Convergence of numerical solutions for the compressible Navier-Stokes system

Bangwei She

based on the work with E. Feireisl, R. Hošek, H. Mizerová, and A. Novotný

Nanjing University

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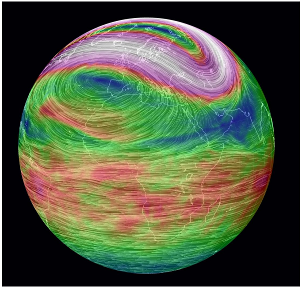
Fluid motion



Galileo Galilei
(1564-1642)

"Mathematics is the language
in which God has written the universe."

Conservation laws

? What is a common way to describe fluid motion?

- Conservation of mass
- Balance of momentum
- Conservation of energy

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$$\partial_t \mathbf{U} + \mathrm{div}_x F(\mathbf{U}) = 0, \quad \mathbf{U}(0,x) = \mathbf{U}_0$$
$$\mathbf{U} = (U_1, \dots, U_M), \quad \mathbf{U} = \mathbf{U}(t,x)$$

$t \in (0,T)$ time, $x \in R^d$ space

$$\begin{cases} \text{density} \dots & \rho(t,\mathbf{x}) \\ \text{velocity} \dots & \mathbf{u}(t,\mathbf{x}) \\ \text{energy} \dots & e(t,\mathbf{x}) \end{cases}$$

Definition 1 (Weak Sols)

Let $\mathbf{U}_0 \in L^1_{loc}(\mathbb{R}^d)$. A function $\mathbf{U}(\mathbf{x},t) \in L^1_{loc}([0,\infty) \times \mathbb{R}^d)$ is a weak solution of the Cauchy problem

$$\frac{\partial \mathbf{U}}{\partial t} + \sum_{k=1}^{d} \frac{\partial \mathbf{F}_k}{\partial x_k}(\mathbf{U}) = 0, \quad \mathbf{U}(\mathbf{x},0) = \mathbf{U}_0,$$

iff $\forall \phi \in C_0^\infty(R_+ \times R^d)$

$$\int_0^\infty \int_{R^d} \left(\mathbf{U} \frac{\partial \phi}{\partial t} + \sum_{k=1}^{d} \mathbf{F}_k(\mathbf{U}) \frac{\partial \phi}{\partial x_k} \right) + \int_{R^d} \mathbf{U}_0 \, \phi(0,\cdot) = 0.$$

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Admissible entropy solution

$$\partial_t S(\mathbf{U}) + \mathrm{div}_x F_S(\mathbf{U}) \; = 0$$

Unique physically admissible solutions

- **discontinuous solutions (shocks)**
  $\Longrightarrow$ weak solutions in the sense of distributions

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  existence and uniqueness of weak entropy solutions
  (Kružkov '70, Lax, Glimm '60, Bressan '90)

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- $m > 1, \; d > 1,$ multi-d systems
  weak entropy solutions are NON-unique
  (De Lellis & Székelyhidi '12-'14)
  (Chiodaroli, Feireisl '14-'15)
  (Chiordaroli, De Lellis, Kreml '16)

  $\Longrightarrow$ **open problem !** *selection criterion ?*

Convergence of numerical solutions

◦ **fundamental question in numerical analysis**

? Does $\mathbf{U}_h \to \mathbf{U}$ as $h \to 0$?

? Rate of convergence ?

◦ **open question for compressible flows**

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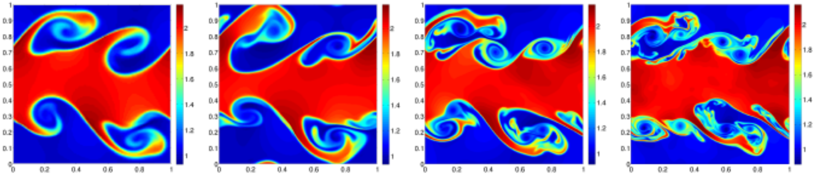
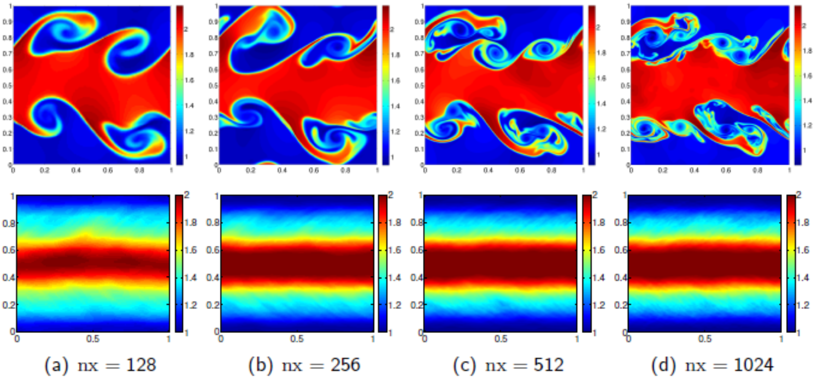
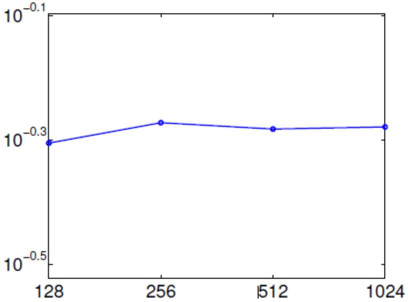
Certain numerical solutions of inviscid problems exhibit scheme independent oscillatory behaviour

Siddhartha Mishra




(a) nx = 128
(b) nx = 256
(c) nx = 512
(d) nx = 1024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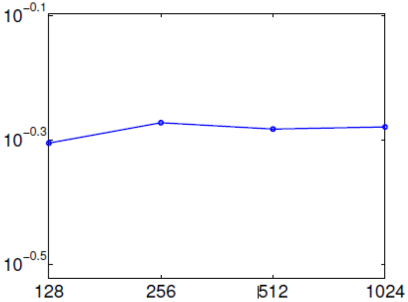
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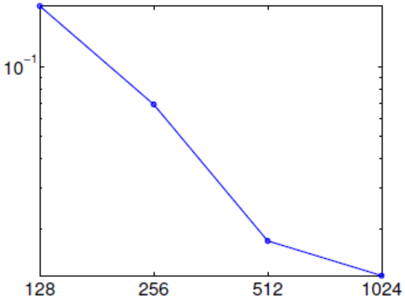
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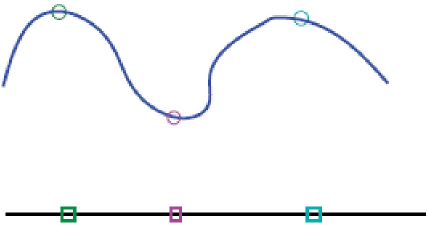

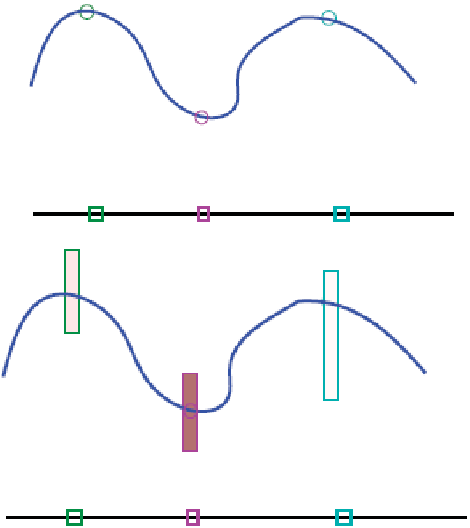

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0. \tag{1a}$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \nabla \cdot \mathbb{S} \tag{1b}$$

p : pressure, $p = a\rho^\gamma$

$\mathbb{S}$: viscous stress, $\mathbb{S} = \mu \nabla \mathbf{u} + (\frac{\mu}{3} + \eta)\operatorname{div}\mathbf{u}, \mu > 0$

Compressible barotropic Navier-Stokes

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Boundary condition for $\mathbf{u}$

$$\mathbf{u}|_{\partial\Omega} = 0 \quad \text{or} \quad \text{periodic} \tag{1c}$$

Initial values

$$\rho(\mathbf{x}, 0) = \rho_0 > 0 \tag{1d}$$

Numerical method of T. Karper(2013)

Discrete time derivative - implicit scheme

$$D_t v_h^k = \frac{v_h^k - v_h^{k-1}}{\Delta t}$$

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Momentum method

$$\int_{\Omega_h} D_t \left(\rho_h^k \left\langle \mathbf{u}_h^k \right\rangle\right) \cdot \boldsymbol{\phi} \mathrm{d}x - \sum_{\Gamma \in \Gamma_{\mathrm{int}}} \int_\Gamma \mathrm{Up}[\rho_h^k \left\langle \mathbf{u}_h^k \right\rangle, \mathbf{u}_h^k] \cdot [\![\langle \boldsymbol{\phi} \rangle]\!] \mathrm{dS}_x - \int_{\Omega_h} p(\rho_h^k) \mathrm{div}_h \boldsymbol{\phi} \mathrm{d}x$$

$$+\mu \int_{\Omega_h} \nabla_h \mathbf{u}_h^k : \nabla_h \boldsymbol{\phi} \mathrm{d}x + \left(\frac{\mu}{3} + \eta\right) \int_{\Omega_h} \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \boldsymbol{\phi} \mathrm{d}x = 0$$

Convergence results for Karper's scheme

Convergence to weak solutions

Karper [2013]: Convergence to a weak solution if $\boxed{\gamma > 3}$

Error estimates

Gallouet, Herbin, Maltese, Novotný [2015]

Convergence to smooth solutions + error estimates if $\boxed{\gamma > 3/2}$

Convergence to strong solutions

Feireisl, Lukáčová [2016]

Convergence via dissipative measure-valued solution for Physical relevant $\gamma \in (1, 2)$

Dissipative measure-valued solution

Definition 2

We say that a parameterized measure $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$,

$$\nu \in L^\infty_{weak}\Big((0,T)\times\Omega; \mathcal{P}\left([0,\infty)\times R^N\right)\Big)$$

is a dissipative measure-valued solution of the Navier-Stokes system in $(0,T)\times\Omega$, if the following holds for a.a. $\tau\in(0,T)$, for any $\psi\in C^1((0,T)\times\Omega;R^d)$

$$\left[\int_\Omega \langle \nu_{\tau,x};\rho\rangle\psi(\tau,\cdot)dx\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega [\langle\nu_{t,x};\rho\rangle\partial_t\psi + \langle\nu_{t,x};\rho\mathbf{u}\rangle\cdot\nabla_x\psi]dxdt$$

$$\left[\int_\Omega \langle \nu_{\tau,x};\rho\mathbf{u}\rangle\psi(\tau,\cdot)dx\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega [\langle\nu_{t,x};\rho\mathbf{u}\rangle\partial_t\psi + \langle\nu_{t,x};\rho\mathbf{u}\otimes\mathbf{u}+p\mathbf{I}\rangle:\nabla_x\psi]dxdt$$
$$-\int_0^\tau\int_\Omega \mathcal{S}(\nabla\mathbf{u}):\nabla_x\psi dxdt + \int_0^\tau\int_\Omega \mathcal{R}:\nabla_x\psi dxdt,$$

$$\left[\int_\Omega \langle \nu_{\tau,x};E\rangle\psi(\tau,\cdot)dx\right]_{t=0}^{t=\tau} + \int_0^\tau\int_\Omega \mathcal{S}(\nabla\mathbf{u}):\nabla_x\psi dxdt + \mathcal{D}(\tau)\le 0,$$

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where $\boxed{\int_0^\tau \|\mathcal{R}\|_{\mathcal{M}(\Omega)}dt \le \int_0^\tau \mathcal{D}(\tau)dt}$

General strategy

Basic properties of numerical scheme

Show stability, consistency

Measure valued solutions

Show convergence of the scheme to a

dissipative measure – valued solution

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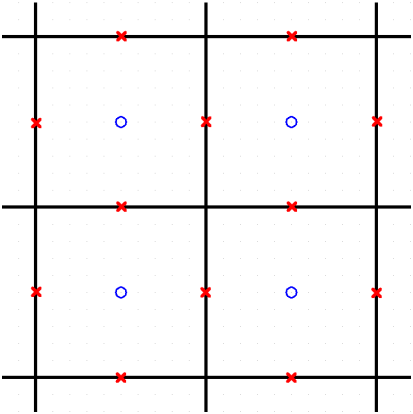
Weak-strong uniqueness

Use the weak-strong uniqueness principle in the class of measure-valued solutions. Strong and measure valued solutions emanating from the same initial data coincide as long as the latter exists

Finite difference MAC scheme

Mesh

- Elements: $\Omega_h = \cup K$
- Faces: $\mathcal{E}$
- Exterior faces: $\mathcal{E}_{ext} = \partial\Omega \cup \mathcal{E}$.
- Interior faces: $\mathcal{E}_{int} = \mathcal{E} \setminus \mathcal{E}_{ext}$
- Interior faces of K: $\mathcal{E}_{int}(K)$
- Interior neighbours of K: $\mathcal{N}(K)$
- $\sigma = \overrightarrow{K|L}$ if $x_L = x_K + \frac{h}{2}\mathbf{e}_s$
  $\sigma_{K,s+}$
- Primary grid o : ρ, p
- Dual grid $\times$: $\mathbf{u}$



Numerical Scheme

$$\partial_h^t \rho_K^n + \mathrm{div}_{\mathrm{Up}}[\rho^n, \mathbf{u}^n]_K - h^\alpha (\Delta_h \rho^n)_K = 0, \tag{3a}$$

$$\{\partial_h^t (\rho \bar{\mathbf{u}})^n\}_\sigma + \{\mathrm{div}_{\mathrm{Up}}[\rho^n \bar{\mathbf{u}}^n, \mathbf{u}^n]\}_\sigma + \left(\partial_h^s p(\rho^n)\right)_\sigma \mathbf{e}_s - \mu\left(\Delta_h \mathbf{u}^n\right)_\sigma - h^\alpha \sum_{r=1}^{d} \left\{\partial_h^r\left(\{\hat{\mathbf{u}}^n\}\partial_h^r \rho^n\right)\right\}_\sigma = 0, \tag{3b}$$

for all $K \in \Omega_h, \sigma \in \mathcal{E}_{int}$ and $n = \{1, \ldots, N\}$, with boundary conditions.

Between grids

$$\{f\}_\sigma = \frac{1}{2}(f_K + f_L), \quad \forall \sigma = K|L \in \mathcal{E}_{int}$$

$$\bar{\mathbf{g}}_K = \frac{1}{2}\begin{pmatrix} g^1_{\sigma_{K,1+}} + g^1_{\sigma_{K,1-}} \\ g^2_{\sigma_{K,2+}} + g^2_{\sigma_{K,2-}} \\ g^3_{\sigma_{K,3+}} + g^3_{\sigma_{K,3-}} \end{pmatrix}$$

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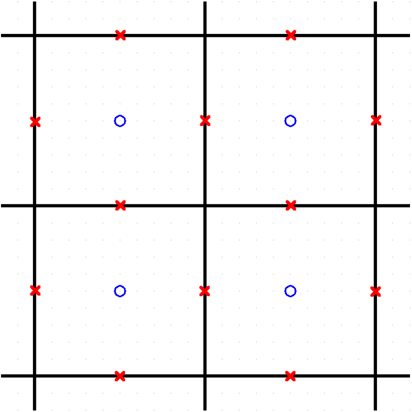
Functional spaces

$X(\Omega_h)$: P0 on primary grid Ω_h

$X(\mathcal{E}_{int})^d$: P0 on dual grid $\mathcal{E}$, and $\mathbf{g}|_{\mathcal{E}_{ext}} = \mathbf{0}$



Time

$$\partial_h^t \phi^n = \frac{\phi^n - \phi^{n-1}}{\mathrm{d}t}$$

Space

Let $f \in X(\Omega_h)$, $\mathbf{g} \in X(\mathcal{E}_{int})^d$

$$(\partial_h^s f)_\sigma = \frac{f_L - f_K}{h}, \ \sigma = \overrightarrow{K|L}$$

$$(\Delta_h f)_K = \frac{1}{h^2} \sum_{L \in \mathcal{N}(K)} (f_L - f_K)$$

$$(\Delta_h g)_\sigma = \frac{1}{h^2} \sum_{s=1}^{d} (g_{\sigma - \mathbf{e}_s} - 2 g_\sigma + g_{\sigma + \mathbf{e}_s}).$$

Upwind flux

$$\mathrm{Up}[f, \mathbf{u}]_\sigma = f_K (u_\sigma^s)^+ + f_L (u_\sigma^s)^-$$
$$f^+ = \max\{0, f\}, \quad f^- = \min\{0, f\}$$

Discrete differential operators II

Upwind flux

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Upwind discrete derivative and upwind divergence

$$\partial_s^{\mathrm{Up}}[f, \mathbf{u}]_K = \frac{\mathrm{Up}[f, \mathbf{u}]_{\sigma_{K,s+}} - \mathrm{Up}[f, \mathbf{u}]_{\sigma_{K,s-}}}{h}$$
$$\mathrm{div}_{\mathrm{Up}}[g, \mathbf{u}]_K = \sum_{s=1}^{d} \partial_s^{\mathrm{Up}}[f, \mathbf{u}]_K$$

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Let $f \in X(\Omega_h), \mathbf{v} = [v^1, \cdots, v^d] \in X(\mathcal{E}_{int})^d$, then $\sum_{K\in\Omega_h} \int_K \mathrm{div}_{\mathrm{Up}}[f,\mathbf{v}]_K = 0.$

Renormalized continuity equation

$$\boxed{\sum_{K\in\Omega_h}\int_K \partial_h^t B(\rho_K^n) + \Big(B'(\rho_K^n)\rho_K^n - B(\rho_K^n)\Big)(\mathrm{div}_h\mathbf{u}^n)_K + \mathcal{P}_K\Big) = 0}$$

$$\mathcal{P}_K = \frac{\mathrm{d}t}{2}B''(\overline{\rho_K^{n-1,n}})|\partial_h^t\rho_K^n|^2 + \frac{1}{2}\sum_{\sigma\in\mathcal{E}(K)}\Big(h|\mathbf{u}_\sigma| + h^\alpha\Big)B''(\rho_\sigma^\star)|(\partial_h\rho)_\sigma|^2$$

$\mathcal{P}_K \geq 0$ provided B is convex.

Existence of num sol

Lemma 3

Let $\rho_h^{n-1} \in X(\Omega_h)$, $\mathbf{u}_h^{n-1} \in X(\mathcal{E}_{int})^d$ *be given;* $\rho_K^{n-1} > 0$ *for all* $K \in \Omega_h$*. Then the numerical scheme (3) admits a solution*

$$\rho_h^n \in X(\Omega_h), \rho_K^n > 0 \text{ for all } K \in \Omega_h, \mathbf{u}_h^n \in X(\mathcal{E}_{int})^d.$$

Moreover, it satisfies the discrete conservation of mass

$$\sum_{K \in \Omega_h} \int_K \rho_K^n = \sum_{K \in \Omega_h} \int_K \rho_K^{n-1}.$$

Mass conservation

$$\partial_h^t \rho_K^n + \mathrm{div}_{\mathrm{Up}}[\rho^n, \mathbf{u}^n]_K - h^\alpha (\Delta_h \rho^n)_K = 0$$

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Positivity–nonnegativity

Recall the renormalized continuity equation

$$\sum_{K \in \Omega_h} \left(\partial_h^t B(\rho_K^n) + \Big(B'(\rho_K^n)\rho_K^n - B(\rho_K^n) \Big) (\mathrm{div}_h \mathbf{u}^n)_K + \mathcal{P}_K \right) = 0,$$

with test function

$$B(z) = \begin{cases} -z & \text{for } z < 0, \\ 0 & \text{for } z \geq 0. \end{cases}$$

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$$B(z) \geq 0$$

$$B'(z)z - B(z) = 0$$

$$\sum_{K\in\Omega_h}\int_K B(\rho_K^n) = \sum_{K\in\Omega_h}\int_K \left(B(\rho_K^{n-1}) - P_K\right) \leq 0$$

$$\boxed{\rho_K^n \geq 0}$$

Positivity–strict positivity

Let $K \in \Omega_h$ satisfy $\rho_K^n \leq \rho_L^n$ for all $L \in \Omega_h$. Then we have

$$\rho_K^n - \rho_K^{n-1} = -\Delta t \mathrm{div}_{\mathrm{Up}}[\rho^n, \mathbf{u}^n]_K + \Delta t h^\alpha (\Delta_h \rho^n)$$

$$\geq -\frac{\Delta t}{h} \sum_{s=1}^{d} \Big(\rho_K^n u_{\sigma_{K,s+}}^s - \rho_K^n u_{\sigma_{K,s-}}^s + (\rho_{K+h\mathbf{e}_s}^n - \rho_K^n) u_{\sigma_{K,s+}}^{s\,-} + (\rho_K^n - \rho_{K-h\mathbf{e}_s}^n) u_{\sigma_{K,s-}}^{s\,+}$$

$$\geq -\Delta t \rho_K^n |(\mathrm{div}_h \mathbf{u}^n)_K|$$

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\begin{aligned}
&\rho_K^n - \rho_K^{n-1} = -\Delta t \mathrm{div}_{\mathrm{Up}}[\rho^n, \mathbf{u}^n]_K + \Delta t h^\alpha (\Delta_h \rho^n) \\
&\geq -\frac{\Delta t}{h} \sum_{s=1}^{d} \Big(\rho_K^n u^s_{\sigma_{K,s+}} - \rho_K^n u^s_{\sigma_{K,s-}} + (\rho^n_{K+h\mathbf{e}_s} - \rho^n_K) u^{s\,-}_{\sigma_{K,s+}} + (\rho^n_K - \rho^n_{K-h\mathbf{e}_s}) u^{s\,+}_{\sigma_{K,s-}} \Big) \\
&\geq -\Delta t \rho_K^n \left| (\mathrm{div}_h \mathbf{u}^n)_K \right|
\end{aligned}
$$

$$
\rho_L^n \geq \rho_K^n \geq \frac{1}{1 + \Delta t \left| (\mathrm{div}_h \mathbf{u}^n)_K \right|} \rho_K^{n-1} > 0, \qquad \text{for any } L \in \Omega_h
$$

Energy stability

Lemma 4

Let $(\rho_h, \mathbf{u}_h)$ *be the numerical solution obtained by the scheme (3) with* $1 < \gamma < 2,\ 1 < \alpha < 2\gamma - 1$. *Then for any* $m = 1, \cdots, N$ *the following estimate holds,*

$$E^m + \Delta t \mu \sum_{n=1}^{m} \sum_{K \in \Omega_h} \int_K \sum_{r=1}^{3} \sum_{s=1}^{3} |(\partial_h^r (u^s)^n)_K|^2 + \sum_{j=1}^{4} \mathcal{N}_j \leq E^0.$$

$$E^m = \sum_{K \in \Omega_h} \int_K \left(\rho_K^m \frac{|\bar{\mathbf{u}}_K^m|^2}{2} + \frac{1}{\gamma - 1} p(\rho_K^m) \right)$$

$$\mathcal{N}_1 = \Delta t \sum_{n=1}^{m} \sum_{K \in \Omega_h} \int_K \sum_{s=1}^{d} \frac{1}{2} \left((h^\alpha + h^2 (u_{\sigma, s\mp}^{s,n})^{\pm}) p''(\rho_{\sigma, s\mp}^{n,\star}) |(\partial_h^s \rho^n)_{\sigma, s\mp}|^2 \right),$$

$$\mathcal{N}_2 = \mathrm{d}t^2 \sum_{n=1}^{m} \sum_{K \in \Omega_h} \int_K \frac{p''(\rho_K^\eta)}{2} |\partial_t^h \rho_K^n|^2, \quad \mathcal{N}_3 = \mathrm{d}t^2 \sum_{n=1}^{m} \sum_{K \in \Omega_h} \int_K \frac{\rho_K^{n-1}}{2} |\partial_t^h \bar{\mathbf{u}}_K^n|^2,$$

$$\mathcal{N}_4 = \Delta t \frac{h}{4} \sum_{n=1}^{m} \sum_{\Gamma \in \mathcal{E}_{int}} \int_\Gamma |Up[\rho^n, \mathbf{u}^n]_\sigma| |(\partial_h^s \bar{\mathbf{u}}^n)_\sigma|^2.$$

Lemma 5

Let $(\rho_h, \mathbf{u}_h)$ *be a numerical solution obtained by the scheme (3).*
Suppose $1 < \gamma < 2,\ 1 < \alpha < 2\gamma - 1$.
Then we have

$$\|\rho_h\|_{L^\infty(L^\gamma(\Omega))} \lesssim 1$$

$$\|p(\rho_h)\|_{L^\infty(L^1(\Omega))} \lesssim 1$$

$$\|\nabla_h \mathbf{u}_h\|_{L^2(L^2(\Omega))} \lesssim 1$$

$$\|\mathbf{u}_h\|_{L^2(L^6(\Omega))} \lesssim 1$$

$$\|\sqrt{\rho_h}\bar{\mathbf{u}}_h\|_{L^\infty(L^2(\Omega))} \lesssim 1$$

$$h\|\sqrt{\rho_h}\|_{L^2(L^\infty(\Omega))} \lesssim h^\theta,\ \theta = 1 - \frac{\alpha + 1}{2\gamma} > 0.$$

Lemma 6

Let $\rho_h^n, \mathbf{u}_h^n$ *be the solution to the numerical scheme (3). Then*

$$\int_\Omega \partial_h^t \rho_h^n \phi dx - \int_\Omega \rho_h^n \mathbf{u}_h^n \cdot \nabla_x \phi dx = \mathcal{O}(h^{\beta_1}), \beta_1 > 0.$$

$$\int_\Omega \partial_h^t (\rho_h \bar{\mathbf{u}}_h)^n \cdot \mathbf{v} dx - \int_\Omega \rho_h^n \bar{\mathbf{u}}_h^n \otimes \bar{\mathbf{u}}_h^n : \nabla_x \mathbf{v} dx - \int_\Omega p(\rho_h^n) div_x \mathbf{v} dx$$
$$+ \mu \int_\Omega (\nabla_h \mathbf{u}_h^n) : \nabla_x \mathbf{v} dx = \mathcal{O}(h^{\beta_2}), \beta_2 > 0.$$

Convergence

Theorem 7

Let $1 < \gamma < 2, \Delta t \approx h,\ 1 < \alpha < 2\gamma - 1$ *and the initial data satisfy*

$$\rho_0 \in L^\infty(R^d),\ \rho_0 \geq \underline{\rho} > 0 \text{ a.a. in } R^d,\ \mathbf{u}_0 \in L^2(R^d).$$

Then any Young measure $\nu_{t,x}$ *generated by the numerical sol of scheme* (3) *represents a dissipative measure-valued solution of NS (1).*

[1]Feireisl et.al. Dissipative measure-valued solutions to the compressible Navier–Stokes system. Calc. Vari. Partial Differ. Equ. 2016

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Applying the weak-strong uniqueness [1] we conclude

Theorem 8

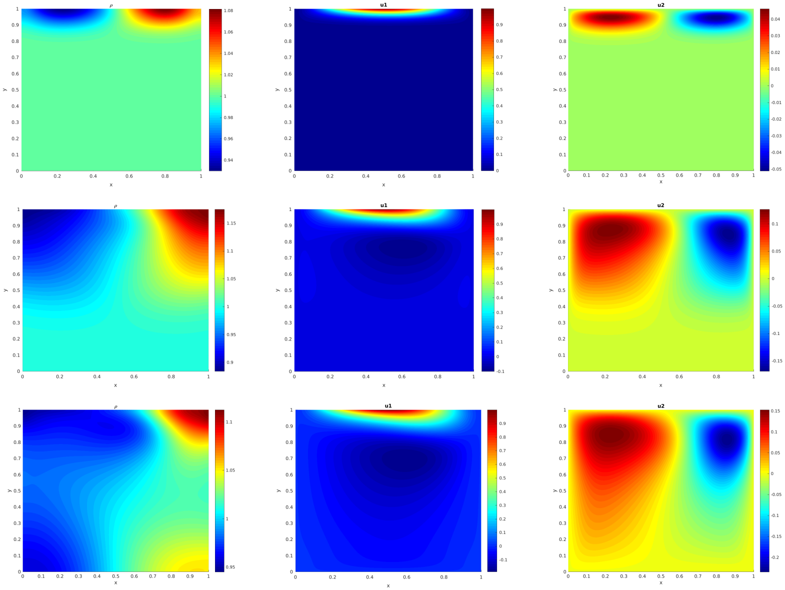
In addition to the hypotheses of Theorem 7, suppose the NS (1) endowed with the periodic boundary condition admits a regular solution.
Then

$$\rho_h \to \rho \text{ (strongly) in } L^\gamma\left((0,T)\times K\right),$$
$$\mathbf{u}_h \to \mathbf{u} \text{ (strongly) in } L^2\left((0,T)\times K; R^d\right)$$

for any compact $K \subset \Omega$.

[1] Feireisl et.al. Dissipative measure-valued solutions to the compressible Navier–Stokes system. Calc. Vari. Partial Differ. Equ. 2016

Test-1 Dirichlet boundary



Test-1 Dirichlet boundary

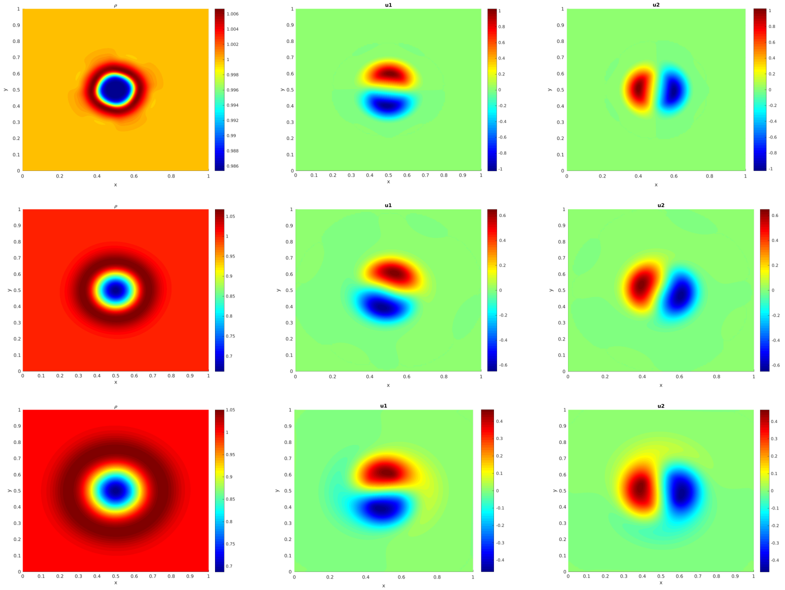
$$\Omega = [0,1]^2, \mu = 0.01, \ a = 1.0, \gamma = 1.4, \alpha = 0.83.$$

Cavity flow, upper boundary $\mathbf{u} = (16x^2(1-x)^2, 0)^T$.

Table: Convergence results

h	$\|\nabla \mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\rho\|_{L^1(L^1)}$	EOC	$\|\rho\|_{L^\infty(L^\gamma)}$	EOC
1/16	6.17e-01	–	4.65e-02	–	7.74e-03	–	4.94e-02	–
1/32	3.08e-01	1.00	2.32e-02	1.00	4.23e-03	0.87	3.19e-02	0.63
1/64	1.51e-01	1.03	1.12e-02	1.05	2.15e-03	0.97	1.96e-02	0.70
1/128	6.60e-02	1.19	4.75e-03	1.23	8.45e-04	1.35	9.97e-03	0.97

Test-2 Periodic boundary



Test-2 Periodic boundary

$$U(0,x,y) = u_r(r) * (y-0.5)/r,$$
$$V(0,x,y) = u_r(r) * (0.5-x)/r.$$

where $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}$ and

$$u_r(r) = \sqrt{\gamma} \begin{cases} 2r/R & \text{if } 0 \le r < R/2, \\ 2(1-r/R) & \text{if } R/2 \le r < R, \\ 0 & \text{if } r \ge R, \end{cases}$$

Table: Convergence results of Gresho vortex test

h	$\|\nabla \mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\rho\|_{L^1(L^1)}$	EOC	$\|\rho\|_{L^\infty(L^\gamma)}$	EOC
1/16	2.23e-01	–	7.84e-03	–	3.19e-06	–	6.66e-03	–
1/32	1.19e-01	0.91	4.09e-03	0.94	1.63e-06	0.97	4.27e-03	0.64
1/64	6.04e-02	0.97	2.01e-03	1.03	5.92e-07	1.46	2.27e-03	0.91
1/128	2.66e-02	1.18	8.98e-03	1.16	2.24e-07	1.40	1.17e-03	0.96

Let $1 < \gamma < 2, \Delta t \approx h,\ 1 < \alpha < 2\gamma - 1$ and the initial data satisfy

$$\rho_0 \in L^\infty(R^d),\ \rho_0 \geq \underline{\rho} > 0 \text{ a.a. in } R^d,\ \mathbf{u}_0 \in L^2(R^d).$$

Then any solution of

$$D_t\varrho^n + \mathrm{div}_{\mathcal{M}}^{\mathrm{up}}(\varrho^n \boldsymbol{u}^n) - h^\alpha \Delta_{\mathcal{M}}\rho^n = 0, \quad 1 < \alpha < 2\gamma - 1,$$

$$D_t(\widehat{\varrho^n}^{(i)} u_i^n) + \mathrm{div}_{\mathcal{E}^{(i)}}^{\mathrm{up}}(\varrho^n \boldsymbol{u}^n u_i^n) - \mu\Delta_{\mathcal{E}}^{(i)} u_i^n$$
$$- (\mu+\lambda)\eth_{\mathcal{E}}^{(i)} \mathrm{div}_{\mathcal{M}}\, \boldsymbol{u}^n + \eth_{\mathcal{E}}^{(i)} p(\varrho^n) - h^\alpha \sum_{j=1}^{d} \eth_{\mathcal{E}}^{(i,j)}(\eth_j \widehat{\varrho^n}^{(i)} u_{i,\epsilon}^n) = 0.$$

generates a dissipative measure-valued solution, and converge to the strong solution if the latter exists.

Let $\gamma > 1, \Delta t \approx h$ and the initial data satisfy

$$\rho_0 \in L^\infty(R^d),\ \rho_0 \geq \underline{\rho} > 0 \text{ a.a. in } R^d,\ \mathbf{u}_0 \in L^2(R^d).$$

Then any solution of

$$\int_\Omega D_t \varrho_h \phi_h \,\mathrm{d}x - \sum_{\sigma \in \Sigma_{int}} \int_\sigma F_h(\varrho_h, \mathbf{u}_h) [\![\phi_h]\!] \,\mathrm{d}Sx = 0 \text{ for any } \phi_h \in Q_h;$$

$$\int_\Omega D_t \mathbf{m}_h \cdot \boldsymbol{\varphi}_h \,\mathrm{d}x - \sum_{\sigma \in \Sigma_{int}} \int_\sigma \mathbf{F}_h(\mathbf{m}_h, \mathbf{u}_h) \cdot [\![\boldsymbol{\varphi}_h]\!] \,\mathrm{d}Sx - \sum_{\sigma \in \Sigma_{int}} \int_\sigma \overline{p_h} \mathbf{n} \cdot [\![\boldsymbol{\varphi}_h]\!] \,\mathrm{d}Sx$$

$$= \mu \sum_{\sigma \in \Sigma_{int}} \int_\sigma \frac{1}{d_\sigma} [\![\mathbf{u}_h]\!] \cdot [\![\boldsymbol{\varphi}_h]\!] \,\mathrm{d}Sx + (\mu + \lambda) \sum_{\sigma \in \Sigma_{int}} \int_\sigma \frac{1}{d_\sigma} [\![\mathbf{u}_h]\!] \cdot \mathbf{n} [\![\boldsymbol{\varphi}_h]\!] \cdot \mathbf{n} \mathrm{d}Sx \text{ for all } \boldsymbol{\varphi}_h \in \mathbf{Q}_h;$$

generates a dissipative measure-valued solution, and converge to the strong solution if the latter exists.

Thank you for your attention!