not adding new reals , such that in the generic extension $\mathcal{A}$ contains a Hausdorff gap.*

For ω_2-p.i.c. see definition 5.2.12.

To prove this theorem, the ideal introduced in next lemma is used.

Lemma 5.2.6. *Let $\mathcal{A} = \{(a_\alpha, b_\alpha) \colon \alpha \in \omega_1\}$ be a gap. Define $\mathcal{I}_\mathcal{A} \subset [\omega_1]^\omega$ by $A \in \mathcal{I}_\mathcal{A}$ iff*

$$|\{\beta \in A \cap \alpha \colon a_\beta \setminus b_\alpha \subset n\}| < \omega \text{ for all } \alpha \in \omega_1 \text{ and } n \in \omega. \tag{*}$$

$\mathcal{I}_\mathcal{A}$ is a p-ideal and for each $B \in [\omega_1]^{\omega_1}$ there exists some $A \in [B]^\omega \cap \mathcal{I}_\mathcal{A}$.

Note that (*) can be equivalently replaced by requiring the condition to hold just for $\alpha \leq \sup A$.

The notion of p-ideal is in this context defined by requirement that for each $\mathcal{C} \in [\mathcal{I}]^\omega$ there exists some $C \in \mathcal{I}$ such that $A \subset^* C$ for each $A \in \mathcal{A}$. This is actually the same definition we had for ideals on ω.

Proof. Take $B \in [\omega_1]^{\omega_1}$ and since $\mathcal{B} = \{(a_\alpha, b_\alpha) \colon \alpha \in B\}$ is a gap, there is a ccc forcing $K_\mathcal{B}$. Take a countable elementary submodel M, $K_\mathcal{B} \in M$ and a M generic filter G on $K_\mathcal{B} \cap M$ (in the groundmodel). There is a set $J \in M[G]$, J cofinal subset of $B \cap M$ such that $\{(a_\alpha, b_\alpha) \colon \alpha \in J\}$ is Hausdorff gap in $M[G]$. Notice that for an in finite $J' \subset J$ bounded in J is $J' \in [B]^\omega \cap \mathcal{I}_\mathcal{A}$. To prove that $\mathcal{I}_\mathcal{A}$ is p-ideal, take $\mathcal{C} = \{A_i \in \mathcal{I}_\mathcal{A} \colon i \in \omega\}$ and enumerate

$$\left\{\beta \leq \sup \bigcup_{i \in \omega} A_i\right\} = \{\beta_i \colon i \in \omega\}.$$

For each $i, j, n \in \omega$ fix finite set

$$F^i_j(n) = \{\gamma \in A_j \cap \beta_i \colon a_\gamma \setminus b_{\beta_i} \subset n\}.$$

Put $A'_j = A_j \setminus \bigcup_{i \leq j} F^i_j(j)$ and $A = \bigcup_{j \in \omega} A'_j$. We have that $A_i \subset^* A$ for all $i \in \omega$ and $A \in \mathcal{I}_\mathcal{A}$ (check that (*) holds for each β_i). $\square$

It is obvious that if there is $I \in [\omega_1]^{\omega_1}$ such that $[I]^\omega \subset \mathcal{I}_\mathcal{A}$, then $\{(a_\alpha, b_\alpha) \colon \alpha \in I\}$ is a Hausdorff gap. So to prove theorem 5.2.5 it is enough to find forcing adding such I for given p-ideal $\mathcal{I}_\mathcal{A}$. Since we are assuming GCH, such p-ideal ideal is generated some $\subset^*$ increasing sequence $\{A_\alpha \colon \alpha \in \omega_1\}$ of countable subsets of ω_1. We will without loss of generality assume that $A_\alpha \subset \alpha$ for each $\alpha \in \omega_1$.

Definition 5.2.7. Let $\mathcal{I}$ be an ideal in $[\omega_1]^{\leq\omega}$ generated by an $\subset^*$ increasing tower $\{A_\alpha \subset \alpha \colon \alpha \in \omega_1\}$. Poset P contains pairs $(x_p, \mathcal{X}_p) = p$ where $x_p \in [\omega_1]^{\leq\omega}$ and

$$\mathcal{X}_p = \{X_n \in [\omega_1]^{\omega_1} \colon n \in \omega\}.$$

For $F \subset \omega_1$ and $p \in P$ we denote

$$E_p(F) = \Big\{X_n^p(F) = \{\alpha \in X_n \colon F \subset A_\alpha\} \colon n \in \omega, X_n \in \mathcal{X}_p\Big\}.$$

We define $q \leq p$ iff $x_p \sqsubset x_q$ (x_q end extends x_p) and $\mathcal{X}_p \cup E_p(x_q \setminus x_p) \subset X_q$ (so all sets in $E_p(x_q \setminus x_p)$ have to be uncountable).

Define $\mathbf{P} = \{p \in P \colon p \leq (\emptyset, \{\omega_1\})\}$, a subposet of P.

To prove theorem 5.2.5 is is enough to show the following.

Lemma 5.2.8. *Let $\mathcal{I} \subset [\omega_1]^{\leq\omega}$ be an ideal generated by $\subset^*$ increasing tower $\{A_\alpha \subset \alpha \colon \alpha \in \omega_1\}$. Suppose moreover that whenever $\omega_1 = \bigcup_{n\in\omega} B_n$ then there exist $n \in \omega$ such that $[B_n]^\omega \cap \mathcal{I} \neq \emptyset$.*

Poset $\mathbf{P}$ associated with $\mathcal{I}$ (see definition 5.2.7) is proper, does not add new countable subsets of groundmodel and has ω_2-p.i.c.

Let us start the proof with a simple density lemma.

Lemma 5.2.9. *For every $p \in \mathbf{P}$ and $\gamma \in \omega_1$ there exists some $q < p$ in $\mathbf{P}$ such that $x_q \setminus \gamma \neq \emptyset$.*

Proof. Suppose that for each $\beta > \gamma, \sup x_p$ there is some n_β such that $X_n^p(\{\beta\}) \in E_p(\{\beta\})$ is at most countable. Put $B_n = \{\beta > \gamma, \sup x_p \colon n_\beta = n\}$. There is some $n \in \omega$ and $\alpha \in \omega_1$ such that $|B_n| > \omega$ and $A_\alpha \cap B_n$ is infinite. Hence there is an uncountable $B \subset X_n$ and infinite $A =^* A_\alpha \cap B_n$ such that $A \subset A_\beta$ for each $\beta \in B$. For each $\beta \in A$ is $B \subset X_n^p(\{\beta\})$ a contradiction. $\square$

The following extension lemma will be used in the proof of properness.

Lemma 5.2.10. *Let $M \prec H(\theta)$ be a countable elementary submodel (for some θ large enough), $\mathcal{I}, \mathbf{P} \in M$, $p \in \mathbf{P} \cap M$. Denote $\varepsilon = \omega_1 \cap M$ and let $\mathcal{D} \in M$ be an open dense subset of $\mathbf{P}$. For any $A =^* A_\varepsilon$ there exists $q \in \mathcal{D} \cap M$, $q < p$ such that $x_q \setminus x_p \subset A$.*

Proof. Suppose that there is no such q. Hence for each $\alpha < \varepsilon$ there is a finite set $F_\alpha \in [\alpha]^{<\omega}$, such that there is no q, for which $x_q \setminus x_p \subset A_\alpha \cap A = A_\alpha \setminus F_\alpha$. Note that set

$$\{\alpha \in \omega_1 \colon \exists F_\alpha \in [\alpha]^{<\omega}, \text{ there is no } q < p \text{ such that } q \in \mathcal{D}, x_q \setminus x_p \subset A_\alpha \setminus F_\alpha\}$$

is defined in M and thus is equal to ω_1. There is a stationary set $S \in [\omega_1]^{\omega_1} \cap M$ such that $F_\alpha = F$ for all $\alpha \in S$. Put $p_1 = (x_p, \mathcal{X}_p \cup \{S\}) < p$. Now use lemma 5.2.9 in M to find $p_2 < p_1$ such that there is some β; $\max F < \beta < \varepsilon$, $x_{p_2} = x_{p_1} \cup \{\beta\}$. Finally find $q < p_2$, $q \in \mathcal{D} \cap M$ and notice that $S_1 = \{\gamma \in S \colon x_q \setminus x_p \subset A_\gamma\} \in \mathcal{X}_q$ is uncountable. We also have $\min(x_q \setminus x_p) = \beta > \max F$ and thus $x_q \setminus x_p \subset A_\gamma \setminus F$ for $\gamma \in S_1$, a contradiction. $\square$

And finally the main part of the proof, the construction of generic conditions.

Lemma 5.2.11. $\mathbf{P}$ *is proper and adds no new countable subsets of groundmodel.*

Proof. Fix $M \prec H(\theta)$ some countable elementary submodel such that $\mathbf{P} \in M$, and $p_0 \in \mathbf{P} \cap M$. Enumerate $\{\mathcal{D}_n : D \in \omega\}$ all dense open subsets of $\mathbf{P}$ in M and denote $\varepsilon = \omega_1 \cap M$. Fix also a bookkeeping bijection $b: \omega \to \omega^2$ such that if $b(i) = (k, l)$ then $k \leq i$.

We will define a descending sequence of conditions $\{p_i \in \mathbf{P} \cap M : i \in \omega\}$ such that $p_{i+1} \in \mathcal{D}_i$ and there is some $q < p_i$ for all $i \in \omega$ and an increasing sequence of finite sets $\{F_i \in [\varepsilon]^{<\omega} : i \in \omega\}$ (Start with $F_0 = \emptyset$).

Suppose p_i and F_i are defined and define p_{i+1} and F_{i+1} in the following way. We have $b(i) = (k, n)$. The set

$$X_n^{p_k}(x_{p_i} \setminus x_{p_k}) \in E_{p_k}(x_{p_i} \setminus x_{p_k}) \subset \mathcal{X}_{p_i}$$

is uncountable hence there is finite set $F'_{i+1} \in [\varepsilon]^{<\omega}$ and an uncountable set

$$X(k, n) \subset X_n^{p_k}(x_{p_i} \setminus x_{p_k}) \subset X_n \in \mathcal{X}_{p_k}$$

such that $A_\varepsilon \setminus F'_{i+1} \subset A_\alpha$ for all $\alpha \in X(k, n)$. Put $F_{i+1} = F_i \cup F'_{i+1}$ and use lemma 5.2.10 to find $p_{i+1} \in \mathcal{D}_i \cap M$ such that $x_{p_{i+1}} \setminus x_{p_i} \subset A_\varepsilon \setminus F_{i+1}$.

Once the sequence is defined put

$$q = \left(x_q = \bigcup_{i \in \omega} x_{p_i}, \mathcal{X}_q = \bigcup_{i \in \omega} \mathcal{X}_{p_i} \cup \bigcup_{i \in \omega} E_{p_i}(x_q \setminus x_{p_i}) \right).$$

To see that q is a condition we only need to check that for each $k \in \omega$ each set in

$$X_n^{p_k}(x_q \setminus x_{p_k}) \in E_{p_k}(x_q \setminus x_{p_k})$$

is uncountable. Find i such that $b(i) = (k, n)$. Hence $x_q \setminus x_{p_i} \subset A_\gamma$ for each $\gamma \in X(k, n)$ and

$$X(k, n) \subset X_n^{p_k}(x_q \setminus x_{p_k}).$$

The statement about not adding countable subsets of grounmodel follows from the fact, that q forces a value for each name from M for a countable subset of groundmodel. $\square$

One problem we may encounter while dealing with iterations of forcing $\mathbf{P}$ is their influence on cardinals of size bigger than ω_1 and on the behavior of the continuum function. The source of possible problems is that the size of $\mathbf{P}$ is ω_2 (or generally 2^{2^ω}). This is however solved by proving that $\mathbf{P}$ satisfies so called properness isomorphism condition – p.i.c. introduced in [She82]. As a reference for results and fact about this condition cited here we refer to [Abr10, Bur98].

We will only mention some basic facts about this condition.

Definition 5.2.12. Poset P has ω_2*-p.i.c. (properness isomorphism condition)* if the following holds.

Suppose we are given sufficiently large θ and two isomorphic countable elementary submodels $M_0, M_1 \prec H(\theta)$, $P \in M_0 \cap M_1$ and an isomorphism $h: M_0 \to M_1$, h is identity in $M_0 \cap M_1$ and for each $\alpha \in M_0 \cap M_1 \cap \omega_2$, $\beta \in (M_0 \setminus M_1) \cap \omega_2$ and $\gamma \in (M_1 \setminus M_0) \cap \omega_2$ is $\alpha < \beta < \gamma$.

Then for each $p_0 \in P \cap M_0$ there exists condition $q < p_0$ which is both M_0 and M_1 generic and for each $q' < q$ and $r \in P \cap M_0$ is $q' < r$ iff $q' < h(r)$.

Fact 5.2.13. *Each proper forcing of size at most ω_1 has ω_2-p.i.c.*

Fact 5.2.14. *Each forcing with ω_2-p.i.c. is ω_2-cc.*

Fact 5.2.15. *Countable support iteration of length $< \omega_2$ of forcing notions with ω_2-p.i.c. has ω_2-p.i.c.*

Fact 5.2.16. *(CH) Countable support iteration of length ω_2 of forcing notions with ω_2-p.i.c. is ω_2-cc.*

Fact 5.2.17. *(CH) Each forcing with ω_2-p.i.c. preserves CH.*

And here we proof that the forcing P has ω_2-p.i.c. indeed.

Lemma 5.2.18. $\mathbf{P}$ *has ω_2-p.i.c.*

Proof. Suppose that the situation is set up as in definition 5.2.12 and we need to find generic q. Note that for $p \in P \cap M_0$ is $h(p) = (x_p, \{h(X_n): X_n \in \mathcal{X}_p, n \in \omega\})$.

We will try to repeat the proof of 5.2.11 not caring about M_1 at first. Once a construction of M_0 generic condition q is done, it would be enough to extend q to q' such that $q' < h(p_i)$ for each $i \in \omega$. So we only need to check that it is possible to add all sets in

$$\{h(X_n): X_n \in \mathcal{X}_{p_i}, n, i \in \omega\} \cup \{X_n^{h(p_i)} \in E_{h(p_i)}(x_q \setminus x_{p_i}): n, i \in \omega\}$$

to $\mathcal{X}_{q'}$. Hence we only need to be sure that every $X_n^{h(p_i)} \in E_{h(p_i)}(x_q \setminus x_{p_i})$ is uncountable.

To ensure this modify the construction of $\{p_i, F_i: i \in \omega\}$ in the following way. When defining F_i' add an additional requirement, that there is not only uncountable $X(k,n) \subset X_n^{p_k}(x_{p_i} \setminus x_{p_k})$, but also some uncountable

$$X'(k,n) \subset h\left(X_n^{p_k}(x_{p_i} \setminus x_{p_k})\right)$$

such that $A_\varepsilon \setminus F_i' \subset A_\alpha$ for all $\alpha \in X(k,n) \cup X'(k,n)$. Thus in the end $X'(k,n) \subset X_n^{h(p_k)}$ and we are done. □

The poset $\mathbf{P}$ has in fact other nice properties not mentioned here. It is even possible to build countable support iteration of such posets without adding any new subsets of ω. For details see [AT97]. However, this results is not essential from our point of view. Our intention is to employ this poset in ${}^\omega\omega$ bounding ω_2-cc proper forcing iteration with countable support.

5.3 Destroying non-trivial automorphisms

In this section we deal with the problem of controlling (i.e. reducing) the set of non-trivial automorphisms on $\mathcal{P}(\omega)/\operatorname{Fin}$. The aim is to introduce a method to destroy nontrivial automorphisms while doing as little as possible which in our context means doing so with a proper ${}^\omega\omega$ bounding forcing. An ultimate result in this direction would be construction of such forcing killing all non-trivial automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$. Unfortunately we can only achieve partial result, forcing killing all automorphisms which are non-trivial on each member of some non-meager p-ideal.

From now on, we will work with an fixed non-trivial automorphism ϕ of $\mathcal{P}(\omega)/\operatorname{Fin}$.

Definition 5.3.1. A map $F: \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ will be called *purely additive* if for all $x, y \subset \omega$, each of the equations $F(x) \cup F(y) = F(x \cup y)$ and $F(x) \cap F(y) = F(x \cap y)$ hold.

The base of topology on $\mathcal{P}(\omega)$ consists of (clopen) sets $[s]$ for $s \in {}^{<\omega}2$. We will also denote this sets as sets $[t; n]$ where $t \subset n$ and $n \in \omega$ where $t; n$ correspond to characteristic function of t as subset of n. If we write just $[t]$, then the value $n = 1 + \max t$ is implicit.

Cohen forcing adding a single Cohen real will be in this section denoted $\mathbf{C}$. As it is defined, conditions of this forcing are elements of ${}^{<\omega}2$. The reader should be aware of the natural correspondence between this forcing and the poset of basic clopen subsets $\mathcal{P}(\omega)$. Following this correspondence, an open dense subset O of $\mathcal{P}(\omega)$ can (and will) be treated as open dense subset of the forcing $\mathbf{C}$ (i.e. the set of all basic open subsets of O) and the other way round.

Proposition 5.3.2. *If F is a purely additive and continuous self-map on $\mathcal{P}(\omega)$, $\mathcal{P}(\omega)/\operatorname{Fin}$, then it is completely additive (see [Far00]) in the sense that $F(x) = \bigcup\{F(\{i\}): i \in x\} \cup F\{\emptyset\}$ for all $x \subset \omega$. In particular, if F is a lifting (see 5.1.4) of an automorphism Φ on $\mathcal{P}(\omega)/\operatorname{Fin}$, then Φ is trivial.*

Proof. Inclusion $\supset$ follows immediately from pure additivity.

For the other inclusion notice, that for $j \in F(\omega) \setminus F(\emptyset)$ the open set $\{x \subset \omega: j \in F(x)\}$ is an ultrafilter, so it is generated by a singleton $\{i\}$. Thus $j \in F(x)$ iff $i \in x$. □

It should be mentioned, that e.g. by G_δ set we in fact mean a set with certain Borel code, so the actual set be can different in various models of set theory. Also if a function is continuous on some Borel set, we will in fact deal with its Borel code (i.e. how it acts on open sets). This will be mainly used for self maps on $\mathcal{P}(\omega)$ continuous on a dense G_δ set Z. Then this function will be defined and continuous in all points of dense set Z (strictly speaking the G_δ set with the same code as Z) in the Cohen extension (e.g. in any real which is Cohen generic).

Next proposition is mostly trivial and not really essential, its purpose is to serve as a showcase of techniques which will be used (often implicitly) in the other places.

Proposition 5.3.3. *Let g be the name for $\mathbf{C}$ generic real, Z dense G_δ subset of $\mathcal{P}(\omega)$ and E, F, G self maps on $\mathcal{P}(\omega)$, all continuous on Z. If*

$$\Vdash_{\mathbf{C}} E(g) \in Z \text{ and } F(g) \neq^* G(g),$$

then there exists X dense G_δ subset of $\mathcal{P}(\omega)$ such that $E(x) \in Z$ and $F(x) \neq^* G(x)$ for each $x \in X$.

Proof. Fix a descending family $\{U_n : n \in \omega\}$ of open dense sets such that its intersection is Z. Put

$$A_n = \bigcup \{[p] : p \in \mathbf{C}, E[[p] \cap Z] \subset U_n\}.$$

A_n is open and $\Vdash_{\mathbf{C}} E(g) \in U_n$ together with continuity of E on Z gives that it is dense.

Let B_n be a dense subset of $\mathbf{C}$ given by $p \in B_n$ iff there exists $k_p > n$ such that

$$p \Vdash k_p \in F(g) \Delta G(g).$$

Now use continuity of F and G to define $T(p) \in \mathbf{C}$ for each $p \in B_n$ to be a condition extending p, such that for each

$$v_0, v_1 \in [T(p)] \cap Z$$

is

$$k_p \in F(v_0) \iff k_p \in F(v_1) \text{ and } k_p \in G(v_0) \iff k_p \in G(v_1).$$

Put $C_n = \bigcup\{[T(p)] : p \in B_n\}$ and it again follows that C_n is dense open subset of $\mathcal{P}(\omega)$.

Hence

$$X = Z \cap \bigcap_{n \in \omega} A_n \cap \bigcap_{n \in \omega} C_n$$

is the desired set. □

Notation. From now on $\Phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ will be a fixed lifting of ϕ and we will moreover assume that $\Phi(x) = \Phi(y)$ for $x =^* y$.

Proposition 5.3.4. *If F is a self-map on $\mathcal{P}(\omega)$ which is continuous on a dense G_δ set Z, then there are $x \subset a \subset \omega$, such that $\omega \setminus a \notin \mathrm{Triv}(\Phi)$ and*

$$\Vdash_{\mathbf{C}} v = x \cup (g \setminus a) \in Z \text{ and } F(v) \cap \Phi(a) \neq^* \Phi(x)$$

(where g is $\mathbf{C}$ generic).

Moreover if $\{I_n : n \in \omega\} \subset \mathcal{P}(\omega)$ are pairwise disjoint sets such that $I_n \notin \mathrm{Triv}(\Phi)$ for each n, then $x \subset a$ can be chosen such that $I_0 \subset a$, $I_n \setminus a \notin \mathrm{Triv}(\Phi)$ for each $n > 0$ and

$$\Vdash_{\mathbf{C}} F(v) \cap \Phi(I_0) \neq^* \Phi(x \cap I_0).$$

Proof. To simplify notation and avoid switching back and forth between characteristic functions and subsets of ω, we will represent the Cohen poset as $\mathbf{C} = [\omega]^{<\omega}$ with the ordering $s < t$ to mean that $s \cap (1 + \max(t)) = t$. Therefore we seek sets $x \subset a \subset \omega$ with $\omega \setminus a \notin \mathrm{Triv}(\Phi)$ and a $\mathbf{C}$-generic filter g such that $x \cup (g \setminus a) \in Z$ and $F(x \cup (g \setminus a)) \cap \Phi(a) \neq^* \Phi(x)$. Technically this statement will be forced by some condition $p = g\restriction\ell$, but we can simply redefine a and x so that $\ell \subset a$ and $x = g \cap \ell$, and we will then have that $\mathbf{C}$ forces the statement that was to be proven.

Fix any descending sequence $\{U_i : i \in \omega\}$ of dense open sets such that the intersection is contained in Z.

Case 1: There are conditions s_0, s_1 such that for <u>no</u> $n_0 > \max s_0 \cup s_1$ and $t \subset [n_0, n_1)$

$$\Vdash F(s_0 \cup t \cup (g \setminus n_1)) \setminus n_1 = F(s_1 \cup t \cup (g \setminus n_1)) \setminus n_1.$$

We will inductively construct an increasing sequence of natural numbers $\{k_i : i \in \omega\}$ and $\{t_i : t_i \subset [k_i, k_{i+1})\}$. Start with $k_0 = n_0$ and when k_i is defined, pick k_{i+1} and t_i such that for each $r \subset [n_0, k_i)$

1. $[s_0 \cup r \cup t_i] \cup [s_1 \cup r \cup t_i] \subset U_i$

2. $\exists j_i \in [k_i, k_{i+1})$ such that for each

$$v_0 \in [s_0 \cup r \cup t_i] \cap Z, v_1 \in [s_1 \cup r \cup t_i] \cap Z \qquad \text{is} \qquad j_i \in F(v_0)\Delta F(v_1).$$

For $A \in [\omega]^{<\omega}$ denote

$$a = \bigcup_{i \in A} [k_i, k_{i+1}) \cup \Phi^{-1}\left[\bigcup_{i \in A} [k_i, k_{i+1})\right] \cup n_0$$

and fix an infinite A such that $\omega \setminus a \notin \mathrm{Triv}(\Phi)$. (This is possible because $\mathrm{Triv}(\Phi)$ is a proper ideal.)

For $h \in 2$ put $x_h = s_h \cup \bigcup_{i \in A} t_i$ and $v_h = x_h \cup (g \setminus a)$. Condition 1. ensures $v_h \in Z$. We have that $\Phi(x_0) =^* \Phi(x_1)$ but

$$\Vdash j_i \in (F(v_0)\Delta F(v_1)) \cap [k_i, k_{i+1}) \neq \emptyset$$

for all $i \in A$ so for at least one $h \in 2$ and infinitely many $i \in A$ is

$$\Vdash (F(v_h)\Delta\Phi(x_h)) \cap [k_i, k_{i+1}) \neq \emptyset.$$

Put $x = x_h$ for this $h \in 2$, and we have $\Vdash F(v_h) \cap \Phi(a) \neq^* \Phi(x)$ (since $[k_i, k_{i+1}) \subset \Phi(a)$ for all but finitely many $i \in A$).

Let g_0, g_1 be $\mathbf{C} \times \mathbf{C}$ generic. Let $\boxplus$ be one of the usual set-theoretic operations $\{\cup, \cap\}$. Notice that $g_0 \boxplus g_1$ is also $\mathbf{C}$ generic.

Case 2: Assume there is a condition (s'_0, s'_1) such that <u>no</u> condition $(s_0, s_1) < (s'_0, s'_1)$ forces $F(g_0) \boxplus F(g_1)$ is almost equal to $F(g_0 \boxplus g_1)$.

Since we can assume case 1 is not true, we take $(s''_0, s''_1) < (s'_0, s'_1)$, $s''_0, s''_1 \subset n$ such that

$$\Vdash F\left(s''_0 \cup (g \setminus n)\right) \setminus n = F\left(s''_1 \cup (g \setminus n)\right) \setminus n = F\left(s''_0 \boxplus s''_1 \cup (g \setminus n)\right) \setminus n.$$

We may without loss of generality take $s'_0 = s'_1 = \emptyset$.

Again, construct inductively an increasing sequence $\{k_i : i \in \omega\}$ of integers and sequence $\{t^0_i, t^1_i \subset [k_i, k_{i+1})\}$. Start with any $k_0 \in \omega$ and when k_i is defined, pick k_{i+1} and t^0_i, t^1_i such that for each $s_0, s_1 \subset k_i$

1.
$$[s_0 \cup t^0_i] \cup [s_0 \cup t^1_i] \cup [s_0 \cup (t^0_i \boxplus t^1_i)] \subset U_i$$

2. $\exists j_i \in [k_i, k_{i+1})$ such that for each

$$v_0 \in [s_0 \cup t^0_i] \cap Z, v_1 \in [s_1 \cup t^1_i] \cap Z \qquad \text{is} \qquad j_i \in (F(v_0) \boxplus F(v_1))\Delta F(v_0 \boxplus v_1).$$

Again, denote

$$a = \bigcup_{i \in A} [k_i, k_{i+1}) \cup \Phi^{-1}\left[\bigcup_{i \in A} [k_i, k_{i+1})\right]$$

and fix an infinite $A \subset \omega$ such that $\omega \setminus a \notin \mathrm{Triv}(\Phi)$. For $h \in 2$ put $x_h = \bigcup_{i \in A} t^h_i$, $x_2 = x_0 \boxplus x_1$ and $v_h = x_h \cup (g \setminus a)$. Condition 1 ensures $v_h \in Z$ for $h \in 3$.

We have that $\Phi(x_0) \boxplus \Phi(x_1) =^* \Phi(x_0 \boxplus x_1)$ but

$$\Vdash j_i \in \big((F(v_0) \boxplus F(v_1))\Delta F(v_0 \boxplus v_1)\big) \cap [k_i, k_{i+1}) \neq \emptyset$$

for all $i \in A$. Hence for at least one $h \in 3$ and infinitely many $i \in A$ is

$$\Vdash (F(v_h)\Delta\Phi(x_h)) \cap [k_i, k_{i+1}) \neq \emptyset.$$

Put $x = x_h$ for this h and we have $\Vdash F(v_h) \cap \Phi(a) \neq^* \Phi(x)$ (since $[k_i, k_{i+1}) \subset \Phi(a)$ for all but finitely many $i \in A$).

Thus, as the last case we assume that we have condition $(s_0, s_1) \in \mathbf{C} \times \mathbf{C}$, $s_0, s_1 \subset n_0$ which forces that

$$\big(F(g_0) \boxplus F(g_1)\big)\Delta F(g_0 \boxplus g_1) \subset n_0$$

for $\boxplus$ being both $\cup$ and $\cap$. Because of case 1 we can also assume

$$\Vdash F\big(s_0 \cup (g \setminus n_0)\big) \setminus n_0 = F\big(s_1 \cup (g \setminus n_0)\big) \setminus n_0.$$

Hence

$$\Vdash \Big(F\big(s_0 \cup (g_0 \setminus n_0)\big) \boxplus F\big(s_0 \cup (g_1 \setminus n_0)\big)\Big) \setminus n_0 = F\big(s_0 \cup (g_0 \boxplus g_1 \setminus n_0)\big) \setminus n_0.$$

In the generic extension define $G\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ by

$$G(x) = \lim_{\ell \in \omega} \Big(F\big(s_0 \cup (x \cap (\ell \setminus n_0)) \cup (g \setminus \ell)\big) \setminus n_0\Big).$$

Continuity of F on Z implies that G is continuous and $G(x) = F(x) \setminus n_0$ for each $x \in [s_0] \cap Z$. Thus G is in fact in ground model and does not depend on the particular choice of g (it is the same mapping for any choice of Cohen generic g). We claim that G is purely additive:

$$G(x) \boxplus G(y) =$$

$$= \lim_{\ell \in \omega} \Bigg(F\Big(s_0 \cup \big(x \cap (\ell \setminus n_0)\big) \cup (g_0 \setminus \ell)\Big) \boxplus F\Big(s_0 \cup \big(y \cap (\ell \setminus n_0)\big) \cup (g_1 \setminus \ell)\Big)\Bigg) \setminus n_0 =$$

$$= \lim_{\ell \in \omega} \Bigg(F\Big(s_0 \cup \big((x \boxplus y) \cap (\ell \setminus n_0)\big) \cup \big((g_0 \boxplus g_1) \setminus \ell\big)\Big)\Bigg) \setminus n_0 = G(x \boxplus y).$$

By Proposition 5.3.2, G is completely additive and we may choose $y \in [s_0]$ such that $G(y) \neq^* \Phi(y)$.

Fix an increasing sequence of natural numbers $\{k_i\colon i \in \omega\}$ and $\{t_i\colon t_i \subset [k_i, k_{i+1})\}$ such that for any $v \in \mathcal{P}(\omega)$, if $v \cap [k_i, k_{i+1}) = t_i$ for infinitely many $i \in \omega$ then $v \in Z$.

We finish by considering two subcases; $\Phi(y) \setminus G(y)$ infinite and $G(y) \setminus \Phi(y)$ infinite.

If $\Phi(y) \setminus G(y)$ is infinite, then choose any infinite $x \in [s_0]$, $x \subset^* y$ so that $\omega \setminus x \notin \mathrm{Triv}(\Phi)$, $x \cap [k_i, k_{i+1}) = \emptyset$ for infinitely many $i \in \omega$, and $\Phi(x) \cap G(y) =^* \emptyset$.

Put $a_1 = \{i\colon G(\{i\}) \cap \Phi(x) \neq \emptyset\}$ and $a = a_1 \cup x \cup n_0$ if a_1 is finite, and $a = x \cup n_0$ otherwise.

Then

$$G\big((g \setminus a) \cup x\big) \cap \Phi(x) \subset \Big(\big(G(g) \setminus G(a)\big) \cap \Phi(x)\Big) \cup \big(G(x) \cap \Phi(x)\big) \subset^* G(g) \setminus G(a) \not\supset^* \Phi(x).$$

The last inclusion follows from $a_1 \subset a$ if a_1 is finite and from the fact, that g misses infinitely many elements of a_1, if a_1 is infinite. Hence $\Vdash v = x \cup (g \setminus a) \in Z$ (from genericity of g and a misses infinitely many intervals $[k_i, k_{i+1})$) and $\Vdash G(v) =^* F(v)$. Thus

$$\Vdash F(v) \cap \Phi(x) =^* G(v) \cap \Phi(x) \not\supset^* \Phi(x)$$

and $\Vdash F(v) \cap \Phi(a) \neq^* \Phi(x)$.

Now suppose that $G(y) \setminus \Phi(y)$ is infinite. Find an infinite $a_1 \subset \omega$ such that

$$\Phi(a_1) \subset^* G(y) \setminus \Phi(y)$$

and put

$$x = \{i \in y\colon G(\{i\}) \cap \Phi(a_1) \neq \emptyset\} \cup s_0$$

and $a = a_1 \cup x \cup n_0$. We may assume $\omega \setminus a \notin \mathrm{Triv}(\Phi)$ and $a \cap [k_i, k_{i+1}) = \emptyset$ for infinitely many $i \in \omega$ (shrink a_1 and x if necessary).

We have $\Phi(x) \cap \Phi(a_1) =^* \emptyset$ but

$$G\big(x \cup (g \setminus a)\big) \cap \Phi(a) \cap \Phi(a_1) \supset^* G(x) \cap \Phi(a_1) =^* \Phi(a_1).$$

Hence

$$\Vdash v = x \cup (g \setminus a) \in Z,\ F(v) =^* G(v) \text{ and } F(v) \cap \Phi(a) \neq^* \Phi(x).$$

To prove the moreover part consider mapping $F \cap \Phi(I_0)$ instead of F and proceed similarly. In cases 1 and 2 do the same construction up to the point where A and a are being defined. Instead choose inductively a $\subset$-decreasing sequence $\{A_j \in [\omega]^\omega : j \in \omega\}$ such that $I_{j+1} \setminus a_j \notin \mathrm{Triv}(\Phi)$ where $a_j = \bigcup_{i \in A_j}[k_i, k_{i+1})$. Then fix infinite $A \subset^* A_j$ for all $j \in \omega$, put $a = \bigcup_{i \in A}[k_i, k_{i+1}) \cup I_0 \cup n_0$ and the proof continues in the same way as in the previous part.

In case 3 choose $y \in [s_0]$, $y \subset^* I_0$, $G(y) \neq^* \Phi(y)$. In the first subcase we only miss the condition $I_0 \subset a$ (we have $a \subset^* I_0$). If a_1 is finite define $x' = x \cup \big((I_0 \setminus a) \cap \bigcup_{i \in \omega} t_i\big)$ and $a' = a \cup I_0$. If a_1 is infinite fix an infinite set $A \subset \omega$ such that $a_1 \setminus \bigcup_{i \in A}[k_i, k_i + 1)$ is still infinite. Define

$$x' = x \cup \big((I_0 \setminus a) \cap \bigcup_{i \in A} t_i\big)$$

and $a' = a \cup I_0$. We still have

$$\Vdash v' = x' \cup (g \setminus a') \in Z \text{ and } F(v') \cap \Phi(I_0) \neq \Phi(I_0 \cap x')$$

(since traces on $\Phi(x)$ still disagree).

For the second subcase choose $a_1 \subset I_0$ and when a and x is found, we again only miss $I_0 \subset a$. Define

$$x' = x \cup \big((I_0 \setminus a) \cap \bigcup_{i \in \omega} t_i\big)$$

and $a' = a \cup I_0$. We have

$$\Vdash v' = x' \cup (g \setminus a) \in Z, \Phi(x') \cap \Phi(a_1) =^* \emptyset \text{ and } F(v') \supset^* \Phi(a_1).$$

$\square$

Remark 5.3.5. *If proposition 5.3.4 holds true for some $x \subset a$ and $c \subset d$ are finite sets disjoint with a, then it still holds for $x \cup c$ and $a \cup d$.*

Next proposition provides an alternative proof of the crucial theorem from [Vel93]. Condition 3 continues the moreover part of proposition 5.3.4.

Proposition 5.3.6. *If $\{F_n : n \in \omega\}$ are Borel self-maps on $\mathcal{P}(\omega)$ and $Z \subset \mathcal{P}(\omega)$ is a dense G_δ, then there is an $x \subset \omega$ such that*

1. *$x^* \in Z$ for each x^* almost equal to x,*
2. *$F_n(x) \neq^* \Phi(x)$ for all n,*
3. *and if, in addition, $\{I_n : n \in \omega\} \subset \mathcal{P}(\omega)$ are pairwise disjoint sets such that $I_n \notin \mathrm{Triv}(\Phi)$ for each n, then x can be chosen so that $F_n(x) \cap \Phi(I_n) \neq^* \Phi(x \cap I_n)$ for each n.*

Proof. Since each F_n is Borel, we may assume that each F_n is continuous on Z and also that $x^* \in Z$ for each x^* which is almost equal to some $x \in Z$ (shrink Z if necessary). Fix a countable $\subset$ descending family $\{U_n : n \in \omega\}$ of dense open subsets of $\mathcal{P}(\omega)$ such that the intersection is Z.

We will use Proposition 5.3.4 repeatedly to construct increasing chains $a_0 \subset a_1 \subset \ldots$ and $x_0 \subset x_1 \subset \ldots$ so that $x_i \cap a_j = x_j$ for $i < j$. The intention is to arrange that $v = \bigcup_i x_i \in Z$ (which allows us to have some connection between the behavior of $F_i(\bigcup_{j<i} x_j)$ and $F_i(v)$), and that $F_i(v) \cap \Phi(a_i) \neq^* \Phi(x_i)$. If we succeed then it will follow that $\Phi(v)$ is not in the set $\{F_i(v) : i \in \omega\}$. In the case of statement 3. of the proposition, we just remark that we may choose a_n so that $I_n \subset a_n$ using the moreover part of Proposition 5.3.4.

We will again use the Cohen forcing $\mathbf{C}$ as in proposition 5.3.4. Select $x_0 = \tilde{x_0} \subset a_0 = \tilde{a_0}$ with $\omega \setminus a_0 \notin \mathrm{Triv}(\Phi)$ such that

$$\Vdash v = x_0 \cup (g \setminus a_0) \in Z \text{ and } F_0(v) \cap \Phi(\tilde{a_0}) \neq^* \Phi(\tilde{x_0}).$$

Fix any $p_0 \subset \ell_0 \in \omega$ so that $p_0 \cap a_0 = \ell_0 \cap x_0$, $[p_0] \subset U_0$ and note that there exists $k_0 < \ell_0$ such that

$$k_0 \in \big((F_0(y) \cap \Phi(\tilde{a_0}))\Delta\Phi(\tilde{x_0})\big)$$

for each $y \in [p_0] \cap Z$. We may assume that $\ell_0 \subset a_0$ and $p_0 = x_0 \cap \ell_0$.

Since $v \cap (\omega \setminus a_0)$ is Cohen generic subset of $\omega \setminus a_0$, using Proposition 5.3.3 we have a dense G_δ set $Z_0 \subset \mathcal{P}(\omega \setminus a_0)$ such that for each $y \in Z_0$

$$\tilde{x}_0 \cup y \in Z \text{ and } F_0(x_0 \cup y) \cap \Phi(\tilde{a}_0) \neq^* \Phi(\tilde{x}_0).$$

Now proceed with inductive construction. Our inductive assumptions are

$$a_i = \bigcup_{j \leq i} \{\tilde{a}_j\} \supset \ell_i \in \omega,$$

Z_i is dense G_δ subset of $\mathcal{P}(\omega \setminus a_i)$, and for each $y \in Z_i$ we have that

$$\forall (j \leq i) F_j(y \cup x_i) \cap \Phi(\tilde{a_j}) \neq^* \Phi(\tilde{x_j}) \text{ and } y \cup x_i \in Z.$$

In the inductive step use Proposition 5.3.4 for Φ restricted to $\omega \setminus a_i$ and mapping

$$F'_{i+1}(y) = F_{i+1}(y \cup x_i) \cap \Phi(\omega \setminus a_i)$$

which is continuous on Z_i. We get $\tilde{x}_{i+1} \subset \tilde{a}_{i+1} \subset \omega \setminus a_i$, $x_{i+1} = x_i \cup \tilde{x}_{i+1}$ and $a_{i+1} = a_i \cup \tilde{a}_{i+1}$ such that

$$\Vdash \tilde{x}_{i+1} \cup (g \setminus \tilde{a}_{i+1}) \in Z_i \text{ and } F_{i+1}(x_{i+1} \cup (g \setminus a_{i+1})) \cap \Phi(\tilde{a}_{i+1}) \neq^* \Phi(\tilde{x}_{i+1})$$

where g is Cohen generic subset of $\omega \setminus a_i$.

Fix any $p_{i+1} \subset \ell_{i+1} \in \omega$ so that $p_{i+1} \cap a_{i+1} = \ell_{i+1} \cap x_{i+1}$, $[p_{i+1}] \subset U_{i+1}$ and for each $j \leq i+1$ there exists $k_{i+1} \in [\ell_i, \ell_{i+1})$ such that

$$k_{i+1} \in \big((F_j(y) \cap \Phi(\tilde{a}_j)) \Delta \Phi(\tilde{x}_j)\big)$$

for each $y \in [p_{i+1}] \cap Z$ (for $j \leq i$ use inductive hypothesis). We may assume that $\ell_{i+1} \subset a_{i+1}$ and $p_{i+1} = x_{i+1} \cap \ell_{i+1}$ (enlarging $\tilde{x}_{i+1}$ and $\tilde{a}_{i+1}$ if necessary).

Use Proposition 5.3.3 to get Z_{i+1} dense G_δ subset of $\mathcal{P}(\omega \setminus a_{i+1})$ such that for each $y \in Z_{i+1}$

$$\tilde{x}_i \cup y \in Z_i \text{ and } F_{i+1}(x_{i+1} \cup y) \cap \Phi(\tilde{a}_{i+1}) \neq^* \Phi(\tilde{x}_{i+1}).$$

Finally after the inductive construction is done put $x = \bigcup_{i \in \omega} x_i$. □

The main use of the next lemma is for cases where the ideal $\mathcal{I}$ is non-meager so there is no harm in reading the lemma as 'for every non-meager p-ideal and non-trivial Φ without any reference to Triv Φ or $\mathcal{F}$.

Lemma 5.3.7. *Let F be function continuous on a dense G_δ set Z, $\mathcal{J}$ a p-ideal such that $\mathcal{J}^\perp$ is countably generated and* $\mathrm{Triv}(\Phi) \cap \mathscr{F} = \emptyset$ *where $\mathscr{F}$ is the dual filter to $\mathcal{J}^\perp$. There are $x \subset a \in \mathcal{J}$ such that for all $m \in \omega$ and $s \subset m$,*

$$\Vdash_{\mathbf{C}} v = s\Delta(x \cup (g \setminus a)) \in Z \text{ and } F(v) \cap \Phi(a) \neq^* \Phi(x)$$

(i.e. mod finite changes to the generic keep the equation true).

Proof. Let $\{U_n : n \in \omega\}$ be a decreasing sequence of dense open subsets of $\mathcal{P}(\omega)$ with intersection Z.

At first assume that $\mathcal{J}$ is tall. Use Proposition 5.3.6 for the countable set of functions

$$F_H(x) = F(x\Delta H) \colon H \in [\omega]^{<\omega}$$

to get $y \subset \omega$ such that $y^* \in Z$ and $F(y^*) \neq^* \Phi(y)$ for each $y^* =^* y$. Using that $\mathcal{J}$ is tall find for each $H \in [\omega]^{<\omega}$ an infinite set $a_H \in \mathcal{J}$ such that $\Phi(a_H) \subset^* F_H(y) \Delta \Phi(y)$. Since $\mathcal{J}$ is an p-ideal, there is $a \in \mathcal{J}$, $a_H \subset^* a$ for each H. We have $\Phi(a) \cap \big(F(y^*) \Delta \Phi(y)\big)$ is infinite for each $y^* =^* y$ and put $x = a \cap y$.

To prove that $\Vdash v \in Z$ for each $s \subset m \in \omega$ take any condition $t \subset m \in \omega$ and $n \in \omega$. Since

$$y^* = s\Delta\Big(\big(y \setminus (m \setminus a)\big) \cup (t \setminus a)\Big) \in U_n,$$

there is some $k \in \omega$ such that $[y^* \cap k] \subset U_n$. Hence $t \cup \big(y \cap [m, k)\big) \Vdash v \in U_n$.

To prove the other part take again $s \subset m \in \omega$ and a condition $t \subset m \in \omega$. We may suppose that

$$(\Phi(a) \cap \Phi(y)) \Delta \Phi(x) \subset m.$$

There exist $j \in \Phi(a) \setminus m$, $j \in \big(F(y^*)\Delta\Phi(y)\big)$ (and hence $j \in \big(F(y^*)\Delta\Phi(x)\big)$) for

$$y^* = s\Delta\big((y \setminus (m \setminus a)) \cup (t \setminus a)\big).$$

Again there is $k \in \omega$ such that for each $u \in [y^* \cap k] \cap Z$ is $j \in (F(u)\Delta\Phi(x))$. The condition $t \cup \big(y \cap [m,k)\big)$ forces that $\big(F(v) \cap \Phi(a)\big)\Delta\Phi(x) \not\subset m$ and we are done.

If $\mathcal{J}^\perp$ is generated by a single set $I \subset \omega$, then we have that $\omega \setminus I \notin \mathrm{Triv}(\Phi)$ and we may work on $\Phi(\omega \setminus I)$ with functions $F(x) \cap \Phi(\omega \setminus I)$ and $\Phi(x) \cap \Phi(\omega \setminus I)$ to get $x \subset a \subset \omega \setminus I$ in the same way as in the previous case.

Let $\{I_n : n \in \omega\} \subset \mathcal{J}^\perp$ enumerate a pairwise disjoint family whose finite unions generate $\mathcal{J}^\perp$. Assume at first that $I_n \in \mathrm{Triv}\,\Phi$ for each $n \in \omega$ and let h_n be a function from I_n to $\Phi(I_n)$ which induces $\Phi \restriction \mathcal{P}(I_n)$. Use Proposition 5.3.6 for set of functions $\{F_{H,n}(x), H \in [\omega]^{<\omega}, n \in \omega\}$, where

$$F_{H,n}(x) = \big(F(x\Delta H) \setminus \bigcup_{k<n} \Phi(I_k)\big) \cup \bigcup_{k<n} h_k[x \cap I_k],$$

to get $y \subset \omega$ such that $y\Delta H \in Z$ and $F_{H,n}(y) \neq^* \Phi(y)$ for each $H \in [\omega]^{<\omega}$ and $n \in \omega$. We have that

$$\big(F(y^*)\Delta\Phi(y)\big) \setminus \bigcup_{k<n} \Phi(I_k)$$

is infinite for all $y^* =^* y$ and $n \in \omega$. Hence for each $H \in [\omega]^{<\omega}$ there is an infinite set $a_H \in \mathcal{J}$ such that $\Phi(a_H) \subset^* F_H(y)\Delta\Phi(y)$ and we may proceed in the same way as in the first case.

If $I_n \notin \mathrm{Triv}\,\Phi$ for only finitely many $n \in \omega$, then we may suppose that only $I_0 \notin \mathrm{Triv}\,\Phi$ and $I_n \in \mathrm{Triv}\,\Phi$ for $n > 0$. We have that $\omega \setminus I_0 \notin \mathrm{Triv}\,\Phi$ and we may again work on $\Phi(\omega \setminus I_0)$ and proceed as in the previous case.

The last case is $I_n \notin \mathrm{Triv}\,\Phi$ for only infinitely many $n \in \omega$ and we may suppose that $I_n \notin \mathrm{Triv}\,\Phi$ for each $n \in \omega$. Use part 3 of Proposition 5.3.6 for system of functions

$$F_{H,n}(x) = F(x\Delta H). \text{ where } H \in [\omega]^{<\omega}, n \in \omega$$

(each function is listed cofinally often) and $\{I_n : n \in \omega\}$ to get $y \subset \omega$ such that for each $y^* =^* y$ is $y^* \in Z$ and there exist infinitely many $n \in \omega$ such that $F(y^*) \cap \Phi(I_n) \neq^* \Phi(y \cap I_n)$. Hence for each H the set $F_H(y)\Delta\Phi(y)$ is not $\subset^*$ contained in $\bigcup_{k<n} \Phi(I_k)$ for each n and there exists an infinite set $a_H \in \mathcal{J}$ such that $\Phi(a_H) \subset^* F_H(y)\Delta\Phi(y)$. The proof now continues again in the same way as in previous cases. □

Corollary 5.3.8. *If $\dot{Y}$ is $\mathbf{C}$ name of a subset of ω and $\mathcal{J}$ is a non-meager p-ideal, then there are $x \subset a \in \mathcal{J}$ and infinite $B_{\dot{Y},x,a} \subset a$ where if $m \in B_{\dot{Y},x,a}$ then for all $s \subset m$ there is $j \in \Phi(a) \setminus m$ and an interval $[m, \bar{m}) \subset a$, such that for $t = s \cup \big(x \cap [m, \bar{m})\big)$ we have $t \Vdash j \in \dot{Y}\Delta\Phi(x)$ and t decides if $j \in \dot{Y}$.*

Proof. There is a self-map F on $\mathcal{P}(\omega)$ which is continuous on a dense G_δ set Z and such that $\Vdash_{\mathbf{C}} F(g) = \dot{Y}[g]$. (Obtained by letting $[t] \subset U_n$ if $t \in [\omega]^{<\omega}$ forces a value on $\dot{Y} \cap n$. Then for $v \in \bigcap_n U_n$, $F(v) = \bigcup_n y_n$ where y_n is the unique subset of n such that, for some $\ell \in \omega$, $[v \cap \ell] \subset U_n$ and $v \cap \ell$ forces the value y_n on $\dot{Y} \cap n$.)

Now we use lemma 5.3.7 for F and $\mathcal{J}$ to get $x \subset a \in \mathcal{J}$.

We will inductively construct an increasing sequence of natural numbers $\{k_i : i \in \omega\}$ and $\{t_i : t_i \subset [k_i, k_{i+1})\}$. Start with any $k_0 \in \omega$ and when k_i is defined, pick k_{i+1} and t_i such that for each $s \subset k_i$ there exists $j_s \in \Phi(a) \setminus k_i$ such that

1. $t_i \cap a = x \cap [k_i, k_{i+1})$
2. for all $v \in Z \cap [s \cup t_i]$, $j_s \in F(v)\Delta\Phi(x)$
3. $[s \cup t_i] \subset U_{j_s+1}$.

To see that such j_s and t_i exist just note, that x and a were chosen to satisfy conclusion of lemma 5.3.7.

Since $\mathcal{J}$ is non-meager, it is possible to choose an infinite $A \subset \omega$ such that $\bigcup_{i \in A}[k_i, k_{i+1}) \in \mathcal{J}$. Let $\bar{x} = x \cup \bigcup_{i \in A} t_i$ and $\bar{a} = a \cup \bigcup_{i \in A}[k_i, k_{i+1})$.

Then choose any $m_0 \in \omega$ such that

$$\big(\Phi(\bar{x}) \cap \Phi(a)\big) \setminus \Phi(x) \subset m_0, \Phi(a) \setminus \Phi(\bar{a}) \subset m_0$$

and put $B_{\dot{Y},\bar{x},\bar{a}} = \{k_i \colon i \in A \setminus m_0\}$.

To prove that these sets satisfy the required conclusion, pick any $m = k_i \in B_{\dot{Y},\bar{x},\bar{a}}$, $s \subset m$ and put $\bar{m} = k_{i+1}$, $t = s \cup t_i = s \cup (\bar{x} \cap [m, \bar{m}))$. We have chosen k_{i+1}, t_i and $j_s (> m_0)$ in such way that $j_s \in \Phi(a) \setminus k_i \subset \Phi(\bar{a}) \setminus m$, t decides if $j_s \in \dot{Y}$ and $t \Vdash j_s \in \dot{Y} \Delta \Phi(x)$ and thus $t \Vdash j_s \in \dot{Y} \Delta \Phi(\bar{x})$. $\square$

Next theorem is the main result of this chapter. It shows that Grigorieff forcing destroys non-trivial automorphisms. It introduces pregap filled with the generic real, image of which is an (unfilled) gap.

Theorem 5.3.9. *If $\mathscr{F}$ is a non-meager p-filter such that $\mathrm{Triv}(\Phi) \cap \mathscr{F}$ is empty, then if G is $G(\mathscr{F})$ generic, then the family*

$$\{\Phi(p^{-1}(1)), \omega \setminus \Phi(p^{-1}(0)) \colon p \in G\}$$

contains an unfilled (ω_1, ω_1^*)-gap (in $V[G]$).

Proof. Let $\dot{Y}_0$ be a $G(\mathscr{F})$ name of a subset of ω and $p_0 \in G(\mathscr{F})$. We will find $q < p_0$ such that

$$q \Vdash \dot{Y}_0 \cap \Phi\big(\mathrm{Dom}(q)\big) \neq^* \Phi\big(g^{-1}(1)\big).$$

Fix countable elementary submodel $M \prec H(\theta)$ such that $p_0, \dot{Y}_0 \in M$. Let $g < p_0$ be $(M, G(\mathscr{F}))$ generic condition (hence $\mathrm{Dom}(p) \subset^* \mathrm{Dom}(g)$ for all $p \in M \cap G(\mathscr{F})$). Let $A = \omega \setminus \mathrm{Dom}(g)$ and define a $\mathbf{C}_A$ name

$$\dot{Y} = \{(n, s) \colon (\exists p \in G(\mathscr{F}) \cap M, p \parallel g)(p \Vdash n \in \dot{Y}_0) \text{ and } p \setminus g \subset s \in \mathbf{C}_A\}.$$

Here $\mathbf{C}_A$ denotes the poset for adding Cohen generic subset of A.

Since $A \notin \mathrm{Triv}(\Phi)$, we can choose $x \subset a \subset A$ with $\omega \setminus a \in \mathscr{F}$ as in corollary 5.3.8. Define $q \supset g$ so that $\mathrm{Dom}(q) = \mathrm{Dom}(g) \cup a$ and $q^{-1}(1) \cap a = x$. Since q is M-generic, q forces that each value of $\dot{Y}_0 \cap n$ gets actually decided by some $p \in M \cap G(\mathscr{F})$ (as a value of $\dot{Y} \cap n$). Hence $q \Vdash \dot{Y}_0 \cap \Phi(a) \neq^* \Phi(x)$ and also $q \Vdash \dot{Y}_0 \cap \Phi\big(\mathrm{Dom}(q)\big) \neq^* \Phi\big(g^{-1}(1)\big)$. $\square$

We used lemma 5.3.7 only for tall ideals $\mathcal{I}$. The reason for general formulation is hope, that further investigation can lead to some more general version of theorem 5.3.9, a similar result for some other forcing notion, which would provide a way of destroying automorphisms which can not be grasped by $G(\mathscr{F})$.

However theorem 5.3.9 is strong enough to give a strengthening of previously mentioned result from [SS02]. Combining theorems 5.3.9 and 5.2.5 provides the following tool.

Proposition 5.3.10. *(GCH) Let $\mathscr{F}$ be a non-meager p-filter. For each non-trivial automorphism φ of $\mathcal{P}(\omega)/\mathrm{Fin}$ such that $\mathrm{Triv}(\varphi) \cap \mathscr{F} = \emptyset$, there exist a proper ${}^\omega\omega$ bounding forcing $\mathbf{D}(\mathscr{F}, \varphi)$ with ω_2-p.i.c. of size ω_2, such that for each generic filter G on $\mathbf{D}(\mathscr{F}, \varphi)$, no automorphism of $\mathcal{P}(\omega)/\mathrm{Fin}$ in $V[G]$ extends φ. Moreover, this remains true in all models of ZFC extending $V[G]$ which preserve $\omega_1^{V[G]}$.*

Proof. The forcing $\mathbf{D}(\mathscr{F}, \varphi)$ is two step iteration Grigorieff forcing $G(\mathscr{F})$ followed by Abraham-Todorcevic gap freezing forcing which freezes the gap demonstrating non-extendability of φ introduced by $G(\mathscr{F})$. $\square$

Theorem 5.3.11. *It is consistent with ZFC that $\mathfrak{d} = \omega_1$ and $\mathrm{Triv}\,\Phi \cap \mathcal{F} \neq \emptyset$ for each automorphism Φ of $\mathcal{P}(\omega)/\mathrm{Fin}$ and each non-meager p-filter $\mathcal{F}$.*

Proof. This is again achieved by starting in a model of GCH and iterating forcings $\mathbf{D}(\mathscr{F}_\alpha, \varphi_\alpha)$ of length ω_2 while following some bookkeeping device to ensure, that each automorphism and each non-meager p-filter is encountered at some stage. In the end argue that each automorphism non-trivial on each element of some non-meager p-filter was already destroyed at some stage of the iteration.

The key elements which keep this construction going are that all forcings are proper and ${}^\omega\omega$ bounding (hence $\mathfrak{d} = \omega_1$ in the extension) and that at each immediate stage we have CH because of ω_2-p.i.c and $2^\omega = \omega_2$ because all forcings are of size ω_2. $\square$

The initial motivation for proposition 5.3.10 was the aim to employ this forcing in iteration used to introduce strong-Q-sequences from chapter 4 and thus possibly building models mimicking the model for $\mathcal{P}(\omega)/\mathrm{Fin} \cong \mathcal{P}(\omega_1)/\mathrm{Fin}$. This is of course possible, this gives results along these lines:

Theorem 5.3.12. *It is consistent with ZFC that $\mathfrak{d} = \omega_1$, there exists a countable like ideal on ω and for each automorphism Φ of $\mathcal{P}(\omega)/\mathrm{Fin}$ and each non-meager p-filter $\mathcal{F}$ is $\mathrm{Triv}\,\Phi \cap \mathcal{F} \neq \emptyset$.*

However, as will be demonstrated in the last section of this chapter, the influence of positive solution of Katowice problem $\mathcal{P}(\omega)/\mathrm{Fin} \cong \mathcal{P}(\omega_1)/\mathrm{Fin}$, does not seem to imply significant restrictions for automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$.

5.4 Non-trivial automorphism under $\mathcal{P}(\omega_1)/\operatorname{Fin} \cong \mathcal{P}(\omega)/\operatorname{Fin}$

The only ambition of this section is to present a recent (and as of now unpublished) result of K. P. Hart. It clarifies the relation between Katowice problem and existence of non-trivial automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$.

Theorem 5.4.1 (K.P. Hart). *If $\mathcal{P}(\omega_1)/\operatorname{Fin} \cong \mathcal{P}(\omega)/\operatorname{Fin}$ then there exist a non-trivial automorphism of $\mathcal{P}(\omega)/\operatorname{Fin}$.*

Proof. Assume that $\mathcal{P}(\omega_1)/\operatorname{Fin} \cong \mathcal{P}(\omega)/\operatorname{Fin}$. If this is true, then there is also an isomorphism

$$f\colon \mathcal{P}(\omega_1 \times \mathbb{Z})/\operatorname{Fin} \to \mathcal{P}(\omega)/\operatorname{Fin}.$$

Denote σ to be the permutation of $\omega_1 \times \mathbb{Z}$ mapping $(\alpha, n) \mapsto (\alpha, n+1)$ for each $n \in \mathbb{Z}$ and $\alpha \in \omega_1$. The automorphism of $\mathcal{P}(\omega_1)/\operatorname{Fin}$ which ϱ naturally generates will be denoted ϱ^*.

Using the isomorphism f we can transfer this automorphism to $\mathcal{P}(\omega)/\operatorname{Fin}$

$$\sigma^*[A] = f \circ \varrho^* \circ f^{-1}[A].$$

Denote $h_n = \omega_1 \times \{n\}$, $v_\alpha = \{\alpha\} \times \mathbb{Z}$ for $n \in \mathbb{Z}$ and $\alpha \in \omega_1$.

Lemma 5.4.2. *For all $\alpha \in \omega_1$ and $n \in \mathbb{Z}$ let V_α and H_n be some subsets of ω such that $[V_\alpha] = f[v_\alpha]$ and $[H_n] = f[h_n]$. Then*

1. *$\{[V_\alpha]\colon \alpha \in \omega_1\}$ is a system of σ^* fixed sets.*
2. *$H_{n+1} =^* \sigma^*[H_n]$ for each $n \in \mathbb{Z}$ and $\{H_n\colon n \in \mathbb{Z}\}$ is an AD system.*
3. *For each $E \subset \omega$ such that $H_n \subset^* E$ for each $n \in \mathbb{Z}$ there is $\alpha \in \omega_1$ such that $V_\beta \subset^* E$ for each $\beta > \alpha$.*
4. *For each $F \subset \omega$ such that for uncountably many $\alpha \in \omega_1$ is $V_\alpha \subset^* F$ there is $S \in [H_0]^\omega$ and $n \in \omega$ such that $\sigma^{*(n)}[S] \subset^* F$.*

Proof. All these statements can be expressed as statements about $V_\alpha, H_n, \sigma^*, \ldots$ as elements of the Boolean algebra $\mathcal{P}(\omega)/\operatorname{Fin}$ so it is enough to prove them for objects in the Boolean algebra $\mathcal{P}(\omega_1 \times \mathbb{Z})/\operatorname{Fin}$ corresponding to these elements via the isomorphism f.

Items 1. and 2. are obvious.

To prove 3. suppose $h_n \subset^* E$ for each $n \in \mathbb{Z}$. The desired α is

$$\sup\{\max\{\beta \in \omega_1\colon (\beta, n) \notin h_n\}\colon n \in \mathbb{Z}\} + 1.$$

To prove 4. suppose $v_\alpha \subset^* F$ for uncountably many $\alpha \in \omega_1$. Use the pigeon hole principle to see that there is $n \in \omega$ and an uncountable $s \subset \omega_1$ such that $(\alpha, n) \in E$ for each $\alpha \in s$. Now $\varrho^{(n)}[s \times \{0\}] \subset F$ and $s \times \{0\}$ is an infinite subset of h_0. ☐

Towards contradiction suppose that σ^* is generated by an almost permutation $\sigma\colon \omega \to \omega$. (If it is not possible to have $\operatorname{Dom}(\sigma) = \omega$ replace ϱ and all derived mappings with their inverses.)

The orbits of σ are classes of the usual equivalence relation on ω; x, y belong to the same orbit iff there is $n \in \omega$ such that $\sigma^{(n)}(x) = y$ or $\sigma^{(n)}(x) = y$. There can possibly exist three types of orbits, *1st kind* are infinite orbits with no initial point, *2nd kind* are infinite orbits with initial point and the *3th kind* are finite (cyclic) orbits. Sets of these orbit will be denoted $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_3$ in the respective order. We have $|\mathcal{O}_1|, |\mathcal{O}_3| \leq \omega$ and $|\mathcal{O}_2| < \omega$ (initial points are precisely elements of $\omega \setminus \operatorname{Rng}(\sigma)$). Denote $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$, $G = \bigcup(\mathcal{O}_1 \cup \mathcal{O}_2)$ and $F = \bigcup \mathcal{O}_3$.

Claim. *For $\alpha \in \omega_1$ let V_α be subsets of ω such that $[V_\alpha] = f[v_\alpha]$. Then $|\{O \in \mathcal{O}\colon \emptyset \neq O \cap V_\alpha \neq O\}| < \omega$. Moreover for $O \in \mathcal{O}_1$ and $i = \pm 1$, or $O \in \mathcal{O}_2$ and $i = 1$, if*

$$|\{n \in \omega\colon \sigma^{(i \cdot n)}(x) \in O\}| = \omega$$

for some $x \in O$ then there is $n_0 \in \omega$ such that $\{\sigma^{(i \cdot n)}(x)\colon n > n_0\} \subset O$.

Proof. Call $x \in \omega$ *entry point* of V_α if $x \notin V_\alpha$ and $\sigma(x) \in V_\alpha$ and *exit point* if $x \in V_\alpha$ and $\sigma(x) \notin V_\alpha$. The claim is consequence of fact, that for each $\alpha \in \omega_1$ the set E of entry points and exit points of V_α is finite. To see this note that $E \subset V_\alpha \triangle \sigma[V_\alpha]$ and remember that $[V_\alpha]$ is σ^* fixed. ☐

From now on, fix sets V_α and H_n for $\alpha \in \omega_1$ and $n \in \mathbb{Z}$ as in previous lemmas. Because of the claim we may suppose that for each $\alpha \in \omega_1$ if $V_\alpha \cap O \neq \emptyset$ then $|V_\alpha \cap O| = |O|$ for all $O \in \mathcal{O}$. We also know that there are at most countably many $\alpha \in \omega_1$ such that $V_\alpha \cap O \neq \emptyset$ for some $O \in \mathcal{O}_1 \cup \mathcal{O}_2$ (Each orbit of 1st kind can intersect at most two V_α and orbits of 2nd kind intersect at most one V_α). Hence there is $\alpha \in \omega_1$ such that $V_\beta \subset F$ for each $\beta > \alpha$.

Claim. *The set $S = F \cap H_0$ is infinite.*

Proof. Use 4. of lemma 5.4.2 to get infinite $S' \subset H_0$ and $n \in \omega$ such that $\sigma^{(n)}[S'] \subset F$. Note that F is σ invariant hence $S' \subset H_0 \cap F$ and S is infinite. □

For each $x \in S$ define $l(x) = \min\{n \in \omega \colon \sigma^{(n+1)}(x) \in S\}$.

Claim. *For each $n \in \omega$ is $\{x \in S \colon l(x) = n\}$ finite.*

Proof. We have $\{\sigma^{(n+1)}(x) \colon x \in S, l(x) = n\} \subset S \cap \sigma^{(n+1)}[S] \subset^* H_0 \cap H_{n+1} =^* \emptyset$. □

Put $E = \{\sigma^{(n)}(x) \colon x \in S, n \in \omega, n < l(x)\}$ (so for each $x \in S$ is $\sigma^{(l(x))} \notin E$) and the claim gives us $\sigma^{(n)}[S] \subset^* E$ for each $n \in \mathbb{Z}$.

For $n \in \mathbb{Z}$ is $H_n \cap F =^* \sigma^{(n)}[H_0] \cap F = \sigma^{(n)}[H_0 \cap F] = \sigma^{(n)}[S] \subset^* E$, hence $H_n \subset^* G \cup E$. Now use 3. of lemma 5.4.2 to get $\alpha \in \omega_1$ such that $V_\beta \subset^* G \cup E$ for each $\alpha < \beta$.

This implies that there is some $\beta \in \omega_1$ such that $V_\beta \subset F$ and $V_\beta \subset^* E$. So V_β has to contain infinitely many finite orbits but no subset of E can contain a whole (finite) orbit, a contradiction. □

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