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There are no P-points in Silver extensions

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THERE ARE NO P-POINTS IN SILVER EXTENSIONS

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ABSTRACT. We prove that after adding a Silver real no ultrafilter from the ground model can be extended to a P-point, and this remains to be the case in any further extension which has the Sacks property. We conclude that there are no P-points in the Silver model. In particular, it is possible to construct a model without P-points by iterating Borel partial orders. This answers a question of Michael Hrušák. We also show that the same argument can be used for the side-by-side product of Silver forcing. This provides a model without P-points with the continuum arbitrary large, answering a question of Wolfgang Wohofsky.

The first author dedicates this work to his teacher, mentor and dear friend Bohuslav Balcar. The crucial result was proved on the day of his passing.

INTRODUCTION

Ultrafilters on countable sets have become of great importance in infinite combinatorics. A non-principal ultrafilter $\mathcal{U}$ is called a *P-point* if every countable subset of $\mathcal{U}$ has a pseudointersection in $\mathcal{U}$. Ultrafilters of this special kind have been extensively studied in set theory and topology. It is possible to construct P-points under the assumption $\mathfrak{d} = \mathfrak{c}$ (see [Bla10]) or if the parametrized diamond principle $\Diamond(\mathfrak{r})$ holds (see [MHD04]). On the other hand, it is a remarkable theorem of Shelah that the existence of P-points can not be proved using only the axioms of ZFC (see [She98]). The model of Shelah is obtained by iterating the Grigorieff forcing with parameters ranging over non-meager P-filters.

By a *canonical model* we understand a model obtained by performing a countable support iteration of Borel proper partial orders of length ω_2. At the *Forcing and its applications retrospective workshop* held at the Fields Institute in 2015, Michael Hrušák posed the following problem.

Problem. Are there P-points in every canonical model?

There will be a P-point if the steps of the iteration add unbounded reals, or if no splitting reals are added—in the resulting model either $\mathfrak{d} = \mathfrak{c}$ or $\Diamond(\mathfrak{r})$ holds. Consequently, one only needs to consider Borel ω^ω-bounding forcing notions which do add splitting reals. The best known examples of this type of forcing are the random and the Silver forcings. We answer the question of Hrušák in negative by Theorem 3; there are no P-points in the Silver model.

In [Coh79] it was claimed that there is a P-point in the random model. Unfortunately, the proof presented in the referenced paper is incorrect and the existence of P-points in the random model is presumably unknown.

Problem. Are there P-points in the random model?

The existence of a model without P-points with the continuum larger that ω_2 was an open question [Woh08]. Theorem 4 states that forcing with the side-by-side product of Silver forcing produces such a model.

Destroying P-points with Silver reals

For a partial function $p;\omega \to 2$ we denote by $\operatorname{cod} p$ the complement of its domain. The *Silver forcing* (also called the *Prikry–Silver forcing*) denoted by **PS** consists of all partial functions $p;\omega \to 2$ such that $\operatorname{cod} p$ is infinite. The relation $p \le q$ is defined as $q \subseteq p$. We will always assume that $p^{-1}(1)$ is infinite for each $p \in \mathbf{PS}$, such conditions form a dense subset of the poset. It is well known that the Silver forcing is proper and has the Sacks property. In particular, this means that for every $p \in \mathbf{PS}$ and each name $\dot{f}$ for a function in ω^ω there exist $q < p$ and $\{X_n : X_n \in [\omega]^{n+1}, n \in \omega\}$ such that $q \Vdash \dot{f}(n) \in X_n$ for each $n \in \omega$.

For a partial (or total) function $p;\omega \to 2$ with $p^{-1}(1)$ infinite denote $e_p\colon \omega \to \omega$ the unique increasing enumeration of $p^{-1}(1)$. Define a decomposition of ω into intervals $\{ I_n(p) \mid n \in \omega \}$, where $I_0(p) = e_p(0)$ and $I_n(p) = e_p(n) \setminus e_p(n-1)$ for $n > 0$.

By $-_n$ and $=_n$ we denote the subtraction operation and congruence relation modulo n. The notation $j \in_n X$ is interpreted as 'there is $x \in X$ such that $j =_n x$.' For $X, Y \subset n$ we write $X -_n Y = \{ x -_n y \mid x \in X, y \in Y \}$.

Lemma 1. *For each $n \in \omega$ there exists $k(n) \in \omega$ such that for each set $C \in [k(n)]^n$ there exists $s \in k(n)$ such that $C \cap \big(C -_{k(n)} \{s\}\big) = \emptyset$.*

Proof. If s does not satisfy the conclusion of the theorem, then $s \in C -_{k(n)} C$. As $\big|C -_{k(n)} C\big| \le n^2$, any choice of $k(n) > n^2$ works as desired. □

Proposition 2. *Let $\mathcal{U}$ be a non-principal ultrafilter and $\dot{\mathrm{Q}}$ be a **PS**-name for a forcing such that $\mathbf{PS} * \dot{\mathrm{Q}}$ has the Sacks property. If $G \subset \mathbf{PS} * \dot{\mathrm{Q}}$ is a generic filter over V, then $\mathcal{U}$ cannot be extended to a P-point in $V[G]$.*

Proof. We first use the function k from Lemma 1 to inductively construct two increasing sequences of integers. Put $v(0) = 0$ and $m(0) = k(2)$. Assume $v(n-1)$, $m(n-1)$ are defined, put $v(n) = \sum\{ m(i) \mid i \in n \}$ and $m(n) = k\big((n+1)(v(n)+2)\big)$.

Let r be the **PS** generic real in $V[G]$ added by the first stage of the iteration. In $V[G]$, for each $n \in \omega$ and $i \in m(n)$ put

$$D_i^n = \bigcup\{ I_j(r) \mid j \in \omega, j =_{m(n)} i \}.$$

For fixed n the sets D^n_i form a finite decomposition of ω. We will show that in $V[G]$, for every function $y\colon \omega \to \omega$, $y(n) < m(n)$ and every pseudointersection Z of $\left\{ D^n_{y(n)} \mid n \in \omega \right\}$ there is a set $U \in \mathcal{U}$ such that $U \cap Z = \emptyset$. This entails that $\mathcal{U}$ cannot be extended to a P-point in $V[G]$.

Let $(p, \dot q)$ be any condition in $\mathbf{PS} * \dot{\mathbb{Q}}$, $\dot Z$ and $\dot y$ be the corresponding names for Z and y. Utilizing the Sacks property, we can assume that there are $f : \omega \to \omega$ and $\left\{ X_n \in [m(n)]^{n+1} \mid n \in \omega \right\}$ in V such that

$$(p, \dot q) \Vdash \left(\dot Z \setminus f(n) \right) \subseteq D^n_{\dot y(n)} \ \text{ and } \ \dot y(n) \in X_n.$$

Choose an interval partition $\{A_n \mid n \in \{-1\} \cup \omega\}$ of ω ordered in the natural way such that

(1) $f(n) < \min A_{2n}$ for each $n \in \omega$, and
(2) $m(n) < \left| A_{2n+j} \cap \operatorname{cod} p \right|$ for each $n \in \omega$, $j \in 2$.

We will assume that $U_0 = \bigcup \{ A_{2n+1} \mid n \in \omega \} \in \mathcal{U}$, otherwise take the interval partition $\langle A_{-1} \cup A_0, A_1, A_2, \dots \rangle$ instead. The plan is to use the trace of extensions of p on the interval A_{2n} to control the set $D^n_{y(n)} \cap A_{2n+1}$.

Let $p_1 \in \mathbf{PS}$ be any extension of p such that $\operatorname{cod} p_1 \cap A_{2n-1} = \emptyset$ and $|\operatorname{cod} p_1 \cap A_{2n}| = m(n)$ for each $n \in \omega$. Note that $|\operatorname{cod} p_1 \cap \min A_{2n}| = v(n)$. Let

$$C_n = \left\{ j \in m(n) \mid j \in_{m(n)} \left(X_n -_{m(n)} \{ i \mid i \in v(n) + 2 \} \right) \right\},$$

and notice that $|C_n| \le (n+1)(v(n)+2)$. For $n \in \omega$ put

$$H_n = A_{2n+1} \cap \bigcup \left\{ I_j(p_1) \mid j \in \omega, j \in_{m(n)} C_n \right\}.$$

We will now distinguish two cases. Case 1; $\bigcup \{ H_n \mid n \in \omega \} \notin \mathcal{U}$, hence $U = \bigcup \{ A_{2n+1} \setminus H_n \mid n \in \omega \} \in \mathcal{U}$. Pick any $p_2 < p_1$, $p_2 \in \mathbf{PS}$ such that $p_2^{-1}(1) = p_1^{-1}(1)$ and $|\operatorname{cod} p_2 \cap A_{2n}| = 1$ for each $n \in \omega$. Now

$$(p_2, \dot q) \Vdash D^n_{\dot y(n)} \cap A_{2n+1} \subset H_n.$$

This together with

$$(p, \dot q) \Vdash (\dot Z \setminus \min A_{2n}) \subseteq D^n_{\dot y(n)}$$

implies that $(p_2, \dot q) \Vdash Z \cap U = \emptyset$.

Case 2; $U = \bigcup \{ H_n \mid n \in \omega \} \in \mathcal{U}$. Applying Lemma 1, for each $n \in \omega$ there exists $s_n \in m(n)$ such that $C_n \cap \left(C_n -_{m(n)} \{ s_n \} \right) = \emptyset$. Pick a condition $p_2 < p_1$, $p_2 \in \mathbf{PS}$ such that $|\operatorname{cod} p_2 \cap A_{2n}| = 1$ and

$$\left| p_2^{-1}(1) \cap A_{2n} \right| = \left| p_1^{-1}(1) \cap A_{2n} \right| + s_n$$

for each $n \in \omega$. Such p_2 exists as $|\operatorname{cod} p_1 \cap A_{2n}| = m(n)$. For $n \in \omega$ put

$$\begin{aligned} \bar H_n = A_{2n+1} \cap \bigcup \left\{ I_j(p_1) \mid j \in \omega, j \in_{m(n)} \left(C_n -_{m(n)} \{ s_n \} \right) \right\} = \\ = A_{2n+1} \cap \bigcup \left\{ I_j(p_2) \mid j \in \omega, j \in_{m(n)} C_n \right\}. \end{aligned}$$

Notice that $H_n \cap \bar{H}_n = \emptyset$. Now

$$(p_2, \dot{q}) \Vdash D^n_{\dot{y}(n)} \cap A_{2n+1} \subset \bar{H}_n.$$

Again, together with

$$(p, \dot{q}) \Vdash (\dot{Z} \setminus \min A_{2n}) \subseteq D^n_{\dot{y}(n)}$$

we get $(p_2, \dot{q}) \Vdash Z \cap U = \emptyset$. □

The Silver model is the result of a countable support iteration of Silver forcing of length ω_2.

Theorem 3. *There are no P-points in the Silver model.*

Proof. Denote by $\mathbf{PS}_\alpha$ the countable support iteration of Silver forcing of length α. Assume V is a model of CH and let $G \subset \mathbf{PS}_{\omega_2}$ be a generic filter. Let $\mathcal{U} \in V[G]$ be a non-principal ultrafilter. For $\alpha < \omega_2$ let $\mathcal{U}_\alpha = \mathcal{U} \cap V[G_\alpha]$, where G_α is the restriction of G to $\mathbf{PS}_\alpha$. By a standard reflection argument, there is $\alpha < \omega_2$ such that $\mathcal{U}_\alpha \in V[G_\alpha]$ and it is an ultrafilter in that model. Since the next step of the iteration adds a Silver real and the tail of the iteration has the Sacks property, Proposition 2 states that $\mathcal{U}_\alpha$ cannot be extended to a P-point in $V[G]$, in particular, $\mathcal{U}$ is not a P-point. □

We show that forcing with the side-by-side product of Silver forcing also produces a model without P-points.

Theorem 4. *Assume* GCH*, let $\kappa > \omega_1$ be a cardinal with uncountable cofinality. If $\bigotimes_\kappa \mathbf{PS}$ is the countable support product of κ many Silver posets and $G \subset \bigotimes_\kappa \mathbf{PS}$ is a generic filter, then*

$$V[G] \models \text{there are no P-points and } \mathfrak{c} = \kappa.$$

Proof. It is well known that under GCH the poset $\bigotimes_\kappa \mathbf{PS}$ is an ω_2-c.c. proper forcing notion, has the Sacks property (see e.g. [Kos92]), and $V[G] \models \mathfrak{c} = \kappa$. Assume $\mathcal{U}$ is an ultrafilter in $V[G]$. Since $\bigotimes_\kappa \mathbf{PS}$ is ω_2-c.c., there is $J \subset \kappa$ of size ω_1 such for every $U \in \mathcal{P}(\omega) \cap V$ and $q \in \bigotimes_\kappa \mathbf{PS}$, the statement $U \in \mathcal{U}$ is decided by a condition with support contained in J and compatible with q. Choose $\alpha \in \kappa \setminus J$ and let r be the $\mathbf{PS}$ generic real added by the α-th coordinate of the product.

The theorem is now proved in the same way as Proposition 2; use the real r to define the partitions $\{D^n_i \mid i \in m(n)\}$, let (p, q) be a condition in $\mathbf{PS} \times \bigotimes_{\kappa \setminus \{\alpha\}} \mathbf{PS}$, and proceed as in the aforementioned proof. When using the assumption $U_0 \in \mathcal{U}$, it might be necessary to strengthen (p, q) to a condition $(p', q') \in \mathbf{PS} \times \bigotimes_{\kappa \setminus \{\alpha\}} \mathbf{PS}$ which forces this assumption. However, as $\alpha \notin J$ and $U_0 \in V$, the strengthening can be chosen so that $p = p'$. The same argument is used once more when assuming $U \in \mathcal{U}$. The rest of the proof works without any changes. □

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