# Dissipative measure-valued solutions and convergent numerical scheme for the compressible Navier-Stokes system

Bangwei She

joint work with E. Feireisl and H. Mizerová

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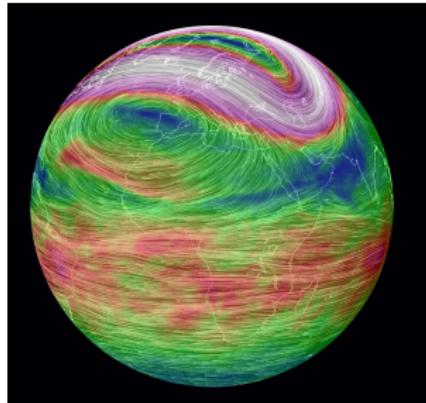
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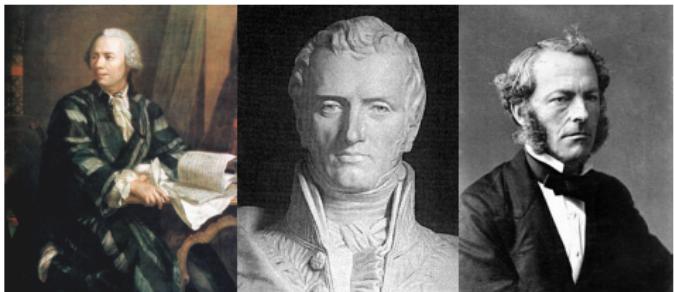
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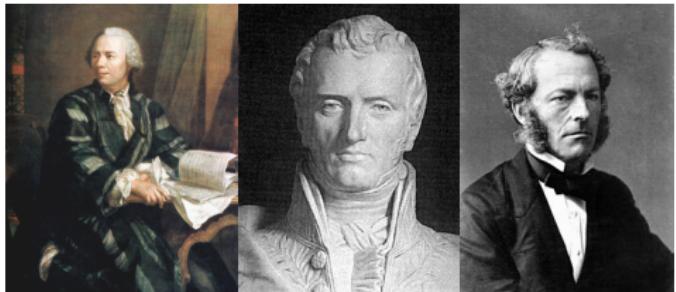
$$\partial_t \mathbf{U} + \mathrm{div}_x F(\mathbf{U}) = 0, \quad \mathbf{U}(0, x) = \mathbf{U}_0$$
$$\mathbf{U} = \mathbf{U}(t, x), \quad t \in (0, T), x \in R^d.$$

$$\begin{cases} \text{density} \dots & \rho(t, \mathbf{x}) \\ \text{velocity} \dots & \mathbf{u}(t, \mathbf{x}) \\ \text{energy} \dots & e(t, \mathbf{x}) \end{cases}$$

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- Discontinuity $\longrightarrow$ *weak solutions*

$$\int_0^\infty \int_{R^d} \left( \mathbf{U} \frac{\partial \phi}{\partial t} + \sum_{k=1}^d \mathbf{F}_k (\mathbf{U}) \frac{\partial \phi}{\partial x_k} \right) + \int_{R^d} \mathbf{U}_0 \, \phi \, (0, \cdot) = 0.$$

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**existence and uniqueness** of weak entropy solutions
(Kružkov '70, Lax, Glimm '60, Bressan '90)

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**existence and uniqueness** of <span style="color:red">weak entropy solutions</span>
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- $m > 1$, or $d > 1$, infinitely many sols
De Lellis & Székelyhidi '12–'14      Chiodaroli, Feireisl '14–'15
Chiordaroli, De Lellis, Kreml '16

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# Convergence of numerical solutions

○ **fundamental question in numerical analysis**

   **?** Does $\mathbf{U}_h \to \mathbf{U}$ as $h \to 0$ ?

○ **open question for compressible flows**

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Certain numerical solutions of inviscid problems exhibit scheme independent oscillatory behaviour

**Siddhartha Mishra**

Dissipative measure-valued solutions & Convergent Schemes for NSE

(a) nx = 128        (b) nx = 256        (c) nx = 512        (d) nx = 1024

Dissipative measure-valued solutions & Convergent Schemes for NSE

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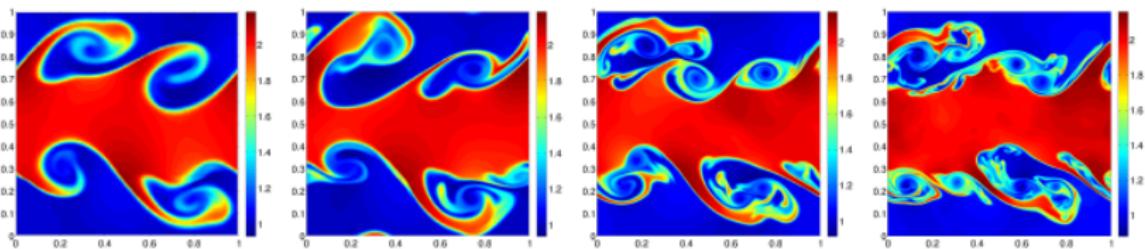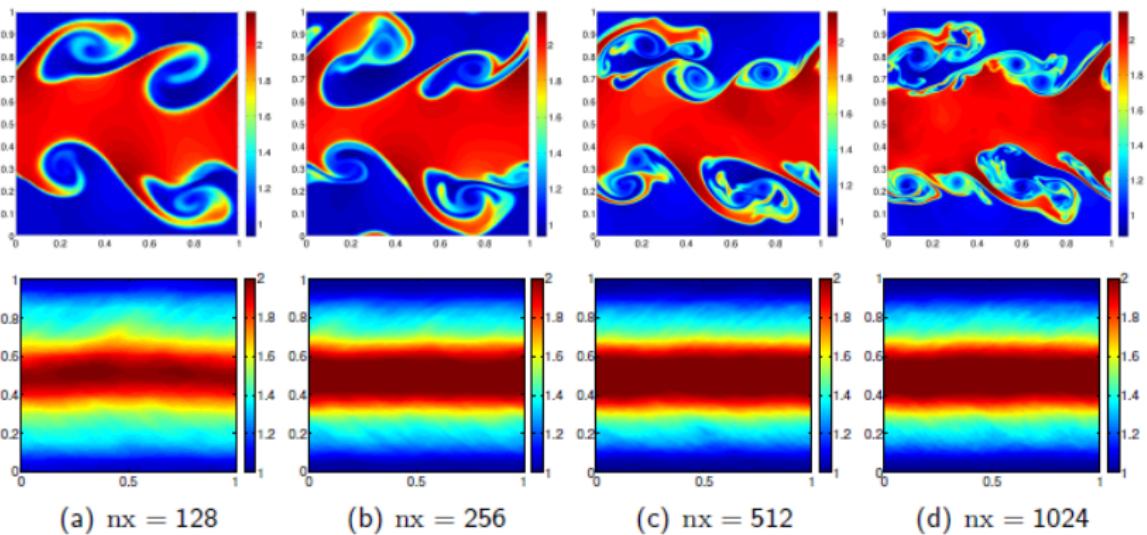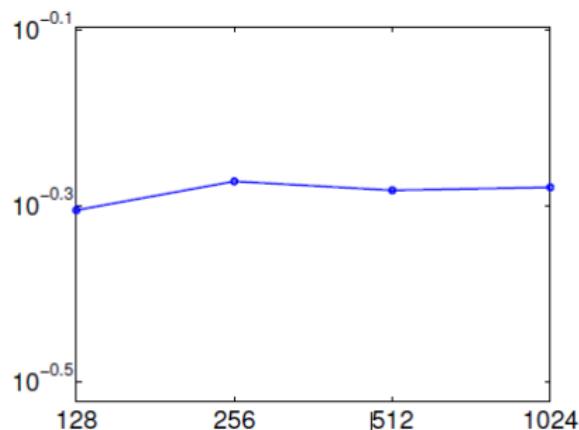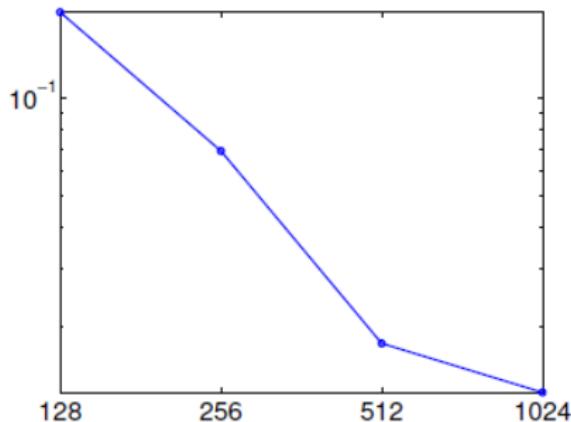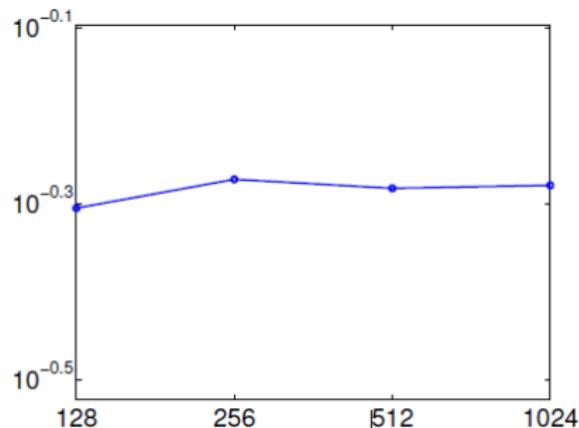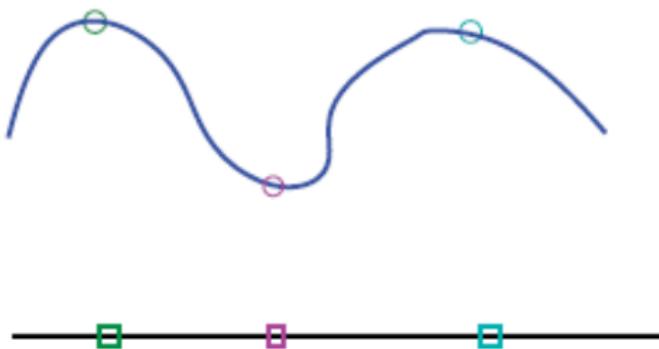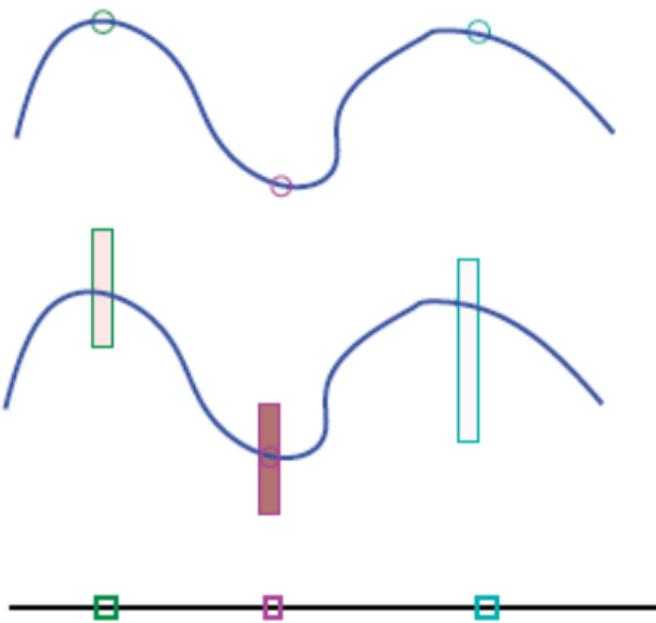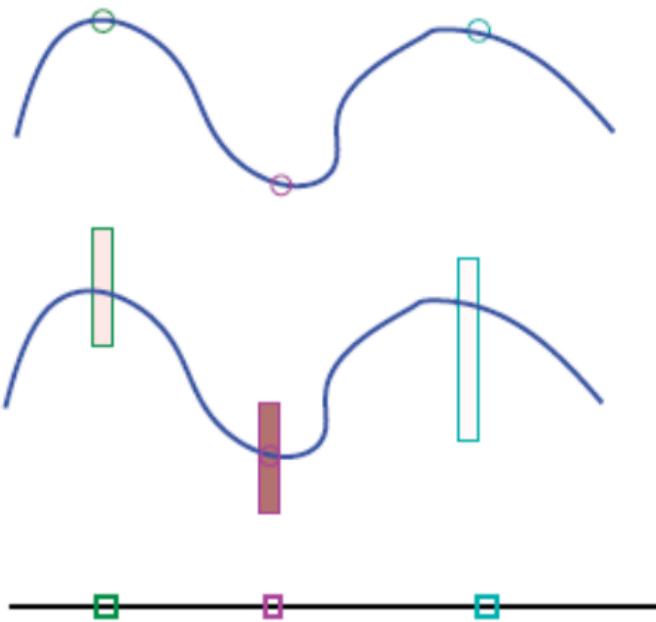
Measure-value solutions!

Feireisl, Lukacova, Mizerova, preprints.

## Compressible isentropic Navier-Stokes

$$\rho_t + \text{div}(\rho\mathbf{u}) = 0. \tag{1a}$$

$$(\rho\mathbf{u})_t + \text{div}(\rho\mathbf{u} \otimes \mathbf{u}) = -\nabla p + \nabla \cdot \mathbb{S} \tag{1b}$$

$p = a\rho^\gamma$
$\mathbb{S} = 2\mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) + \eta\text{div}\mathbf{u}\mathbb{I},$

$$\rho(\mathbf{x}, 0) = \rho_0 > 0, \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0.$$

## Convergence to weak solutions

**Karper [2013]:** Convergence to a weak solution if $\boxed{\gamma > 3}$

## Error estimates

**Gallouet, Herbin, Maltese, Novotný [2015]**

Convergence to smooth solutions + error estimates if $\boxed{\gamma > 3/2}$

## Convergence to strong solutions

**Feireisl, Lukáčová [2016]**

Convergence via dissipative measure-valued solution for Physical relevant $\gamma \in (1, 2)$

## Dissipative measure-valued solution

We say that a parameterized measure $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$,
$$\nu \in L^\infty_{weak}\Big((0,T)\times\Omega; \mathcal{P}\left([0,\infty)\times R^N\right)\Big)$$
is a dissipative measure-valued solution of the Navier-Stokes system in $(0,T)\times\Omega$, if the following holds for a.a. $\tau \in (0,T)$, for any $\psi \in C^1\big((0,T)\times\Omega; R^d\big)$

$$\left[\int_\Omega \langle \nu_{t,x}; \rho\rangle \psi(t,\cdot)dx\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big(\langle\nu_{t,x};\rho\rangle\partial_t\psi + \langle\nu_{t,x};\rho\mathbf{u}\rangle\cdot\nabla_x\psi\Big)dxdt$$

$$\left[\int_\Omega \langle \nu_{t,x}; \rho\mathbf{u}\rangle \psi(t,\cdot)dx\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big(\langle\nu_{t,x};\rho\mathbf{u}\rangle\partial_t\psi - \mathcal{S}(\nabla\mathbf{u}):\nabla_x\psi dxdt$$

$$+\langle\nu_{t,x};\rho\mathbf{u}\otimes\mathbf{u}+p\mathbb{I}\rangle:\nabla_x\psi\Big)dxdt + \int_0^\tau\int_\Omega \mathcal{R}:\nabla_x\psi dxdt,$$

$$\left[\int_\Omega \langle \nu_{t,x}; E\rangle \psi(t,\cdot)dx\right]_{t=0}^{t=\tau} + \int_0^\tau\int_\Omega \mathcal{S}(\nabla\mathbf{u}):\nabla_x\psi dxdt + \mathcal{D}(\tau) \le 0,$$

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where $\boxed{\displaystyle\int_0^\tau\|\mathcal{R}\|_{\mathcal{M}(\Omega)}dt \le \int_0^\tau \mathcal{D}(\tau)dt}$

# Mesh



**Figure :** Grid

# Mesh



**Figure :** Dual grid

# Discrete operators
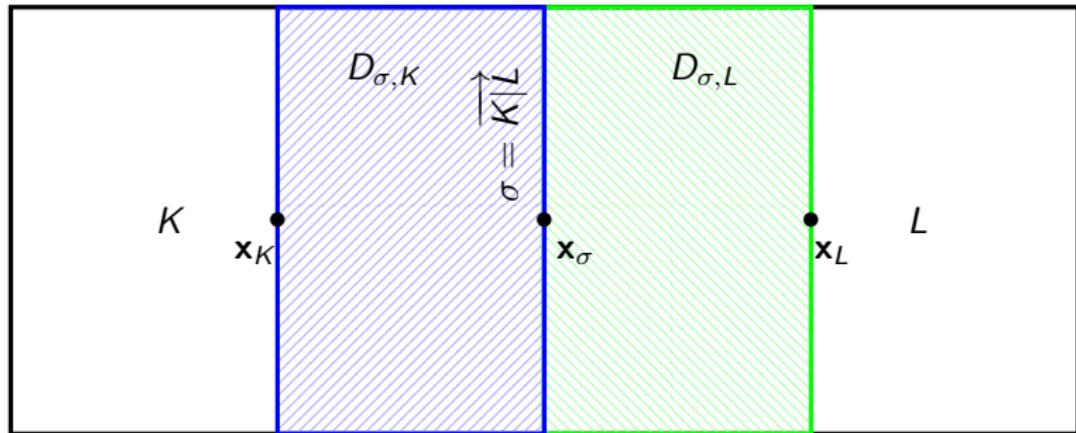
**Jump and Average**

$$v^{\mathrm{out}}(x) = \lim_{\delta \to 0+} v(x + \delta \boldsymbol{n}), \ \ v^{\mathrm{in}}(x) = \lim_{\delta \to 0+} v(x - \delta \boldsymbol{n}),$$

$$\overline{v}(x) = \frac{v^{\mathrm{in}}(x) + v^{\mathrm{out}}(x)}{2}, \ \ [\![v]\!] = v^{\mathrm{out}}(x) - v^{\mathrm{in}}(x)$$

**Divergence**

$$\mathrm{div}_h \mathbf{v}(\mathbf{x}) = \sum_{K \in \mathcal{T}_h} (\mathrm{div}_h \mathbf{v})_K 1_K, \quad (\mathrm{div}_h \mathbf{v})_K = \frac{1}{|K|} \sum_{\sigma \in \partial K} |\sigma| \overline{\mathbf{v}}_\sigma \cdot \boldsymbol{n}_\sigma, \ \forall \, \mathbf{v} \in Q_h.$$

**Upwind flux**

$$\mathrm{Up}[r, \mathbf{u}]_\sigma = r^{\mathrm{in}} (\mathbf{u} \cdot n)_\sigma^+ + r^{\mathrm{out}} (\mathbf{u} \cdot n)_\sigma^-$$

$$r^+ = \max\{0, f\}, \quad r^- = \min\{0, f\}$$

$$Up[r, \mathbf{v}] = r^{\mathrm{up}} \mathbf{v} \cdot \boldsymbol{n} = r^{\mathrm{in}} [\overline{\mathbf{v}} \cdot \boldsymbol{n}]^+ + r^{\mathrm{out}} [\overline{\mathbf{v}} \cdot \boldsymbol{n}]^- = \overline{r} \, \overline{\mathbf{v}} \cdot \boldsymbol{n} - \frac{1}{2} |\overline{\mathbf{v}} \cdot \boldsymbol{n}| [\![r]\!],$$

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## Implicit FVM

$$\int_\Omega D_t \rho_h \phi_h - \sum_{\sigma \in \mathcal{E}} \int_\sigma F_h(\rho_h, \mathbf{u}_h) [\![\phi_h]\!] \mathrm{d}Sx = 0, \tag{2a}$$

$$\int_\Omega D_t(\rho_h \mathbf{u}_h) \cdot \phi_h - \sum_{\sigma \in \mathcal{E}} \int_\sigma \mathbf{F}_h(\rho_h \mathbf{u}_h, \mathbf{u}_h) \cdot [\![\phi_h]\!] \mathrm{d}Sx - \sum_{\sigma \in \mathcal{E}} \int_\sigma \overline{p_h} \mathbf{n} \cdot [\![\phi_h]\!] \mathrm{d}Sx$$

$$= -\mu \sum_{\sigma \in \mathcal{E}} \int_\sigma \frac{1}{d_\sigma} [\![\mathbf{u}_h]\!] \cdot [\![\phi_h]\!] \mathrm{d}Sx - (\mu + \lambda) \int_\Omega \mathrm{div}_h \mathbf{u}_h \, \mathrm{div}_h \phi_h \mathrm{d}x. \tag{2b}$$

## Implicit FVM

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$$\int_\Omega D_t(\rho_h \mathbf{u}_h) \cdot \phi_h - \sum_{\sigma \in \mathcal{E}} \int_\sigma \mathbf{F}_h(\rho_h \mathbf{u}_h, \mathbf{u}_h) \cdot [\![\phi_h]\!] \mathrm{d}Sx - \sum_{\sigma \in \mathcal{E}} \int_\sigma \overline{p_h} \mathbf{n} \cdot [\![\phi_h]\!] \mathrm{d}Sx$$

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$$\eth^{(i)}_\mathcal{E} r_h(\mathbf{x}) := \sum_{\sigma \in \mathcal{E}_i} \mathbb{1}_{D_\sigma} \left( \eth^{(i)}_\mathcal{E} r_h \right)_{D_\sigma}, \quad \left( \eth^{(i)}_\mathcal{E} r_h \right)_{D_\sigma} := \frac{r_L - r_K}{d_\sigma}, \ \forall \ \sigma = \overrightarrow{K|L} \in \mathcal{E}_i,$$

$$\nabla_\mathcal{E} \mathbf{u}_h := (\nabla_\mathcal{E} u_{1,h}, \dots, \nabla_\mathcal{E} u_{d,h})(\mathbf{x}), \quad \nabla_\mathcal{E} r_h(\mathbf{x}) := (\eth^{(1)}_\mathcal{E} r_h, \dots, \eth^{(d)}_\mathcal{E} r_h)(\mathbf{x}).$$

Dissipative measure-valued solutions & Convergent Schemes for NSE

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## Lemma 1 (Internal energy balance)

Let $(\rho_h, \mathbf{u}_h)$ satisfy the discrete continuity equation (2a), then there exists $\xi \in \mathrm{co}\{\rho_h^{k-1}, \rho_h^k\}$ and $\zeta \in \mathrm{co}\{\rho_K^k, \rho_L^k\}$ for any $\sigma = K|L \in \mathcal{E}$ such that

$$\int_\Omega D_t \mathcal{H}(\rho_h^k) - \sum_{\sigma \in \mathcal{E}} \int_\sigma \overline{\mathbf{u}_h^k} \cdot \mathbf{n} [\![ p(\rho_h^k) ]\!] \mathrm{d}Sx$$

$$= -\frac{\Delta t}{2} \int_\Omega \mathcal{H}''(\xi) |D_t \rho_h^k|^2 \mathrm{d}x - \frac{1}{2} \sum_{\sigma \in \mathcal{E}} \int_\sigma \mathcal{H}''(\zeta) [\![ \rho_h^k ]\!]^2 \left( h^\varepsilon + |\overline{\mathbf{u}_h} \cdot \mathbf{n}| \right) \mathrm{d}Sx,$$

where $\mathcal{H}(\rho) = \frac{p(\rho)}{\gamma - 1}$.

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where $\mathcal{H}(\rho) = \frac{p(\rho)}{\gamma - 1}$.

- Positivity of numerical density
- Existence of numerical solution

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## Theorem 2 (Discrete energy inequality)

Let $(\rho_h, \mathbf{u}_h)$ be a numerical solution obtained from the scheme (2). Then for any $k = 1, \ldots, N_t$, there exists $\xi \in \mathrm{co}\{\rho_h^{k-1}, \rho_h^k\}$ and $\zeta \in \mathrm{co}\{\rho_K^k, \rho_L^k\}$ for any $\sigma = K|L \in \mathcal{E}$ such that

$$D_t \int_\Omega \left( \frac{1}{2} \rho_h^k |\mathbf{u}_h^k|^2 + \mathcal{H}(\rho_h^k) \right) \mathrm{d}x$$

$$+ \mu \int_\Omega |\nabla_\mathcal{E} \mathbf{u}_h^k|^2 \mathrm{d}x + (\mu + \lambda) \int_\Omega |\mathrm{div}_h \mathbf{u}_h^k|^2 \mathrm{d}x$$

$$= -\frac{\Delta t}{2} \int_\Omega \mathcal{H}''(\xi) |D_t \rho_h^k|^2 - \sum_{\sigma \in \mathcal{E}} \int_\sigma \frac{h^\varepsilon}{2} \overline{\rho_h^k} [\![\mathbf{u}_h^k]\!]^2 \mathrm{d}Sx$$

$$- \frac{\Delta t}{2} \int_\Omega \rho_h^{k-1} |D_t \mathbf{u}_h^k|^2 \mathrm{d}x - \frac{1}{2} \sum_{\sigma \in \mathcal{E}} \int_\sigma (\rho_h^k)^{\mathrm{up}} |\overline{\mathbf{u}_h^k} \cdot \mathbf{n}| [\![\mathbf{u}_h^k]\!]^2$$

$$- \frac{1}{2} \sum_{\sigma \in \mathcal{E}} \int_\sigma \mathcal{H}''(\zeta) [\![\rho_h^k]\!]^2 \left( h^\varepsilon + |\overline{\mathbf{u}_h^k} \cdot \mathbf{n}| \right) \mathrm{d}Sx \leq 0.$$

## Uniform bounds

Let $(\rho_h, \mathbf{u}_h)$ be a numerical solution obtained by the scheme (2) with $\gamma > 1$, $0 < \varepsilon < 1$. Then we have

$$\|\rho_h \mathbf{u}_h^2\|_{L^\infty(L^1(\Omega))} \lesssim 1 \qquad \|\nabla_\varepsilon \mathbf{u}_h\|_{L^2 L^2} \lesssim 1$$
$$\|\rho_h\|_{L^\infty(L^\gamma(\Omega))} \lesssim 1 \qquad \|\mathrm{div}_h \mathbf{u}_h\|_{L^2 L^2} \lesssim 1$$
$$\|p(\rho_h)\|_{L^\infty(L^1(\Omega))} \lesssim 1 \qquad \|\mathrm{div}_h \mathbf{u}_h\|_{L^2 L^2} \lesssim 1$$

$$h^\varepsilon \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma \overline{\rho_h} [\![\mathbf{u}_h]\!]^2 \mathrm{d}S x \mathrm{d}t \lesssim 1$$

$$\int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma \mathcal{H}''(\zeta) [\![\rho_h]\!]^2 (h^\varepsilon + |\overline{\mathbf{u}_h} \cdot \boldsymbol{n}|) \mathrm{d}S x \mathrm{d}t \lesssim 1$$

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$$\|\rho_h\|_{L^\infty(L^\gamma(\Omega))} \lesssim 1 \qquad \|\mathrm{div}_h \mathbf{u}_h\|_{L^2 L^2} \lesssim 1$$
$$\|p(\rho_h)\|_{L^\infty(L^1(\Omega))} \lesssim 1 \qquad \|\mathrm{div}_h \mathbf{u}_h\|_{L^2 L^2} \lesssim 1$$

$$h^\varepsilon \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma \overline{\rho_h} [\![\mathbf{u}_h]\!]^2 \, \mathrm{d}S x \mathrm{d}t \lesssim 1$$

$$\int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma \mathcal{H}''(\zeta) [\![\rho_h]\!]^2 (h^\varepsilon + |\overline{\mathbf{u}_h} \cdot \boldsymbol{n}|) \, \mathrm{d}S x \mathrm{d}t \lesssim 1$$

$$h\|\rho_h\|_{L^2 L^2} \lesssim h^\beta, \quad h\|\rho_h \mathbf{u}_h\|_{L^2 L^2} \lesssim h^\beta.$$

$$\beta = 1 - \frac{\varepsilon + 2}{2\gamma} > 0 \text{ if } \varepsilon < 2(\gamma - 1).$$

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## Lemma 3 (Consistency)

Let $\rho_h^k, \mathbf{u}_h^k$ be the solution to the numerical scheme (2). Then

$$\int_\Omega \partial_h^t \rho_h^k \phi \, dx - \int_\Omega \rho_h^k \mathbf{u}_h^k \cdot \nabla_x \phi \, dx = \mathcal{O}(h^{\beta_1}), \beta_1 > 0.$$

$$\int_\Omega \partial_h^t (\rho_h \mathbf{u}_h)^k \cdot \mathbf{v} \, dx - \int_\Omega \rho_h^k \mathbf{u}_h^k \otimes \mathbf{u}_h^k : \nabla_x \mathbf{v} \, dx - \int_\Omega p(\rho_h^k) div_x \mathbf{v} \, dx$$

$$+\mu \int_\Omega (\nabla_{\mathcal{E}} \mathbf{u}_h^k) : \nabla_x \mathbf{v} \, dx = \mathcal{O}(h^{\beta_2}), \beta_2 > 0.$$

## Limit: setting $h \to 0$

$$\left[\int_\Omega \langle \nu_{t,x}; \rho \rangle \psi(t,\cdot) dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left( \langle \nu_{t,x}; \rho \rangle \partial_t \psi + \langle \nu_{t,x}; \rho \mathbf{u} \rangle \cdot \nabla_x \psi \right) dx dt$$

$$\left[\int_\Omega \langle \nu_{t,x}; \rho \mathbf{u} \rangle \psi(t,\cdot) dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left( \langle \nu_{t,x}; \rho \mathbf{u} \rangle \partial_t \psi - \mathcal{S}(\nabla \mathbf{u}) : \nabla_x \psi dx dt \right.$$

$$\left. + \langle \nu_{t,x}; \rho \mathbf{u} \otimes \mathbf{u} + p \mathbb{I} \rangle : \nabla_x \psi \right) dx dt + \int_0^\tau \int_\Omega \mathcal{R} : \nabla_x \psi dx dt,$$

$$\left[\int_\Omega \langle \nu_{t,x}; E \rangle \psi(t,\cdot) dx \right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \mathcal{S}(\nabla \mathbf{u}) : \nabla_x \psi dx dt + \mathcal{D}(\tau) \leq 0,$$

where $\boxed{\int_0^\tau \|\mathcal{R}\|_{\mathcal{M}(\Omega)} dt \leq \int_0^\tau \mathcal{D}(\tau) dt}$

$$\left[ \int_\Omega \langle \nu_{t,x}; \rho \rangle \psi(t, \cdot) dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left( \langle \nu_{t,x}; \rho \rangle \partial_t \psi + \langle \nu_{t,x}; \rho \mathbf{u} \rangle \cdot \nabla_x \psi \right) dx dt$$

$$\left[ \int_\Omega \langle \nu_{t,x}; \rho \mathbf{u} \rangle \psi(t, \cdot) dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left( \langle \nu_{t,x}; \rho \mathbf{u} \rangle \partial_t \psi - \mathcal{S}(\nabla \mathbf{u}) : \nabla_x \psi dx dt \right.$$

$$\left. + \langle \nu_{t,x}; \rho \mathbf{u} \otimes \mathbf{u} + p \mathbb{I} \rangle : \nabla_x \psi \right) dx dt + \int_0^\tau \int_\Omega \textcolor{red}{\mathcal{R} : \nabla_x \psi} dx dt,$$

$$\left[ \int_\Omega \langle \nu_{t,x}; E \rangle \psi(t, \cdot) dx \right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \mathcal{S}(\nabla \mathbf{u}) : \nabla_x \psi dx dt + \textcolor{blue}{\mathcal{D}(\tau)} \leq 0,$$

where $\boxed{\int_0^\tau \|\mathcal{R}\|_{\mathcal{M}(\Omega)} dt \leq \int_0^\tau \mathcal{D}(\tau) dt}$

$$\mathcal{R} = \{ \rho \mathbf{u} \otimes \mathbf{u} + p(\rho) \mathbb{I} \} - \left\langle \nu_{t,x}; \widetilde{\rho \mathbf{u} \otimes \mathbf{u}} + \widetilde{p(\rho)} \mathbb{I} \right\rangle \in [L^\infty(0, T; \mathcal{M}(\Omega))]^{d \times d}$$

# Convergence

### Theorem 4

Let $\gamma > 1$, $\Delta t \approx h$, $0 < \varepsilon < 1$ and the initial data satisfy

$$\rho_0 \in L^\infty(R^d), \ \rho_0 \geq \underline{\rho} > 0 \ \text{a.a. in } R^d, \ \mathbf{u}_0 \in L^2(R^d).$$

Then any Young measure $\nu_{t,x}$ generated by the numerical sol of scheme (2) represents a dissipative measure-valued solution of NS (1).

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[1] Feireisl et al. Dissipative measure-valued solutions to the compressible Navier–Stokes system. Calc. Vari. Partial Differ. Equ., 2016

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# Convergence

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*Then any Young measure $\nu_{t,x}$ generated by the numerical sol of scheme (2) represents a dissipative measure-valued solution of NS (1).*

Applying the weak-strong uniqueness[1] we conclude

## Theorem 5

*In addition to the hypotheses of Theorem 4, suppose the NS (1) endowed with the periodic boundary condition admits a regular solution. Then*

$$\rho_h \to \rho \ (strongly) \ in \ L^\gamma \left( (0, T) \times K \right),$$

$$\mathbf{u}_h \to \mathbf{u} \ (strongly) \ in \ L^2 \left( (0, T) \times K; R^d \right)$$

*for any compact $K \subset \Omega$.*

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## Remark 6

*Suppose the numerical density is uniformly bounded.*

1. *The above Theorem 4, 6 also holds for no-slip boundary condition on smooth boundary. Then more general mesh (unstructured, any shape) can be used.*

2. *Applying the conditional regularity result[2], we conclude the global existence of regular solution.*

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[2]Sun et al., A Beale–Kato–Majda blow–up criterion for the 3–D compressible Navier–Stokes equations. J. Math. Pures. Appl., 2011.

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## Thank you for your attention!

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