# EVOLUTIONARY EQUATIONS

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# Chapter 1

# First order transport equation

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As we have seen, a general form of a *balance law* reads

$$\partial_t d + \mathrm{div}_x \mathbf{J} = s,$$

where $d$ is the density of a (macroscopic) quantity, $\mathbf{J}$ denotes the flux, and $s$ is a source term. In this chapter, we consider a very simple example of a balance law

$$\partial_t u(t,x) + \partial_x f(u(t,x)) = s(u(t,x)), \tag{1.1}$$ F1

where the unknown function $u$ depends on the scalar variables $t$ and $x$.

## 1.1 A linear equation with constant speed of propagation

The simplest example of equation (1.1) reads

$$\partial_t u + c\partial_x u = 0, \tag{1.2}$$ F2

where $c$ is a (positive) constant. In addition, we suppose that the initial distribution of $u$ is determined by a given function $u_0$,

$$u(0,x) = u_0(x),\ x \in \mathbb{R}. \tag{1.3}$$ F3

Equation (1.2) asserts that $u$ is constant along the straight lines

$$t \mapsto [t, x + ct],$$

in particular

$$u(t, x + ct) = u_0(x) \text{ for all } t, x \in \mathbb{R}. \tag{1.4}$$ F4

Setting $y = x + ct$ we immediately deduce a solution formula

$$u(t, y) = u_0(y - ct) \text{ for all } t, y \in \mathbb{R}. \tag{1.5}$$ F5

The initial shape $u_0$ is simply shifted along the straight lines $[t, x + ct]$ called *characteristics*

## 1.2 Blow-up for a non-linear equation

Consider a non-linear equation

$$\partial_t u + c\partial_x u = u^2 \text{ for } t, x \in \mathbb{R}, \ u(0, x) = u_0(x), \ x \in \mathbb{R}, \tag{1.6}$$ F6

that may be viewed as a variant of (1.2), supplemented with a source term proportional to $u^2$.

Similarly to problem (1.2), (1.3), equation (1.6) may be integrated along characteristics. More specifically, introducing

$$v(\tau) = u(\tau, x + c\tau),$$

we observe that $v$ must satisfy a simple differential equation

$$\partial_\tau v(\tau) = v^2(\tau), \ v(0) = u_0(x).$$

In other words, $v$ is given through formula

$$v(\tau) = \frac{u_0(x)}{1 - \tau u_0(x)},$$

therefore

$$u(t, x + ct) = \frac{u_0(x)}{1 - tu_0(x)}. \tag{1.7}$$ F7

Formula (1.7) makes sense as long as $1 - tu_0(x)$, in other words, problem (1.2) admits a unique smooth solution defined on the time interval

$$t \in [0, T_{\max}), \ T_{\max} = \infty \text{ provided } u_0 \leq 0, \ T_{\max} = \frac{1}{\sup_{x\in\mathbb{R}} u_0(x)} \text{ otherwise.}$$

Moreover, in the latter case,

$$\sup_{x\in\mathbb{R}} u(t, x) \to \infty \text{ as } t \to T_{\max}.$$

This phenomenon is usually called *blow-up*. In such a case, there does not seem to be any "sensible" way how to continue the solution after the blow-up time. *Non-linear* equations therefore may not admit (smooth) solutions on an arbitrary time interval. A more sophisticated and also more realistic example will be given in the next section.

## 1.3 Shock waves

We now consider a more realistic example transport represented by *Burger's equation*

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) = 0,\ t > 0,\ x \in \mathbb{R}, \tag{1.8}$$ F8

supplemented with the initial condition

$$u(0,x) = u_0(x),\ x \in \mathbb{R}. \tag{1.9}$$ F9

Equation (1.9) may be viewed as a drastically simplified model of the unidirectional motion of, say, cars that move with the speed determined initially by $u_0(x)$. The cars are modeled as a continuum and they are not allowed to "overtake" each other. Intuitively, there are two possibilities:

- The function $u_0$ is monotonically *increasing*, meaning, the cars ahead move with a higher speed, and, accordingly, the motion is smooth.
- There are two points, say, $x_1 < x_2$ such that $u_0(x_1) > u_0(x_2)$, meaning the car occupying initially the position $x_1$ moves faster than that one at $x_2$. It is intuitively clear that such a situation must inevitably provide a "collision" in a finite time.

Motivated by formula (1.4), we suppose that the solution is constant along straight lines, with the slope proportional to $u_0$. Indeed it is easy to check that a function $u$ satisfying

$$u(t, x + u_0(x)t) = u_0(x) \tag{1.10}$$ F10

solves (1.8), (1.9).

Let $x_1 < x_2$ be two different points on the real line. Supposing that $u_0(x_1) > u_0(x_2)$ we find a critical time $t_{\rm crit}$, for which

$$t_{\rm crit} = \frac{x_2 - x_1}{u_0(x_1) - u_0(x_2)},$$

in other words

$$x_1 + u_0(x_1)t_{\rm crit} = x_2 + u_0(x_2)t_{\rm crit} = y.$$

Accordingly, formula (1.10) yields two *different* values of $u(t_{\rm crit}, y)$, namely, $u_0(x_1)$ and $u_0(x_2)$ ! As the solution becomes discontinuous at $t_{\rm crit}$, the life-span of classical solutions to problem (1.8), (1.9) does not exceed, in general, the time

$$T_{\rm max} = \sup_{x\in\mathbb{R}} \frac{1}{|\partial_x u_0(x)|}.$$

Of course, solutions remain smooth for any $t > 0$ as soon as $u_0$ is non-decreasing. This is a mathematical counterpart of the previous discussion concerning the road traffic. The singularity now appears in the derivative of $u$ - a phenomenon called *shock wave* by analogy with problems arising in mathematical fluid mechanics.

## 1.4 Generalized solutions

The almost ubiquitous presence of various singularities in solutions to nonlinear problems initiated a thorough discussion about suitability of the classical concepts of continuum mechanics, based on smooth functions solving systems of partial differential equations at any point of the space-time. The present "crisis" of the theory has apparently two ways out:

- The models based on continuum hypothesis apply with certain limitations. Their success in many numerical experiments is due to the fact that the situations considered were not "extremal". After all, cars never form a continuum no matter how close each other they are. New models or even conceptual approaches are necessary in order to handle a broader class of problems.
- The models are essentially good but the hypothesis of smoothness of solutions is not justified. A new concept of generalized solutions to differential equations is needed.

Pursuing the latter idea we introduce the concept of *weak solution* based on the theory of generalized derivatives (distributions). The leading idea of this approach is to replace the point-wise values of the unknown functions by their integral averages. Practically this means that we multiply the equations by suitable smooth *test function* with compact support and perform by-parts integration.

Thus a weak formulation of equation (1.8) is represented by the integral identity

$$\int_0^\infty \int_{\mathbb{R}} \Big( u(t,x)\partial_t\varphi(t,x) + \frac{1}{2}u^2(t,x)\partial_x\varphi(t,x) \Big) \,\mathrm{d}x\ \mathrm{d}t = 0 \tag{1.11}$$ F11

for any test function $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$. It is worth-noting that (1.11) makes sense if $u$ and $u^2$ are merely locally integrable functions.

Let us examine how the new formulation accommodates possible singularities in the solutions to equation (1.8). Consider the simplest case when the solution $u$ has a jump along a curve $\chi : t \mapsto [t, \chi(t)]$ and is smooth otherwise. Writing

$$0 = \int_0^\infty \int_{\mathbb{R}} \Big( u(t,x)\partial_t\varphi(t,x) + \frac{1}{2}u^2(t,x)\partial_x\varphi(t,x) \Big) \,\mathrm{d}x\ \mathrm{d}t$$

$$= \int_0^\infty \int_{\{x<\chi(t)\}} \Big( u(t,x)\partial_t\varphi(t,x) + \frac{1}{2}u^2(t,x)\partial_x\varphi(t,x) \Big) \,\mathrm{d}x\ \mathrm{d}t$$

$$+ \int_0^\infty \int_{\{x>\chi(t)\}} \Big( u(t,x)\partial_t\varphi(t,x) + \frac{1}{2}u^2(t,x)\partial_x\varphi(t,x) \Big) \,\mathrm{d}x\ \mathrm{d}t$$

and using Gauss-Green theorem we may infer that

$$\int_0^\infty \int_{\{x=\chi(t)\}} \Big( [u(t,\chi(t)-) - u(t,\chi(t)+)]n_t$$

$$+\frac{1}{2}[u^2(t,\chi(t)-) - u^2(t,\chi(t)+)]n_x\Big)\varphi(t,\chi(t))\ \mathrm{d}S_x = 0$$

for any $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$, where $[n_t, n_x]$ is the normal vector to $\chi$. Consequently, the solution $u$ must satisfy the *Rankine-Hugeniot jump conditions*:

$$[u(t,\chi(t)-) - u(t,\chi(t)+)]n_t + \frac{1}{2}[u^2(t,\chi(t)-) - u^2(t,\chi(t)+)]n_x = 0. \quad (1.12) \quad \boxed{\texttt{F12}}$$

### 1.4.1 Example I - Riemann problem

We consider the standard example of discontinuous initial function $u_0$,

$$u_0(x) = \begin{cases} 1 \text{ if } x \le 0, \\ \\ 0 \text{ if } x > 0 \end{cases} \quad (1.13) \quad \boxed{\texttt{F13}}$$

Finding solutions of equation (1.8) emanating from the initial datum specified in (1.13) is usually called *Riemann problem.*

We suppose that the points of discontinuity of the weak solution are located on a curve $\chi : t \mapsto [t, \chi(t)]$ such that $\chi(0) = 0$, while

$$u(t,x) = \begin{cases} 1 \text{ if } x < \chi(t), \\ \\ 0 \text{ if } x > \chi(t) \end{cases}$$

Thus $u$ is a weak solution in the sense of (1.11) as soon as the Rankine-Hugeniot conditions (1.12) hold. However, (1.12) reduces to a simple relation $n_t + \frac{1}{2}n_x = 0$, in other words, $\chi$ is a straight-line

$$\chi : t \mapsto [t, \frac{1}{2}t], t \ge 0.$$

Note that shock waves - solutions emanating from the initial datum $\lambda u_0$, $\lambda > 0$ - propagate with different speeds related to the value of $\lambda$.

### 1.4.2 Example II - uniqueness failure

The class of weak solutions is apparently larger than the class of classical ones. Consider the initial datum in the form

$$u_0(x) = \begin{cases} 0 \text{ if } x \le 0, \\ \\ 1 \text{ if } x > 0. \end{cases} \quad (1.14) \quad \boxed{\texttt{F14}}$$

In virtue of the arguments used in Example I, problem (1.8), (1.14) admits a piecewise constant (discontinuous) solution

$$u_1(t,x) = \begin{cases} 0 \text{ if } x < t/2, \\ \\ 1 \text{ if } x > t/2. \end{cases}$$

On the other hand, the same problem possesses a continuous solution

$$u_2(t,x) = \begin{cases} 0 \text{ if } x \le 0, \\ \lambda \text{ if } x = \lambda t,\ 0 < \lambda < 1 \\ 1 \text{ if } x \ge 1. \end{cases}$$

We have constructed *two* different weak solutions to the same problem! Obviously, one is tempted to say that $u_2$ is the "correct" solution as it is more regular, however, the selection problem is more subtle and requires a deep understanding of the physical background of the concrete model.

## 1.5 Selection criteria

We examine various criteria how identify the "physical" solution of equation (1.8).

### 1.5.1 Maximal energy dissipation

Revoking the analogy with cars, we compute the kinetic energy density of the system proportional to $\frac{1}{2}|u|^2$. Multiplying equation (1.8) on $u$ we deduce the kinetic energy balance in the form

$$\partial_t \left( \frac{1}{2} u^2 \right) + \partial_x \left( \frac{1}{3} u^3 \right) = 0. \tag{1.15}$$ F15

Equations (1.8), (1.15) are equivalent provided the solution $u$ is regular. This may not be the case in the class of discontinuous weak solutions. Note that the weak formulation of (1.15) reads

$$\int_0^\infty \int_{\mathbb{R}} \left( \left( \frac{1}{2} u^2 \right) \partial_t \varphi + \left( \frac{1}{3} u^3 \right) \partial_x \varphi \right) \ \mathrm{d}x \ \mathrm{d}t = 0 \tag{1.16}$$ F16

for any $\varphi \in C_c^\infty((0,\infty) \times \mathbb{R})$.

Let us examine the situation discussed in Example II. It is easy to check that solution $u_2$ "conserves" the kinetic energy as $u_2$ clearly satisfies (1.16).

On the other hand, however, we get

$$\int_0^\infty \int_{\mathbb{R}} \left( \left( \frac{1}{2} u_1^2 \right) \partial_t \varphi + \left( \frac{1}{3} u_1^3 \right) \partial_x \varphi \right) \ \mathrm{d}x \ \mathrm{d}t = -\frac{1}{6} \int_{\{x = t/2\}} \varphi(t, t/2) \mathrm{d}S_x \tag{1.17}$$ F17

for any $\varphi \in C_c^\infty((0,\infty) \times \mathbb{R})$. Relation (1.17) may be interpreted as

$$\partial_t \left( \frac{1}{2} u_1^2 \right) + \partial_x \left( \frac{1}{3} u_1^3 \right) = \frac{1}{6} \times [\text{positive measure supported on } \chi],$$

or, simply,

$$\int_0^\infty \int_{\mathbb{R}} \left( \left( \frac{1}{2} u_1^2 \right) \partial_t \varphi + \left( \frac{1}{3} u_1^3 \right) \partial_x \varphi \right) \ \mathrm{d}x \ \mathrm{d}t \le 0$$

for any $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$, $\varphi \geq 0$ that may be viewed as a weak formulation of

$$\partial_t \left(\frac{1}{2}u_1^2\right) + \partial_x \left(\frac{1}{3}u_1^3\right) \geq 0.$$

We conclude that solution $u_1$ produces energy in contrast with the commonly accepted physical principles. Obviously, it is the function $u_2$ that represents a physically admissible solution.

Finally, it is interesting to observe that the shock wave solution $u$ constructed in Example I dissipates energy, specifically,

$$\partial_t \left(\frac{1}{2}u^2\right) + \partial_x \left(\frac{1}{3}u^3\right) \leq 0.$$

In the presence of shocks, the kinetic energy is converted into heat, in accordance with the second law of thermodynamics.

### 1.5.2 Entropy solutions

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Consider a more general equation

$$\partial_t u(t,x) + \partial_x f(u(t,x)) = 0 \text{ in } (0,\infty)\times\mathbb{R}, \tag{1.18}$$ F18

with the initial condition

$$u(0,x) = u_0(x),\ x \in \mathbb{R}. \tag{1.19}$$ F19

Motivated by the previous discussion, we take a convex function $E$ and multiply (1.18) on $E'$ to obtain

$$\partial_t E(u(t,x)) + \partial_x F(u(t,x)) = 0 \text{ where } F'(u) = f'(u)E'(u).$$

The function $E$ is termed *entropy* and $F$ is the entropy *flux*.

We shall say that $u$ is an *entropy solution* to (1.19) if

$$\int_0^\infty \int_{\mathbb{R}^1} \Big(E(u)\partial_t\varphi + F(u)\partial_x\varphi\Big)\mathrm{d}x\ \mathrm{d}t \geq 0 \tag{1.20}$$ F20

for any $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$, $\varphi \geq 0$, and any convex entropy $E$, with the associated flux $F$.

Relation (1.20) can be interpreted as

$$\partial_t E(u) + \partial_x F(u) \leq 0 \text{ in the sense of generalized derivatives.}$$

Clearly, (1.20) provides more information than just the weak formulation of (1.19) that is included as a special case $E(u) = \pm u$. It is interesting to note that for the *linear* equation (1.2), the entropy formulation yields

$$\partial_t E(u) + c\partial_x E(u) = 0,$$

which is nothing other that (1.2) as all solutions are constant along characteristic lines. As we shall see in the next section, however, the fact that the entropy is different from the flux in non-linear equations produces an important compactification effect. Unlike (1.2), the sign "$\leq$" in (1.20) specifies the arrow of time in many physical problems. To avoid confusion, the *physical* entropies are, by definition, concave therefore satisfying (1.20) in the opposite sense. In other words, the physical entropy is being produced rather than dissipated.

### 1.5.3 Viscosity solutions

Another possibility how to identify the relevant class of weak solutions is to regard equation (1.18) as a limit case of a more complex problem. It turns out that a suitable approximation is provided by adding *artificial viscosity* to the problem. The resulting equation reads

$$\partial_t u(t,x) + \partial_x f(u(t,x)) = \varepsilon \Delta u(t,x), \ \Delta = \partial^2_{x,x}, \tag{1.21}$$ F21

where $\varepsilon > 0$ is a small parameter. Equation (1.18) is then formally recovered as the asymptotic limit $\varepsilon \to 0$.

We recall that the solutions of the linear equation

$$\partial_t v(t,x) = \Delta v(t,x), \ v(0,x) = v_0(x) \tag{1.22}$$ F22

are given by the explicit formula

$$v(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{|x-y|^2}{4t}\right) v_0(y) \ \mathrm{d}y, \tag{1.23}$$ F23

in particular, they are smooth even for rather irregular initial data. We may therefore conjecture that the same property is likely to hold also for problem (1.21).

Note in passing that the artificial viscosity regularization, in general, does not prevent blow-up of solutions in the supremum norm. Indeed consider

$$\partial_t v + c\partial_x v = \varepsilon \Delta v + v^2. \tag{1.24}$$ F24

Now, spatially homogeneous functions $v = v(t)$ satisfying

$$\partial_t v(t) = v^2(t)$$

represent particular solutions of (1.24).

Now, similarly to Section 1.5.2, let us multiply (1.21) by a function $E'(u)$ to obtain

$$\partial_t E(u) + \partial_x F(u) = \varepsilon \partial_x \Big( E'(u)\partial_x u \Big) - E''(u)|\partial_x u|^2.$$

If $E$ is convex, the last term is non-positive, and we obtain

$$\partial_t E(u) + \partial_x F(u) \leq \varepsilon \partial_x \Big( E'(u)\partial_x u \Big) \tag{1.25}$$ F25

for any entropy-flux pair $E$, $F$. Relation (1.25) is, of course, reminiscent of (1.20). More specifically, property (1.20) is likely to be inherited by any weak solution obtained by means of the asymptotic limit $\varepsilon \to 0$. In the ideal case, both solutions coincide and (1.20) would represent an *intrinsic* definition of a viscosity solutions. However, such a property has been rigorously verified only for some special classes of equations including (1.20).

## 1.6 Weak sequentially stability of bounded sets of entropy solutions

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In order to avoid problems with "large" domains, we consider problems supplemented with the periodic boundary conditions. Given a non-linear equation

$$\partial_t u(t,x) + \partial_x f(u(t,x)) = 0, \tag{1.26}$$ w1

we look for solutions satisfying

$$u(t, x + 2\pi) = u(t,x) \text{ for all } t > 0, \ x \in \mathbb{R}.$$

Accordingly, the initial data $u_0(x)$ obey the same periodicity condition

$$u_0(x + 2\pi) = u_0(x).$$

The specific choice of the period length $2\pi$ is for convenience, clearly, any other period can be treated in the same way. We may also say that equation (1.26) is considered on the "flat" torus

$$\mathbb{T}^1 = [0, 2\pi]|_{\{0,2\pi\}}.$$

### 1.6.1 Uniform bounds on entropy solutions

*Entropy solution* of equation (1.26), considered on $\mathbb{T}^1$, can be defined in the same way as in Section 1.5.2. Specifically, the integral identity

$$\int_0^\infty \int_{\mathbb{T}^1} \Big( E(u)\partial_t \varphi + F(u)\partial_x \varphi \Big) \mathrm{d}x \ \mathrm{d}t \geq 0 \tag{1.27}$$ w2

for any $\varphi \in C_c^\infty((0,\infty) \times \mathbb{T}^1)$, $\varphi \geq 0$, and any convex entropy $E$, with the associated flux $F$.

It seems convenient to incorporate the initial condition $u(0,x) = u_0(x)$ in the weak formulation. This can be achieved by extending conveniently the set of admissible test functions.

We shall say that $u$ is an *entropy solution* of equation (1.26), supplemented with the periodic boundary conditions and the initial condition $u(0,x) = u_0(x)$, if the integral identity

$$\int_0^\infty \int_{\mathbb{T}^1} \Big( E(u)\partial_t \varphi + F(u)\partial_x \varphi \Big) \mathrm{d}x \ \mathrm{d}t \geq - \int_{\mathbb{T}^1} E(u_0(x))\varphi(0,x) \ \mathrm{d}x \tag{1.28}$$ w3

holds for any $\varphi \in C_c^\infty([0,\infty) \times \mathbb{T}^1)$, $\varphi \geq 0$, and any convex entropy $E$, with the associated flux $F$.

The system of integral identities (1.28) provides a uniform bound for any entropy solution in terms of the data. Given $T > 0$, consider a family of functions

$$\varphi_\varepsilon = \varphi_\varepsilon(t) = \begin{cases} 1 \text{ for } t < T - \varepsilon \\ \\ \text{non-increasing and smooth in } [T-\varepsilon, T] \\ \\ 0 \text{ for } t > T. \end{cases}$$

Taking $\varphi_\varepsilon$ as a test function in (1.28) and letting $\varepsilon \to 0$ we obtain

$$\int_{\mathbb{T}^1} E(u)(t,\cdot)\ \mathrm{d}x \leq \int_{\mathbb{T}^1} E(u_0)\ \mathrm{d}x \text{ for a.a. } t \in (0,T). \tag{1.29}$$ w4

Relation (1.29) yields boundedness of entropy solutions in terms of the initial data.

Pw1 **Proposition 1.1** *Let $u \in L^\infty((0,T) \times \mathbb{T}^1)$ satisfy (1.28). Then*

$$\operatorname{ess}\inf_{x\in\mathbb{T}^1} u_0(x) \leq u(t,x) \leq \operatorname{ess}\sup_{x\in\mathbb{T}^1} u_0(x) \textit{ for a.a. } t \in (0,\infty),\ x \in \mathbb{T}^1. \tag{1.30}$$ w5

**Proof:** We consider a convex function $u \mapsto |u-c|^+$. It follows from (1.29) that

$$|u(t,x) - c|^+ = 0 \text{ provided } |u_0(x) - c| \leq 0 \text{ for a.a. } t > 0.$$

Taking $c = \operatorname{ess}\sup_{x\in\mathbb{R}} u_0(x)$ we therefore obtain the upper bound in (1.30). The lower bound can be deduced in a similar way replacing $|u-c|^+$ by $|u-c|^-$.

q.e.d.

### 1.6.2 Weak continuity in time

It follows from (1.28) that the function

$$\left[t \mapsto \int_{\mathbb{T}^1} u(t,x)\varphi(x)\ \mathrm{d}x\right] \text{ is continuous in time for any fixed } \varphi \in C_c^\infty(\mathbb{R}^1). \tag{1.31}$$ w6

Relation (1.31) is termed *weak continuity* in time. Using density of smooth functions in $L^p(\mathbb{T}^1)$ we may infer that

$$u \in C_{\mathrm{weak}}([0,T]; L^p(\mathbb{T}^1)) \text{ for any } 1 \leq p < \infty \text{ whenever } u \in L^\infty((0,T) \times \mathbb{T}^1). \tag{1.32}$$ w7

In other words, the mapping $t \mapsto u(t,\cdot)$ is continuous as a mapping of the time interval $[0,T]$ into the Lebesgue space $L^p(\mathbb{T}^1)$ endowed with the weak topology. Accordingly, we may speak about *instantaneous values* of $u(t,\cdot)$ for *any* time $t \geq 0$.

### 1.6.3 Weak compactness of bounded families of solutions

Our main goal in this section is to show the following remarkable result.

Tw1 **Theorem 1.1** *Let* $\{u_n\}_{n=1}^{\infty}$ *be a family of entropy solutions to equation (1.26) such that*

$$\|u_n\|_{L^\infty((0,T)\times\mathbb{T}^1)} \leq M \text{ uniformly for all } n = 1, 2, \dots$$

*Assume that* $f$ *is a twice continuously differentiable function such that* $f''(u) > 0$ *for all* $u \in \mathbb{R}^1$.

*Then there exists a subsequence (not relabeled) such that*

$$u_n \to u \text{ a.a. in } (0,T) \times \mathbb{T}^1.$$

Theorem 1.1 is a remarkable results. To begin, it shows compactness of bounded sets of entropy solutions without any specific constraints imposed on the initial data. Such a result *does not* hold for linear equations and leans heavily on the strict convexity of the flux function $f$. This observation fits well into the abstract framework of the weak solutions proposed by DiPerna [1]. In many problems of mathematical physics, the linear, in terms of the entropy and flux, character of the balance laws interferes with non-linear constitutive relations represented here by the flux function $f$. Different speed of shock waves leads eventually to their mutual cancellation and the solution set becomes compact. Unfortunately, however, such a nice property is known to hold for an extremely narrow class of problems.

The rest of this section is devoted to the proof of Theorem 1.1.

### 1.6.4 Compensated compactness

It is well known that Sobolev spaces $W^{k,p}(\Omega)$ of functions having $k$ generalized derivatives in the Lebesgue space $L^p$ enjoy certain compactness properties stated in Rellich-Kondrashev theorem. Compensated compactness assumes boundedness of generalized derivatives only in certain specific direction whereas the results are weaker than for classical Sobolev spaces. The best known result in this direction is the celebrated Div-Curl lemma by Murat and Tartar [4], [6].

Div-Curl lemma - soft version

Lw1 **Lemma 1.1** *Assume that* $B \subset \mathbb{R}^N$ *is an open set. Let*

$$\mathbf{U}_n \to \mathbf{U} \text{ weakly in } L^2(B;\mathbb{R}^N),\ \mathbf{V}_n \to \mathbf{V} \text{ weakly in } L^2(B;\mathbb{R}^N),$$

$$\mathrm{div}_x \mathbf{U}_n = 0,\ \mathbf{curl} \mathbf{V}_n = 0 \text{ in } B.$$

*Then*

$$\int_B \mathbf{U}_n \cdot \mathbf{V}_n \varphi \, \mathrm{d}x \to \int_B \mathbf{U} \cdot \mathbf{V} \varphi \mathrm{d}x \text{ for any } \varphi \in C_c^\infty(B).$$

**Proof:**

Without loss of generality, we may assume that $B$ is bounded, regular, and simply connected. In such a case, as $\mathbf{curl V}_n = 0$, there exists a potential

$$\Phi_n,\ \mathbf{V}_n = \nabla_x \Phi_n,\ \{\Phi_n\}_{n=1}^\infty \text{ bounded in } W^{1,2}(B).$$

By virtue of the standard compactness embedding relations, we may assume that

$$\Phi_n \to \Phi \text{ (strongly) in } L^2(B; \mathbb{R}^N),\ \text{ where } \nabla_x \Phi = \mathbf{V}.$$

Now, write,

$$\int_B \mathbf{U}_n \cdot \mathbf{V}_n \varphi \, \mathrm{d}x = \int_B \mathbf{U}_n \cdot \nabla_x \Phi_n \varphi \, \mathrm{d}x = - \int_B \Phi_n \mathbf{U}_n \cdot \nabla_x \varphi \, \mathrm{d}x,$$

where

$$\int_B \Phi_n \mathbf{U}_n \cdot \nabla_x \varphi \, \mathrm{d}x \to \int_B \Phi \mathbf{U} \cdot \nabla_x \varphi \, \mathrm{d}x = - \int_B \mathbf{U} \cdot \mathbf{V} \varphi \, \mathrm{d}x$$

q.e.d.

Unfortunately, the hypotheses of Lemma 1.1 are to strong to be applicable to our problem.

Div-Curl lemma - hard version

Lw2 **Lemma 1.2** *Assume that $B \subset R^N$ is a bounded domain. Let*

$$\mathbf{U}_n \to \mathbf{U} \text{ weakly in } L^2(B; \mathbb{R}^N),\ \mathbf{V}_n \to \mathbf{V} \text{ weakly in } L^2(B; \mathbb{R}^N),$$

$$\{\mathrm{div}_x \mathbf{U}_n\}_{n=1}^\infty,\ \{\mathbf{curl V}_n\}_{n=1}^\infty \text{ precompact in } W^{-1,2}(B).$$

*Then*

$$\int_B \mathbf{U}_n \cdot \mathbf{V}_n \varphi \, \mathrm{d}x \to \int_B \mathbf{U} \cdot \mathbf{V} \varphi \mathrm{d}x \text{ for any } \varphi \in C_c^\infty(B).$$

**Remark** Since the result is local, we can assume that $B$ is any open set supposing precompactness in $W^{-1,2}_{\mathrm{loc}}(B)$.

**Proof:**

Write

$$\mathbf{U}_n = \Big( \mathbf{U}_n - \nabla_x \Delta_D^{-1} \mathrm{div}_x \mathbf{U}_n \Big) + \nabla_x \Delta_D^{-1} \mathrm{div}_x \mathbf{U}_n,$$

and, similarly,

$$\mathbf{V}_n = \Big( \mathbf{V}_n - \nabla_x \Delta_D^{-1} \mathrm{div}_x \mathbf{V}_n \Big) + \nabla_x \Delta_D^{-1} \mathrm{div}_x \mathbf{V}_n,$$

where the symbol $\Delta_D$ denotes the Laplace operator endowed with the homogeneous Dirichlet boundary condition on $\partial B$.

It follows from the standard elliptic regularity theory that

$$\{\nabla_x \Delta_D^{-1} \mathrm{div}_x \mathbf{U}_n\}_{n=1}^{\infty} \text{ is precompact in } L^2(B; R^N),$$

while,

$$\mathrm{div}_x \Big( \mathbf{V}_n - \nabla_x \Delta^{-1} \mathrm{div}_x \mathbf{V}_n \Big) = 0, \ \mathbf{curl} \Big( \mathbf{V}_n - \nabla_x \Delta^{-1} \mathrm{div}_x \mathbf{V}_n \Big) = \mathbf{curl} \mathbf{V}_n;$$

whence, by the same token,

$$\{\mathbf{V}_n - \nabla_x \Delta^{-1} \mathrm{div}_x \mathbf{V}_n\}_{n=1}^{\infty} \text{ is precompact in } L^2(B; R^N).$$

On the other hand,

$$\mathrm{div}_x \Big( \mathbf{U}_n - \nabla_x \Delta_D^{-1} \mathrm{div}_x \mathbf{U}_n \Big) = 0, \ \mathbf{curl} \nabla_x \Delta_D^{-1} \mathrm{div}_x \mathbf{V}_n = 0,$$

and we may apply Lemma 1.1 to conclude the proof.

q.e.d

What Lemma 1.1 says is that product of two weakly converging sequences tends to the product of their corresponding weak limits as soon as we can show that their possible oscillations direction are orthogonal.

### 1.6.5 Reformulation of the entropy formulation in terms of "defect" measures

The entropy inequality (1.27) may be interpreted in terms of a "defect" measure supported by possible discontinuities of the solution. Indeed (1.27) expresses the fact that the distribution

$$\partial_t E(u) + \partial_x F(u) \text{ is non-positive.}$$

In accordance with the well known observation, non-negative distributions may be represented by a Radon measure $\mu_F$, specifically,

$$\partial_t E(u) + \partial_x F(u) = -\mu_F,$$

or, more precisely,

$$\int_0^\infty \int_{\mathbb{T}^1} \Big( E(u) \partial_t \varphi + F(u) \partial_x \varphi \Big) \mathrm{d}x \ \mathrm{d}t = < \mu_F; \varphi > \tag{1.33}$$ w8

for any $\varphi \in C_c^\infty((0, \infty) \times \mathbb{T}^1)$.

### 1.6.6 Tartar's equation

Consider a bounded (regular) domain $B \subset (0,T) \times \mathbb{T}^1$. Since the Sobolev space $W_0^{1,p}(B)$ is *compactly* embedded into the space $C(\overline{B})$ of continuous functions for any $p > 2$ we may infer that

$$\{\partial_t E(u_n) + \partial_x F(u_n)\}_{n=1}^{\infty} \text{ is precompact in } W^{-1,q}(B) \text{ for any } 1 \leq q < 2, \tag{1.34}$$ w9

where $\{u_n\}_{n=1}^{\infty}$ is the bounded family of entropy solutions introduced in Theorem 1.1.

On the other hand, since $u_n$ are uniformly bounded in $L^\infty$, we have

$$\{\partial_t E(u_n) + \partial_x F(u_n)\}_{n=1}^{\infty} \text{ bounded } W^{-1,r}(B) \text{ for any } r \geq 1. \tag{1.35}$$ w10

Interpolating (1.34), (1.35) we conclude that

$$\{\partial_t E(u_n) + \partial_x F(u_n)\}_{n=1}^{\infty} \text{ is precompact in } W^{-1,2}(B) \tag{1.36}$$ w11

for any entropy-flux pair $E, F$. Thus we are allowed to apply Div-Curl lemma (Lemma 1.2) to deduce that

$$E_1(u_n)F_2(u_n) - E_2(u_n)F_1(u_n) \to \overline{E_1(u)}\ \overline{F_2(u)} - \overline{E_2(u)}\ \overline{F_1(u)} \text{ weakly in } L^2(B) \tag{1.37}$$ w12

for any choice of entropy flux pairs $E_i$, $F_i$, $i = 1, 2$, where we have used the standard notation $\overline{B(u)}$ for a weak limit of the sequence of compositions $\{B(u_n)\}_{n=1}^{\infty}$.

Relation (1.37) is the celebrated *Tartar's equation*

$$\overline{E_1(u)F_2(u) - E_2(u)F_1(u)} = \overline{E_1(u)}\ \overline{F_2(u)} - \overline{E_2(u)}\ \overline{F_1(u)} \tag{1.38}$$ w13

for any convex $E_1$, $E_2$ and the corresponding fluxes $F_1' = E_1' f$, $F_2' = E_2' f$.

Given the rich variety of entropy-flux pairs - $E$ may be any convex function - relation (1.38) is so restrictive that it implies strong (a.a. pointwise) compactness of the sequence $\{u_n\}_{n=1}^{\infty}$. Indeed, for fixed $(t,x)$, we take

$$E_1(z) = z,\ F_1(z) = f(z),$$

$$E_2(z) = |z - U|,\ F_2(z) = \mathrm{sgn}(z - U)(f(z) - f(U)),\ U \text{ constant},$$

where $u$ denotes a weak limit of the sequence $\{u_n\}_{n=1}^{\infty}$. Applying (1.38) we obtain

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{T}^1} \Big(\overline{f(u)} - f(U)\Big)|u_n - U|\varphi \ \mathrm{d}x \ \mathrm{d}t \tag{1.39}$$ w14

$$= \int_0^T \int_{\mathbb{T}^1} (u - U)\overline{\mathrm{sgn}(u - U)(f(u) - f(U))} \ \mathrm{d}x \ \mathrm{d}t.$$

for any $U$ and any $\varphi \in C_c^\infty((0,T) \times \mathbb{T}^1)$. In particular, relation (1.39) holds for $U = u(\tau, y)$.

Now, we may deduce

$$\lim_{r\to 0}\frac{1}{|B_r(\tau,y)|}\left(\lim_{n\to\infty}\int_{B_r(\tau,y)}\Big(\overline{f(u)}-f(u(\tau,y))\Big)|u_n-u(\tau,y)|\ \mathrm{d}x\ \mathrm{d}t\right)=0;$$

whence, at any Lebesgue point $(\tau, y)$ of the function $u$,

$$\lim_{r\to 0}\frac{1}{|B_r(\tau,y)|}\left(\lim_{n\to\infty}\int_{B_r(\tau,y)}\Big(\overline{f(u)}-f(u)\Big)|u_n-u|\ \mathrm{d}x\ \mathrm{d}t\right)=0. \tag{1.40}$$ w15

We recall that $(\tau, y)$ is a Lebesgue point of $u$ if

$$\lim_{r\to 0}\frac{1}{|B_r(\tau,y)|}\int_{B_r(\tau,y)}|u-u(\tau,y)|\ \mathrm{d}x\ \mathrm{d}t=0.$$

Relation (1.40) in turn implies that

$$(\overline{f(u)}-f(u))w=0 \text{ a.a. in } (0,T)\times\mathbb{T}^1,$$

where $w$ denotes a weak limit of $\{|u_n-u|\}_{n=1}^{\infty}$. In particular,

$$u_n\to u \text{ a.a. in the set } \{(t,x)\mid \overline{f(u)}\neq f(u)\}. \tag{1.41}$$ w16

Finally, since $f$ is strictly convex, we have

$$f(u_n)-f(u)=f'(u)(u_n-n)+f''(\xi)|u_n-u|^2$$

for a certain $\xi$. As $u_n$, $u$ are bounded, we deduce that

$$u_n\to u \text{ whenever } f(u)=\overline{f(u)}. \tag{1.42}$$ w17

Relations (1.41), (1.42) imply pre-compactness of the sequence $\{u_n\}_{n=1}^{\infty}$ in $L^1((0,T)\times\mathbb{T}^1$ and complete the proof of Theorem 1.1.

## 1.7 Exercises

ex

**Exercise 1:** Let $u\in L^\infty((0,T)\times\mathbb{R}^1)$ be a weak (distributional) solution of equation (1.18), with continuously differentiable flux function $f$. Show that $u$ is weakly continuous in time, specifically, after modification in a zero set of times if necessary, the function

$$t\mapsto\int_{\mathbb{R}^1}u(t,x)\varphi(x)\ \mathrm{d}x$$

is continuous in $[0,T]$ for any $\varphi\in C_c^\infty(\mathbb{R}^1)$.

---

**Exercise 2:** Let $u \in L^\infty((0,T) \times \mathbb{T}^1)$ be a weak (distributional) space periodic solution of equation (1.18). Show that the $L^1$-norm is a constant of motion, meaning

$$\int_{\mathbb{T}^1} u(t_1, x) \, \mathrm{d}x = \int_{\mathbb{T}^1} u(t_2, x) \, \mathrm{d}x$$

for any $t_1, t_2$.

---

**Exercise 3:** Let $u_\varepsilon$ be smooth solutions of the viscous regularization

$$\partial_t u_\varepsilon(t,x) + \partial_x f(u_\varepsilon(t,x)) = \varepsilon \Delta_x u_\varepsilon(t,x) \text{ in } (0,T) \times \mathbb{T}^1$$

such that

$$\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{T}^1)} \leq M$$

Show that Div-Curl lemma can be used to establish *precompactness* of the family $\{u_\varepsilon\}_{\varepsilon>0}$ in the Lebesgue space $L^1((0,T) \times \mathbb{T}^1)$.

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**Exercise 4:** Use Div-Curl lemma to show the following "Lions-Aubin" like compactness result:

Let $u_\varepsilon$ be a weak solution of

$$\partial_t u_\varepsilon(t,x) = \Delta_x u_\varepsilon(t,x) \text{ in } (0,T) \times \mathbb{T}^1$$

such that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{T}^1)} + \int_0^T \|\partial_x u_\varepsilon\|^2_{L^2(\mathbb{T}^1)} \, \mathrm{d}t \leq M$$

uniformly for $\varepsilon \to 0$.

Then $\{u_\varepsilon\}_{\varepsilon>0}$ is precompact in $L^2((0,T) \times \mathbb{T}^1)$.

# Chapter 2

# A brief introduction to the theory of dynamical systems

D

The central object of mathematical modeling is a *system* to be described. Systems may originate in physics, chemistry, biology and other areas as the case may be. The *state* of a system at an instant $t \in R$ is (believed to be) described as a point $U(t, )$ in an abstract *phase space* $X$. In *deterministic models*, the state at any time $t > 0$ is uniquely determined by the state $U_0$ at $t = 0$ and the time $t$; we write $U = U(t, U_0)$.

## 2.1 Standard approach via semigroups

For simplicity, we restrict ourselves to autonomous system whose characteristics do not change in time. Accordingly, the dynamical system $\{U(t, \cdot)\}_{t>0}$ enjoys the following properties:

- $U(0, \cdot) = \mathbb{I}$, in other words, $U(0, U_0) = U_0$ for any $U_0 \in X$;
- $U(t + s, U_0) = U(t, U(s, U_0))$ for any $s, t \geq 0$ yielding $U(t + s, U_0) = U(s, U(t, U_0))$, meaning the mappings $U(t, \cdot)$ commute;
- the mapping $t \mapsto U(t, U_0)$ is continuous for any fixed $U_0$; the mapping $U_0 \mapsto U(t, U_0)$ is continuous for any fixed $t \geq 0$.

In the case when $U(t, U_0)$ represents the value at the time $t$ of a solution of an evolutionary differential equation or system emanating from the initial state $U_0$, the mappings $\{U(t, \cdot)\}_{t \geq 0}$ are called *solution semigroup*. Note that the property of continuity requires a kind of "distance" or topology to be defined in the phase space $X$. Typically, $X$ is a Banach space endowed with a norm topology. However it may happen, and we have seen several examples in the previous text, that it is convenient to consider the *weak* topology that replaces the standard "coordinate-wise" continuity. Moreover, the semigroup property obviously includes *uniqueness* of solutions in terms of the initial data. Although uniqueness is an indispensable attribute of any deterministic system, in many important applications solutions are not (known to be) unique. On the other hand, as we shall see below, many of the classical concepts of the theory of dynamical systems as absorbing set or even attractor can be developed with a minimum piece of information available concerning the associated solution semigroup.

### 2.1.1 Conservative vs. dissipative dynamical system

In order to fix ideas, assume that $X$ is a Banach space endowed with norm $\|\cdot\|_X$. We say that a dynamical system is *conservative* if

$$\|U(t, U_0)\|_X = \|U_0\| \text{ for any } t \geq 0,\ U_0 \in X.$$

**Example 2.1**
Consider the linear transport equation

$$\partial_t u(t, x) + c\partial_x u(t, x) = 0,\ u(0, x) = u_0(x)$$

endowed with the space-periodic boundary conditions discussed in Chapter 1. AS we have seen, the (unique) solution $u$ can be written as

$$u(t, x) = u_0(x - ct).$$

Clearly, the solution is classical provided $u_0$ is continuously differentiable. In such a case, it is easy to check that

$$\int_{\mathbb{T}^1} F(u(t, x))\ \mathrm{d}x = \int_{\mathbb{T}^1} F(u_0(x))\ \mathrm{d}x.$$

Thus the solution semigroup $U(t, u_0) = u(t, \cdot)$ can be extended as a conservative dynamical system on any space $L^p(\mathbb{T}^1)$ and also $C(\mathbb{T}^1)$, in particular, it conserves the "energy" norm in $L^2(\mathbb{T}^1)$.

Typically, conservative systems may be extended even for $t \leq 0$ forming a *group*.

---

A dynamical system is called *dissipative* provided it possesses a *bounded absorbing set* $\mathcal{B}_a$. A subset $\mathcal{B}_a \subset X$ is absorbing if for any bounded set $B$ there is a time $t_0(B)$ such that

$$U(t, U_0) \in \mathcal{B}_0 \text{ for any } U_0 \in B \text{ and any } t \geq t_0(B).$$

**Example 2.2**

Consider a parabolic equation

$$\partial_t u(t,x) - \partial^2_{x,x} u(t,x) = f(x),\ u(0,x) = u_0(x),$$

with the Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0 \text{ for all } t.$$

Multiplying the equation by $u$ and integrating by parts, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}u^2(t,x)\ \mathrm{d}x + \int_0^1 |\partial_x u|^2\ \mathrm{d}x = \int_0^1 f(x)u(t,x)\ \mathrm{d}x.$$

Now, we can use *Poincaré's inequality*

$$\int_0^1 |\partial_x u|^2\ \mathrm{d}x \geq a\int_0^1 u^2\ \mathrm{d}x,\ a > 0,$$

together with the Hölder's inequality

$$ab \leq \frac{\varepsilon^2}{2}a^2 + \frac{1}{2\varepsilon^2}b^2$$

to conclude that

$$\chi'(t) + d\chi(t) \leq B,\ \text{where we have set } \chi(t) = \int_0^1 u^2(t,x)\ \mathrm{d}x$$

Thus the associated dynamical system is dissipative in $X = L^2(0,1)$. Of course, we left open the problem of existence and uniqueness of solutions with the data in the aforementioned space.

It is worth-noting that the concept of absorbing set can be introduced without continuity and even without the semigroup property of the underlying dynamical system.

## 2.2 Attractors

The notion of attractors have been originally introduced in problems connected with fluid dynamics, in particular in meteorological models. There was a strong hope attractors may shed some light on the complex phenomena related to turbulence, see Eckmann and Ruelle [2], Robinson [5]. However, we should always keep in mind that attractors describe the behavior of the corresponding system for a very large time comparable to the "age of universe", while the real world applications take place on much shorter time scales.

Attractor is a *compact set* in the phase space $X$ that "attracts" all solutions emanating from a bounded set of initial data. Clearly, the existence of an attractor is conditioned by the existence of a bounded absorbing set. Let us introduce *attractor* in an intuitive way. Suppose we know that a dynamical system $\{U(t,\cdot)\}_{t\geq 0}$ possesses a bounded absorbing set $\lfloor_a \subset X$ and that it is *asymptotically compact*, meaning

$$\{U(t_n, U_{0,n})\}_{n=1}^{\infty} \text{ is precompact in } X$$

whenever $t_n \to \infty$ and $\{U_{0,n}\}_{n=1}^{\infty} \subset X$ is bounded. We set

$$\mathcal{A} = \{z \in X \mid \text{there exist } t_n \to \infty,\ \{U_{0,n}\}_{n=1}^{\infty} \subset \mathcal{B}_a,\ U(t_n, U_{0,n}) \to z\}.$$

Since $\mathcal{B}_a$ is bounded and absorbing, $\mathcal{A}$ is a bounded subset of $X$. Moreover, it is easy to observe that $\mathcal{A}$ is, in fact, *compact* in $X$. Indeed consider a sequence $\{z_n\}_{n=1}^{\infty} \subset \mathcal{A}$. It follows from the proper definition of $\mathcal{A}$ that there is a sequence $t_n \to \infty$ and $U_{0,n} \in \mathcal{B}_a$ such that

$$\|U(t_n, U_{0,n}) - z_n\|_X < \frac{1}{n}.$$

However $\{U(t_n, U_{0,n})\}_{n=1}^{\infty}$ is precompact in $X$ and so is $z_n$. Moreover, each accummulation point of $\{z_n\}_{n=1}^{\infty}$ belongs to $\mathcal{A}$.

Similarly, arguing by contradiction, we can show that $\mathcal{A}$ attracts bounded sets in $X$, specifically, for any bounded set $B$ and any $\varepsilon > 0$, there exists a time $t_0(B, \varepsilon)$ such that

$$\text{dist}[U(t, U_0); \mathcal{A}] < \varepsilon \text{ for all } U_0 \in B,\ t > t_0(\varepsilon, B).$$

Finally, it follows from continuity of $U$ that $\mathcal{A}$ is invariant, meaning,

$$U(t, \mathcal{A}) = \mathcal{A}$$

Indeed, if $U(t_n, U_{0,n}) \to z$, then $U(t_n \pm \tau, U_{0,n}) \to U(\pm\tau, z)$ for any $\tau \geq 0$.

Motivated by the previous discussion, we introduce the concept of attractor as follows.

**Definition 2.1**

Let $\{U(t,\cdot)\}_{t\geq 0}$ be a dynamical system in a Banach space $X$. A set $\mathcal{A} \subset X$ is called *attractor* if:

- $\mathcal{A}$ is compact in $X$;
- $\mathcal{A}$ is invariant, meaning $U(t, \mathcal{A}) = \mathcal{A}$ for any $t \geq 0$;
- $\mathcal{A}$ attracts bounded sets in $X$, that means for any bounded $B \subset X$ and any $\varepsilon > 0$ there exists $t_0 = t_0(\varepsilon, B)$ such that

$$\text{dist}[U(t, U_0); \mathcal{A}] < \varepsilon \text{ for all } t \geq t_0 \text{ and all } U_0 \in B.$$

# Chapter 3

# Time discretization, accretive operators

a Consider an abstract differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) + \mathcal{A}[U(t)] = 0,\ t \in (0,T), \tag{3.1}$$ a1

supplemented with the initial condition

$$U(0) = U_0. \tag{3.2}$$ a2

Here $\mathcal{A}$ denotes a mapping acting in a Banach space $X$,

$$\mathcal{A} : \mathcal{D}[\mathcal{A}] \subset X \mapsto X. \tag{3.3}$$ a3

Our aim is to solve problem (3.1), (3.2) by the method of *time discretization.* Accordingly, fixing $h > 0$, we replace (3.1) by

$$U(t+h) + h\mathcal{A}[U(t+h)] = U(t),\ U(0) = U_0. \tag{3.4}$$ a4

Accordingly, in order to solve (3.4) for an arbitrary choice of initial conditions, we need

$$\mathcal{R}[\mathrm{Id} + h\mathcal{A}] = X. \tag{3.5}$$ a5

Moreover, a certain kind of continuity of the inverse $[\mathrm{Id} + h\mathcal{A}]^{-1}$ is needed in order to make the scheme stable. Specifically, we suppose that

$$\|y_1 + h\mathcal{A}[y_1] - (y_2 + h\mathcal{A}[y_2])\|_X \geq \|y_1 - y_2\|_X \tag{3.6}$$ a6

for any $y_1$, $y_2 \in \mathcal{D}[\mathcal{A}]$. This motivates the following definition.

**Definition 3.1** *Let $X$ be a Banach space, and let $\mathcal{A} : \mathcal{D}(\mathcal{A}) : X \to X$ be a mapping. We say that $\mathcal{A}$ is* accretive *if*

$$\|y_1 + h\mathcal{A}[y_1] - (y_2 + h\mathcal{A}[y_2])\|_X \geq \|y_1 - y_2\|_X$$

*holds for any* $y_1$, $y_2 \in \mathcal{D}[\mathcal{A}]$ *and any* $h > 0$.

*The mapping* $\mathcal{A}$ *is called* m-accretive *if it is accretive and*

$$\mathcal{R}[\mathrm{Id} + h\mathcal{A}] = X \text{ for any } h > 0.$$

## 3.1 Construction of solutions

Our first goal is to construct solutions to problem (3.1), (3.2) by means of the approximation scheme based on (3.4). To this end, fix

$$\lambda = \frac{T}{N}$$

and set, recursively,

$$U_\lambda^0 = U_0,\ U_\lambda^n + \lambda\mathcal{A}[U_\lambda^n] = U_\lambda^{n-1},\ n = 1, \dots, N.$$

- **Step 1:**

  Obviously,

$$U_\lambda^n + h\mathcal{A}[U_\lambda^n] - U_\lambda^{n-1} - h\mathcal{A}[U_\lambda^{n-1}] = U_\lambda^{n-1} - U_\lambda^{n-2},$$

therefore, as $\mathcal{A}$ is $m-$accretive,

$$\|U_\lambda^n - U_\lambda^{n-1}\|_X \le \|U_\lambda^{n-1} - U_\lambda^{n-2}\|_X.$$

Since

$$\|U_\lambda^n - U_\lambda^{n-1}\|_X \le \|U_\lambda^1 - U_0\|_X,$$

where, by accretivity of $\mathcal{A}$,

$$\|U_\lambda^1 - U_0\|_X \le \lambda\|\mathcal{A}[U_0]\|_X,$$

we may infer that

$$\|U_\lambda^n - U_0\|_X \le n\lambda\|\mathcal{A}[U_0]\|_X = \frac{n}{N}T\|\mathcal{A}[U_0]\|_X. \tag{3.7}$$ a7

The family $U_0, U_\lambda^1, \dots, U_\lambda^N$ is bounded uniformly for $\lambda \to 0$ provided $U_0$ belongs to the domain of $\mathcal{A}$.

- **Step 2:**

  Setting

$$\mu = \frac{T}{M},\ U_\mu^m + \mu\mathcal{A}[U_\mu^m] = U_\mu^{m-1},\ U_\mu^0 = U_0,\ m = 1, \dots, M$$

we compare the family $U_0, U_\mu^1, \dots, U_\mu^M$ with $U_0, U_\lambda^1, \dots, U_\lambda^N$ constructed in the previous step. We have

$$U_\mu^m - U_\lambda^n = U_\mu^{m-1} - \mu\mathcal{A}[U_\mu^m] - U_\lambda^{n-1} + \lambda\mathcal{A}[U_\lambda^n]$$

$$= U_\mu^{m-1} - U_\lambda^{n-1} - \mu\Big(\mathcal{A}[U_\mu^m] - \mathcal{A}[U_\lambda^n]\Big) + (\lambda-\mu)\mathcal{A}[U_\lambda^n],$$

where

$$(\lambda-\mu)\mathcal{A}[U_\lambda^n] = \frac{\lambda-\mu}{\lambda}\Big(U_\lambda^{n-1} - U_\lambda^n\Big).$$

Consequently,

$$U_\mu^m + \mu\mathcal{A}[U_\mu^m] - U_\lambda^n - \mu\mathcal{A}[U_\lambda^n] = U_\mu^{m-1} - U_\lambda^n - \frac{\mu}{\lambda}\Big(U_\lambda^{n-1} - U_\lambda^n\Big) \tag{3.8}$$ a8

$$= \theta\Big(U_\mu^{m-1} - U_\lambda^{n-1}\Big) + (1-\theta)\Big(U_\mu^{m-1} - U_\lambda^n\Big),$$

for

$$\theta = \frac{\mu}{\lambda},\ n = 0,\dots,N,\ m = 0,\dots,M.$$

Without loss of generality, we will assume that $\mu < \lambda$, meaning, $\theta \in (0,1)$. By virtue of the accretivity hypothesis (3.6), relation (3.8) implies that

$$a_{m,n} \equiv \|U_\mu^m - U_\lambda^n\|_X \le \theta\|U_\mu^{m-1} - U_\lambda^{n-1}\|_X + (1-\theta)\|U_\mu^{m-1} - U_\lambda^n\|_X \tag{3.9}$$ a9

$$= \theta a_{m-1,n-1} + (1-\theta)a_{m-1,n}.$$

Moreover, as we have shown in (3.7),

$$a_{0,n} \le n\lambda\|\mathcal{A}[U_0]\|_X,\ a_{m,0} \le m\mu\|\mathcal{A}[U_0]\|_X. \tag{3.10}$$ a10

It can be deduced from (3.9), (3.10) that

$$\left\|U_\lambda^n - U_\mu^m\right\|_X \le \left(m\mu(\lambda-\mu) + (n\lambda - m\mu)^2\right)^{1/2}\|\mathcal{A}[U_0]\|_X. \tag{3.11}$$ a11

Following [3] we show (3.10) in several steps:

**(1)**

$$a_{m,n} \le b_{m,n} \text{ for all } 1 \le m \le M,\ 1 \le n \le N$$

provided

$$a_{m,0} \le b_{m,0},\ a_{0,n} \le b_{0,n}$$

and

$$b_{m,n} = \theta b_{m-1,n-1} + (1-\theta)b_{m-1,n}. \tag{3.12}$$ a12

**(2)** Let $c_{m,n}$, $d_{m,n}$ solve the scheme (3.12), and, in addition,

$$c_{m,0} \ge d_{m,0}^2,\ c_{0,n} \ge d_{0,n}^2.$$

Then $c_{m,n} \ge d_{m,n}^2$ for all $m,n$.

Indeed it follows from (3.12) that

$$c_{m,n} = \sum_{i=0\vee j=0}\theta_{i,j}c_{i,j},\ d_{m,n} = \sum_{i=0\vee j=0}\theta_{i,j}d_{i,j},\ \sum\theta_{i,j} = 1,\ \theta_{i,j} \ge 0.$$

Thus, by virtue of Jensen's inequality,

$$d_{m,n}^2 = \left( \sum_{i=0 \vee j=0} \theta_{i,j} d_{i,j} \right)^2 \leq \sum_{i=0 \vee j=0} \theta_{i,j} d_{i,j}^2$$

$$\leq \sum_{i=0 \vee j=0} \theta_{i,j} c_{i,j} = c_{m,n}.$$

**(3)**

Consider the solution $c_{m,n}$ of the scheme (3.12) with

$$c_{m,0} = (m\mu)^2, c_{0,n} = (n\lambda)^2.$$

We claim that

$$\left| c_{m,n} - (m\mu - n\lambda)^2 \right| \leq m\mu(\lambda - \mu). \tag{3.13}$$ a13

In order to see (3.13), we set

$$r_{m,n} = c_{m,n} - (m\mu - n\lambda)^2.$$

Obviously $r_{m,0} = r_{0,n} = 0$, and, by direct computation,

$$r_{m,n} = \theta r_{m-1,n-1} + (1 - \theta) r_{m-1,n} + \mu(\lambda - \mu);$$

whence

$$\max_{n \leq N} |r_{m,n}| \leq \max_{n \leq N} |r_{m-1,n}| + \mu(\lambda - \mu),$$

and (3.13) follows.

**(4)**

Setting

$$b_{m,0} = m\mu \|\mathcal{A}[U_0]\|_X, \ b_{0,n} = n\lambda \|\mathcal{A}[U_0]\|_X,$$

we combine steps **(2)**, **(3)** to obtain the desired conclusion (3.11).

**Step 3:**

We define approximate solutions as a piecewise linear function in the following way:

Let

$$t = n\lambda + \xi = m\mu + \eta, \ \xi \in [0, \lambda), \ \eta \in [0, \mu).$$

We set

$$U_N(t) = U_\lambda^n + (U_\lambda^{n+1} - U_\lambda^n)\frac{\xi}{\lambda}, \ U_M(t) = U_\mu^m + (U_\mu^{m+1} - U_\mu^m)\frac{\eta}{\mu}.$$

Thus

$$\|U_N(t) - U_M(t)\|_X \leq \|U_\lambda^n - U_\mu^m\|_X + \|U_\lambda^{n+1} - U_\lambda^n\|_X + \|U_\mu^{m+1} - U_\mu^m\|_X, \tag{3.14}$$ a11a

where, in accordance with **Step 1** and relation (3.10),

$$\|U_\lambda^n - U_\mu^m\|_X + \|U_\lambda^{n+1} - U_\lambda^n\|_X + \|U_\mu^{m+1} - U_\mu^m\|_X \tag{3.15}$$

$$\leq \Big( T(\lambda - \mu) + |\xi - \eta|^2 \Big)^{1/2} \|\mathcal{A}[U_0]\|_X + (\lambda + \mu)\|\mathcal{A}[U_0]\|_X.$$

a12a

As a direct consequence of (3.14), (3.15), we get

$$U_N(t) \to U(t) \text{ as } N \to \infty \text{ uniformly for } t \in [0, T]. \tag{3.16}$$

a13a

## 3.2 Solution semigroup

We *define* a solution semigroup by means of the formula obtained in the previous section, specifically,

$$S_t[U_0](t) = \lim_{N \to \infty} U_N[U_0](t),$$

where

$$U_N[U_1](t) - U_N[U_2](t) = \left( \mathrm{Id} + \frac{T}{N} \right)^{-n} [U_1] - \left( \mathrm{Id} + \frac{T}{N} \right)^{-n} [U_2] \text{ for } t = n\frac{T}{N}.$$

Since $\mathcal{A}$ is accretive, we deduce that

$$\|U_N[U_1](t) - U_N[U_2](t)\|_X \leq \|U_1 - U_2\|_X \text{ whenever } t = n\frac{T}{N} \text{ for a certain } n. \tag{3.17}$$

a14a

Letting $N \to \infty$ in (3.17) and using continuity of the limit $U(t)$ we conclude that

$$\|S_t[U_1] - S_t[U_2]\|_X \leq \|U_1 - U_2\|_X \text{ for any } t \in [0, T], \tag{3.18}$$

a15a

meaning we have constructed a *contractive semigroup* of solutions to problem (3.1), (3.2).

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