Guaranteed eigenvalue bounds for elliptic partial differential operators

Tomáš Vejchodský (vejchod@math.cas.cz)

Institute of Mathematics
Czech Academy of Sciences

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Reliable numerical methods

To compute (approximate) solution is not sufficient.
We should provide an information about the error.

Can we provide
a guaranteed upper bound?

$$\|u - u_h\| \leq \eta$$

Sinking of the Sleipner A offshore platform in 1991, Norway. The failure resulted from inaccurate NASTRAN calculations.

Babuška, Verfürth, Ainsworth, Rannacher, Repin, . . .

Eigenvalue problems

Laplace eigenvalue problem

$$\begin{aligned} -\Delta u_n &= \lambda_n u_n \quad && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega \end{aligned}$$

Finite element method

- Very flexible (various domains, high order, various problems, ...)
- Converges with optimal speed
- Adaptive mesh refinement
- Nice theory

Guaranteed upper bound

$$\lambda_n \leq \lambda_{h,n}$$

Can we dream about anything else?

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Laplace eigenvalue problem

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Can we dream about anything else? Lower bounds!

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Can we dream about anything else? Lower bounds!

Guaranteed error bounds on eigenfunctions: $\|u_n - u_{h,n}\| \leq \eta$

Outline

1. Motivation
2. Theory
   2.1 Existence
   2.2 Min-max principle
   2.3 Optimal constants
3. Rayleigh–Ritz (Galerkin) method
4. Lower bounds on eigenvalues
   4.1 Weinstein's bound
   4.2 Lehmann–Goerisch method
   4.3 Interpolation constant based methods
5. Guaranteed bounds on eigenfunctions
6. Literature

2. Theory

2.1 Existence

Abstract formulation

Eigenvalue problem: Find eigenvalue λ_n and eigenfunction $u_n \in V \setminus \{0\}$ such that

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V.$$

- V is a Hilbert space.
- $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are two bilinear forms on V.

Example

$$\begin{aligned} -\Delta u_n &= \lambda_n u_n && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega \end{aligned}$$

Weak formulation

$$(\nabla u_n, \nabla v) = \lambda_n (u_n, v) \quad \forall v \in V$$

- $V = H_0^1(\Omega)$
- $a(u, v) = (\nabla u, \nabla v)$
- $b(u, v) = (u, v)$

$$(u, v) = \int_\Omega uv \,\mathrm{d}x$$

Hilbert–Schmidt theorem

$$Su_n = \mu_n u_n$$

Let

- V be a Hilbert space
- $S : V \to V$ be linear, bounded, compact, self-adjoint operator

Then

- there is (at most) countable sequence of nonzero real eigenvalues of S (repeated according to their multiplicity): $|\mu_1| \geq |\mu_2| \geq |\mu_3| \geq \cdots > 0$,
  and if the sequence is infinite then $\lim_{n\to\infty} \mu_n = 0$
- eigenfunctions u_n corresponding to these μ_n form a complete orthonormal system in $\mathcal{M}$ and

$$V = (\ker S) \oplus \mathcal{M}$$

Note: $\mathcal{M} = \overline{\operatorname{range} S}$

Assumptions

Find $\lambda_n \in \mathbb{R}$, $u_n \in V \setminus \{0\}$: $\quad a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$

- V is a real Hilbert space
- $a(\cdot,\cdot)$ is continuous, bilinear, symmetric, V-elliptic
- $b(\cdot,\cdot)$ is continuous, bilinear, symmetric, positive semidefinite
- $\|v\|_a = a(v,v)^{1/2}$ is the norm induced by $a(\cdot,\cdot)$
- $|v|_b = b(v,v)^{1/2}$ is the seminorm induced by $b(\cdot,\cdot)$
- $|\cdot|_b$ is compact with respect to $\|\cdot\|_a$,
  i.e. from any sequence bounded in $\|\cdot\|_a$, we can extract a subsequence which is Cauchy in $|\cdot|_b$

Existence

Theorem. There exists (at most) countable sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

and the corresponding eigenfunctions can be normalized to satisfy

$$b(u_i, u_j) = \delta_{ij} \quad \forall i, j = 1, 2, \ldots .$$

Proof

- Solution operator $S : V \to V$: $a(Su, v) = b(u, v) \quad \forall v \in V$
- $a(u_n, v) = \lambda_n \underbrace{b(u_n, v)}_{a(Su_n, v)} \quad \forall v \in V \quad \Leftrightarrow \quad Su_n = \frac{1}{\lambda_n} u_n$
- Exercise: compactness of $|\cdot|_b$ with respect to $\|\cdot\|_a$ is equivalent to compactness of S
- Hilbert–Schmidt theorem: $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots > 0$, $\lambda_n = 1/\mu_n$ because $0 < \|u_n\|_a^2 = \lambda_n |u_n|_b^2$.

Existence

Theorem. There exists (at most) countable sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

and the corresponding eigenfunctions can be normalized to satisfy

$$b(u_i, u_j) = \delta_{ij} \quad \forall i, j = 1, 2, \ldots .$$

Note

$$\frac{1}{\lambda_i} a(u_i, u_j) = \delta_{ij} \quad \forall i, j = 1, 2, \ldots$$

Orthonormal basis of eigenfunctions

Theorem. The space V can be decomposed as

$$V = \mathcal{K} \oplus \mathcal{M},$$

where $\mathcal{K} = \{v \in V : |v|_b = 0\}$ and $\mathcal{M} = \text{span}\{u_1, u_2, \dots\}$.
Moreover,

$$\begin{aligned} a(u,v) &= 0 \quad \forall u \in \mathcal{K},\ \forall v \in \mathcal{M}, \\ b(u,v) &= 0 \quad \forall u \in \mathcal{K},\ \forall v \in V. \qquad (*) \end{aligned}$$

Proof

- $(*)$ follows from $|b(u,v)| \le |u|_b |v|_b = 0$
- Hilbert–Schmidt theorem: $V = (\ker S) \oplus \mathcal{M}$
  Now, $\ker S = \mathcal{K}$, because
  (a) $u \in \mathcal{K} \Rightarrow 0 = b(u,v) = a(Su,v)\ \forall v \in V$
  $\Rightarrow Su = 0 \Rightarrow u \in \ker S$
  (b) $u \in \ker S \Rightarrow 0 = a(Su,u) = b(u,u) = |u|_b^2 \Rightarrow u \in \mathcal{K}$

Orthonormal basis of eigenfunctions

Theorem. The space V can be decomposed as

$$V = \mathcal{K} \oplus \mathcal{M},$$

where $\mathcal{K} = \{v \in V : |v|_b = 0\}$ and $\mathcal{M} = \text{span}\{u_1, u_2, \dots\}$.
Moreover,

$$\begin{aligned} a(u,v) &= 0 \quad \forall u \in \mathcal{K},\ \forall v \in \mathcal{M}, \\ b(u,v) &= 0 \quad \forall u \in \mathcal{K},\ \forall v \in V. \qquad (*) \end{aligned}$$

Proof

- Express $v \in \mathcal{M}$ as $v = \sum_{n=1}^{\infty} c_n u_n$ and

$$a(u,v) = \sum_{n=1}^{\infty} c_n a(u,u_n) = \sum_{n=1}^{\infty} c_n \lambda_n b(u,u_n) \overset{(*)}{=} 0.$$

□

Parseval's identities

Theorem. For all $v \in V$, there are unique $v^{\mathcal{K}} \in \mathcal{K}$ and $v^{\mathcal{M}} \in \mathcal{M}$ such that

$$v = v^{\mathcal{K}} + v^{\mathcal{M}}, \quad v^{\mathcal{M}} = \sum_{n=1}^{\infty} c_n u_n, \quad c_n = b(v^{\mathcal{M}}, u_n) = b(v, u_n)$$

$$|v|_b^2 = \sum_{n=1}^{\infty} |b(v, u_n)|^2,$$

$$\|v\|_a^2 = \|v^{\mathcal{K}}\|_a^2 + \|v^{\mathcal{M}}\|_a^2 \quad \text{with } \|v^{\mathcal{M}}\|_a^2 = \sum_{n=1}^{\infty} \lambda_n |b(v, u_n)|^2.$$

Proof

- $v = v^{\mathcal{K}} + v^{\mathcal{M}} = v^{\mathcal{K}} + \sum_{n=1}^{\infty} c_n u_n$
- $|v|_b^2 = b(v, v^{\mathcal{K}} + \sum_{n=1}^{\infty} c_n u_n) = \sum_{n=1}^{\infty} c_n b(v, u_n)$
- $\|v\|_a^2 = \|v^{\mathcal{M}}\|_a^2 + \|v^{\mathcal{K}}\|_a^2$ and $\|v^{\mathcal{M}}\|_a^2 = \sum_{n=1}^{\infty} \lambda_n c_n^2$

□

Example 1: Dirichlet Laplacian

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega$$
$$u_n = 0 \qquad \text{on } \partial\Omega$$

Weak formulation: Find $\lambda_n \in \mathbb{R}$, $u_n \in H_0^1(\Omega) \setminus \{0\}$:

$$(\nabla u_n, \nabla v) = \lambda_n (I u_n, I v) \quad \forall v \in H_0^1(\Omega),$$

where $I : H_0^1(\Omega) \to L^2(\Omega)$ is the identity operator.

- $V = H_0^1(\Omega)$
- $a(u,v) = (\nabla u, \nabla v)$ ... cont., bilin., sym., V-elliptic
- $b(u,v) = (Iu, Iv)$ ... cont., bilin., sym., pos. def.
- Compactness: I is a compact operator by Rellich theorem.
  Definition: I is compact if from a sequence $\{v_i\} \subset H_0^1(\Omega)$ bounded in $\|\nabla v\|_{L^2(\Omega)} \le C$ we can extract a subsequence such that $\{I v_{i_j}\}$ is Cauchy in $L^2(\Omega)$.

Example 1: Dirichlet Laplacian

$$
\begin{aligned}
-\Delta u_n &= \lambda_n u_n \quad && \text{in } \Omega \\
u_n &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Exact solution for an interval $\Omega = (0, L)$

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad u_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

Easy to verify

$$u_n'(x) = \frac{n\pi}{L}\cos\frac{n\pi x}{L}$$

$$u_n''(x) = -\frac{n^2\pi^2}{L^2}\sin\frac{n\pi x}{L} = -\frac{n^2\pi^2}{L^2}u_n(x)$$

Is it complete?

Example 1: Dirichlet Laplacian

$$\begin{aligned} -\Delta u_n &= \lambda_n u_n \quad \text{in } \Omega \\ u_n &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

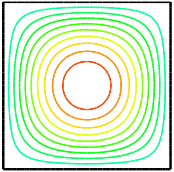
Exact solution for a square $\Omega = (0, \pi)^2$

$$\lambda_{k,\ell} = k^2 + \ell^2, \quad u_{k,\ell}(x,y) = \sin(kx)\sin(\ell y), \quad k, \ell = 1, 2, \ldots$$

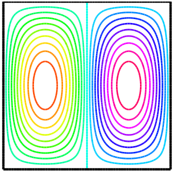
$$\begin{aligned}
\lambda_1 &= 2\ (k=1,\ \ell=1) & \lambda_6 &= 10\ (k=1,\ \ell=3) \\
\lambda_2 &= 5\ (k=2,\ \ell=1) & \lambda_7 &= 13\ (k=3,\ \ell=2) \\
\lambda_3 &= 5\ (k=1,\ \ell=2) & \lambda_8 &= 13\ (k=2,\ \ell=3) \\
\lambda_4 &= 8\ (k=2,\ \ell=2) & \lambda_9 &= 17\ (k=4,\ \ell=1) \\
\lambda_5 &= 10\ (k=3,\ \ell=1) & \lambda_{10} &= 17\ (k=1,\ \ell=4)
\end{aligned}$$

Example: Square

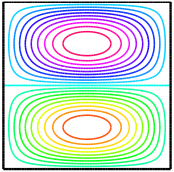
$\lambda_1 = 2, \; u_1(x, y) = \sin(x)\sin(y)$

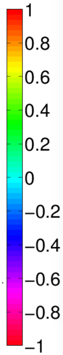


$\lambda_2 = 5, \; u_2(x, y) = \sin(2x)\sin(y)$



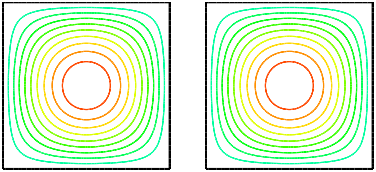
$\lambda_3 = 5, \; u_3(x, y) = \sin(x)\sin(2y)$



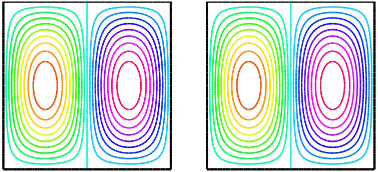


Example: Two squares

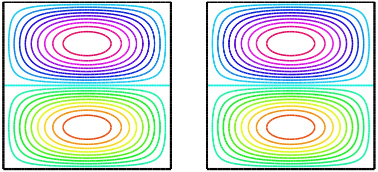
$\lambda_1 = 2$

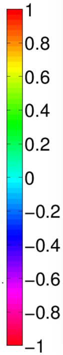


$\lambda_2 = 5$

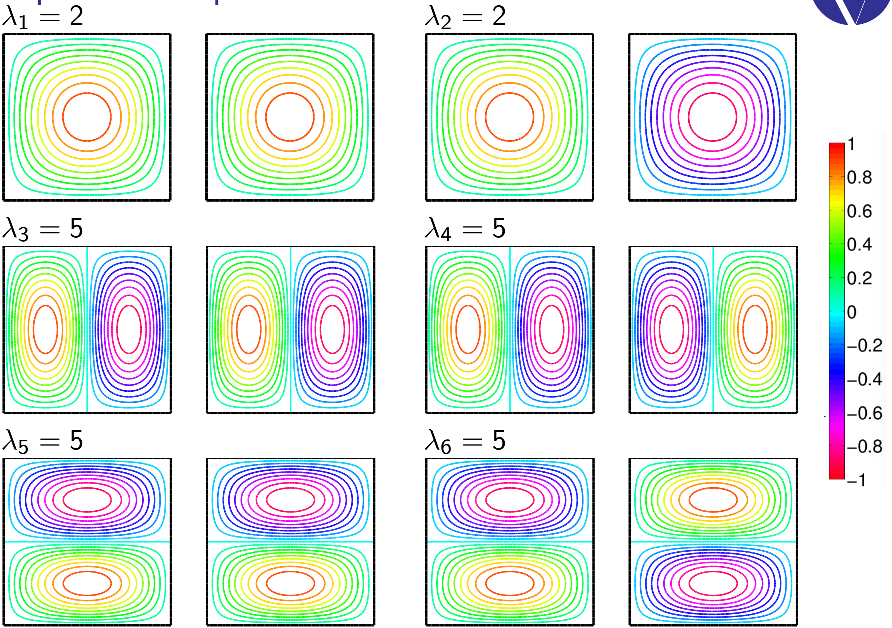


$\lambda_3 = 5$

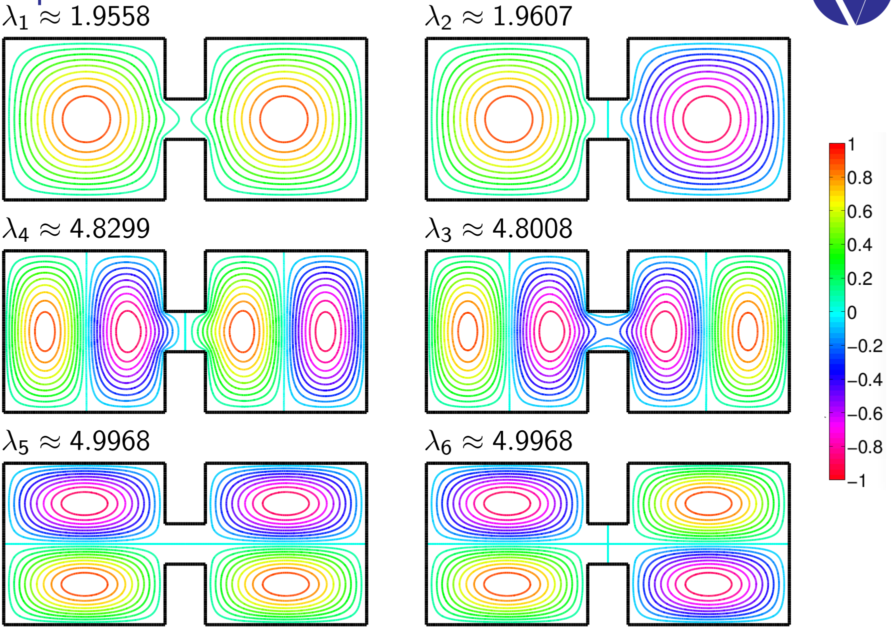




Example: Two squares



Example: Dumbbell



2. Theory

2.2 Min-max principle

Minimum principle

Rayleigh quotien: $R(v) = \dfrac{a(v,v)}{b(v,v)} = \dfrac{\|v\|_a^2}{|v|_b^2}$

Theorem. Numbers $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and functions $u_1, u_2, \cdots \in V \setminus \{0\}$ are eigenpairs of

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

if and only if

$$\lambda_1 = \min_{v \in V,\ |v|_b \neq 0} R(v) \qquad u_1 = \operatorname*{arg\,min}_{v \in V,\ |v|_b \neq 0} R(v),$$

$$\lambda_n = \min_{v \in \mathcal{M}_{n-1}^{\perp}} R(v) \qquad u_n = \operatorname*{arg\,min}_{v \in \mathcal{M}_{n-1}^{\perp}} R(v),$$

where $\mathcal{M}_{n-1} = \operatorname{span}\{u_1, u_2, \ldots, u_{n-1}\}$,

$$\begin{aligned} \mathcal{M}_{n-1}^{\perp} &= \{v \in \mathcal{M} : b(v, u_i) = 0,\ \forall i = 1, 2, \ldots, n-1\} \\ &= \{v \in V : b(v, u_i) = 0,\ \forall i = 1, 2, \ldots, n-1 \\ &\qquad\qquad \text{and } |v|_b \neq 0\}. \end{aligned}$$

Minimum principle

Proof. (Including $n = 1$).

$\Rightarrow$ Let $a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$.

Then $u_n \in \mathcal{M}_{n-1}^{\perp}$, $\lambda_n = R(u_n)$, and thus $\inf_{\mathcal{M}_{n-1}^{\perp}} R(v) \le \lambda_n$.

If $v \in \mathcal{M}_{n-1}^{\perp}$ then $v^{\mathcal{K}} = 0$, $c_i = b(v, u_i) = 0$ for $i = 1, \dots, n-1$, and

$$R(v) = \frac{\|v\|_a^2}{|v|_b^2} = \frac{\sum_{i=n}^{\infty} \lambda_i c_i^2}{\sum_{i=n}^{\infty} c_i^2} \ge \lambda_n \frac{\sum_{i=n}^{\infty} c_i^2}{\sum_{i=n}^{\infty} c_i^2} = \lambda_n$$

$\Leftarrow$ The minimum is attained: $\exists u_n \in \mathcal{M}_{n-1}^{\perp} : \quad \lambda_n = R(u_n)$.

Let $t \in \mathbb{R}$, $v \in \mathcal{M}_{n-1}^{\perp}$ and $\varphi(t) = R(u_n + tv)$.

Derivative $\varphi'(0)$ exists and

$$\varphi'(0) = \frac{2}{|u_n|_b} \left(a(u_n, v) - \frac{\|u_n\|_a^2}{|u_n|_b^2} b(u_n, v) \right)$$

Since $\varphi(t)$ has a minimum at $t = 0$, we have $\varphi'(0) = 0$.

If $v = u_i$, $i = 1, 2, \dots, n-1$, then

$$b(u_n, u_i) = 0 \text{ and } a(u_n, u_i) = \lambda_i b(u_n, u_i) = 0.$$

(Courant–Fischer–Weyl) Min-max principle

Theorem.

$$\lambda_n = \min_{v \in \mathcal{M}_{n-1}^{\perp}} R(v) = \min_{E \in \mathcal{V}^{(n)}} \max_{v \in E} R(v)$$

where $\mathcal{V}^{(n)}$ is the set of all n-dimensional subspaces of $\mathcal{M}$.
Moreover, the mininum is attaind for $E = \operatorname{span}\{u_1, \ldots, u_n\}$.

Proof. (Induction over n.)

$n = 1$: Since $R(\alpha v) = R(v)$ for all $\alpha \neq 0$, we have

$$\min_{E \in \mathcal{V}^{(1)}} \max_{v \in E} R(v) = \min_{v \in \mathcal{M}} R(v) = \min_{v \in V,\ |v|_b \neq 0} R(v)$$

(Courant–Fischer–Weyl) Min-max principle

Theorem.

$$\lambda_n = \min_{v \in \mathcal{M}_{n-1}^{\perp}} R(v) = \min_{E \in \mathcal{V}^{(n)}} \max_{v \in E} R(v)$$

where $\mathcal{V}^{(n)}$ is the set of all n-dimensional subspaces of $\mathcal{M}$.
Moreover, the mininum is attaind for $E = \operatorname{span}\{u_1, \ldots, u_n\}$.
Proof. (Induction over n.)
$n > 1$: Let $\widetilde{\mathcal{V}}^{(n)} \subset \mathcal{V}^{(n)}$ be a set of all spaces
$\widetilde{E}^z = \operatorname{span}\{u_1, \ldots, u_{n-1}, z\}$, where $b(z, u_i) = 0$ for $i = 1, \ldots, n-1$.

$$\min_{E \in \mathcal{V}^{(n)}} \max_{v \in E} R(v) \le \min_{\widetilde{E}^z \in \widetilde{\mathcal{V}}^{(n)}} \max_{v \in \widetilde{E}^z} R(v) = \min_{z \in \mathcal{M}_{n-1}^{\perp}} \max_{v \in \widetilde{E}^z} R(v) \overset{(!)}{=} \min_{z \in \mathcal{M}_{n-1}^{\perp}} R(z)$$

To prove (!), let $v \in \widetilde{E}^z$, $|v|_b = |z|_b = 1$. Thus,
$v = \alpha z + \sum_{i=1}^{n-1} c_i u_i$, $|v|_b^2 = \alpha^2 + \sum_{i=1}^{n-1} c_i^2 = 1$, and

$$R(v) = \|v\|_a^2 = \alpha^2 \|z\|_a^2 + \sum_{i=1}^{n-1} c_i^2 \|u_i\|_a^2 \le \left(\alpha^2 + \sum_{i=1}^{n-1} c_i^2\right) \|z\|_a^2 = R(z),$$

because $z \in \mathcal{M}_{i-1}^{\perp}$ for all $i = 1, 2, \ldots, n-1$ and $R(u_i) \le R(z)$.

(Courant–Fischer–Weyl) Min-max principle

Theorem.

$$\lambda_n = \min_{v \in \mathcal{M}_{n-1}^{\perp}} R(v) = \min_{E \in \mathcal{V}^{(n)}} \max_{v \in E} R(v)$$

where $\mathcal{V}^{(n)}$ is the set of all n-dimensional subspaces of $\mathcal{M}$.
Moreover, the mininum is attaind for $E = \text{span}\{u_1, \ldots, u_n\}$.
Proof. (Induction over n.)
$n > 1$: (cont'd)
Let $E \in \mathcal{V}^{(n)}$.
There exists $z \in E$: $|z|_b \neq 0$ and $b(z, u_i) = 0$ for $i = 1, 2, \ldots, n-1$.

$$\max_{v \in E} R(v) \geq R(z) \geq \min_{z \in \mathcal{M}_{n-1}^{\perp}} R(z)$$

□

Example 2: Neumann Laplacian

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega$$
$$\frac{\partial u_n}{\partial \nu} = 0 \qquad \text{on } \partial\Omega$$

Weak formulation: Find $\lambda_n \in \mathbb{R}$, $u_n \in H^1(\Omega) \setminus \{0\}$:

$$(\nabla u_n, \nabla v) = \lambda_n (u_n, v) \quad \forall v \in H^1(\Omega)$$

Problem: $u_0 \equiv 1$, $\lambda_0 = 0$
$\Rightarrow$ bilinear form $a(u,v) = (\nabla u, \nabla v)$ is not $H^1(\Omega)$-elliptic.

- $V = \{v \in H^1(\Omega) : \int_\Omega v = 0\}$
- $a(u,v) = (\nabla u, \nabla v)$ ... cont., bilin., sym., V-elliptic
- $b(u,v) = (u,v)$ ... cont., bilin., sym., pos. def.
- Compactness: by Rellich theorem.

Example 2: Neumann Laplacian

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega$$

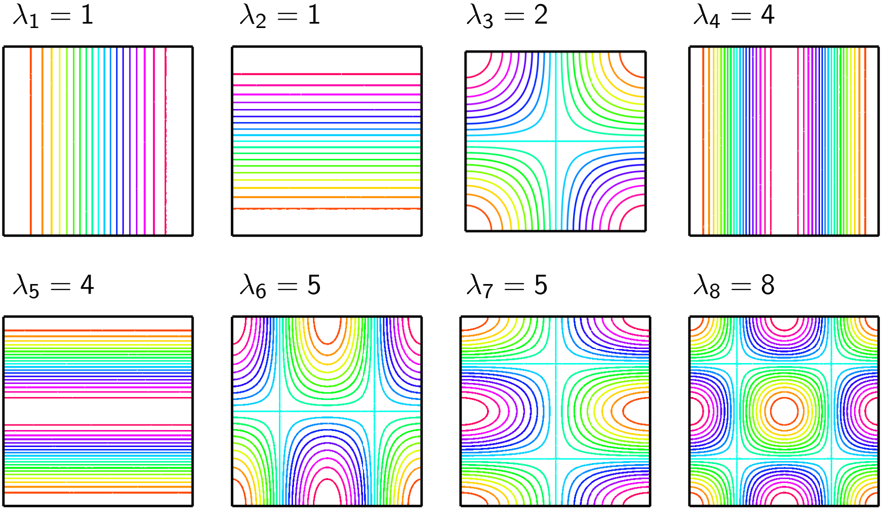
$$\frac{\partial u_n}{\partial \nu} = 0 \qquad \text{on } \partial\Omega$$

Exact solution for a square $\Omega = (0, \pi)^2$

$$\lambda_{k,\ell} = k^2 + \ell^2, \quad u_{k,\ell}(x,y) = \cos(kx)\cos(\ell y), \quad k, \ell = 0, 1, 2, \ldots$$

$$\begin{aligned}
\lambda_0 &= 0\ (k = 0,\ \ell = 0) & \lambda_5 &= 4\ (k = 0,\ \ell = 2) \\
\lambda_1 &= 1\ (k = 1,\ \ell = 0) & \lambda_6 &= 5\ (k = 2,\ \ell = 1) \\
\lambda_2 &= 1\ (k = 0,\ \ell = 1) & \lambda_7 &= 5\ (k = 1,\ \ell = 2) \\
\lambda_3 &= 2\ (k = 1,\ \ell = 1) & \lambda_8 &= 8\ (k = 2,\ \ell = 2) \\
\lambda_4 &= 4\ (k = 2,\ \ell = 0) & \lambda_9 &= 9\ (k = 3,\ \ell = 0)
\end{aligned}$$

Example 2: Neumann Laplacian



Example 3: Steklov eigenvalue problem

$$-\Delta u_n + u_n = 0 \qquad \text{in } \Omega$$
$$\frac{\partial u_n}{\partial \nu} = \lambda_n u_n \quad \text{on } \partial\Omega$$

Weak formulation: Find $u_n \in H^1(\Omega)$, $\|u_n\|_{L^2(\partial\Omega)} \neq 0$, and $\lambda_n \in \mathbb{R}$:

$$(\nabla u_n, \nabla v) + (u_n, v) = \lambda_n (\gamma u_n, \gamma v)_{\partial\Omega} \quad \forall v \in H^1(\Omega)$$

- $V = H^1(\Omega)$, $V = \mathcal{K} \oplus \mathcal{M}$, $\mathcal{K} = \{v \in H^1(\Omega) : \gamma v = 0 \text{ on } \partial\Omega\}$
  $\mathcal{M} = \{v \in H^1(\Omega) : \gamma v \neq 0 \text{ on } \partial\Omega\}$
- $a(u, v) = (\nabla u, \nabla v) + (u, v)$ ... cont., bilin., sym., V-elliptic
- $b(u, v) = (u, v)_{\partial\Omega}$ ... cont., bilin., sym., pos. semidefinite
- Compactness:
  Trace operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ is compact
  [Kufner, John, Fučík 1997], [Biegert 2009]

Example 3: Steklov eigenvalue problem

$$-\Delta u_n + u_n = 0 \qquad \text{in } \Omega$$

$$\frac{\partial u_n}{\partial \nu} = \lambda_n u_n \quad \text{on } \partial\Omega$$

Exact solution for a square $\Omega = (-L, L)^2$

$$\lambda_1 = \frac{\sqrt{2}}{2} \tanh\left(\frac{\sqrt{2}}{2} L\right), \quad u_1(x,y) = \cosh\left(\frac{\sqrt{2}}{2} x\right) \cosh\left(\frac{\sqrt{2}}{2} y\right)$$

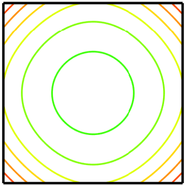
$$\lambda_2 = ?$$

$$\lambda_3 = ?$$

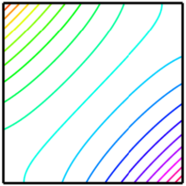
$$\lambda_4 = \frac{\sqrt{2}}{2} \coth\left(\frac{\sqrt{2}}{2} L\right), \quad u_4(x,y) = \sinh\left(\frac{\sqrt{2}}{2} x\right) \sinh\left(\frac{\sqrt{2}}{2} y\right)$$

Example 3: Steklov eigenvalue problem ($L = \pi/2$)

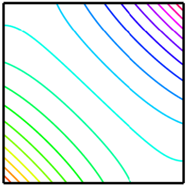
$\lambda_1 = 0.5687$



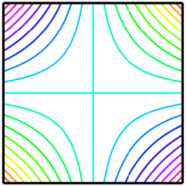
$\lambda_2 = 0.7610$



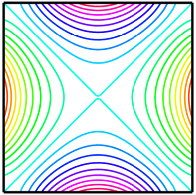
$\lambda_3 = 0.7610$



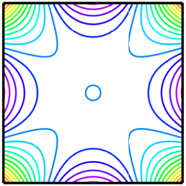
$\lambda_4 = 0.8791$



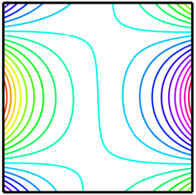
$\lambda_5 = 1.739$



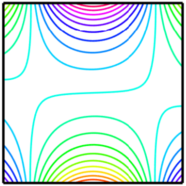
$\lambda_6 = 1.739$



$\lambda_7 = 1.763$



$\lambda_8 = 1.763$



2. Theory

2.3 Optimal constants

Optimal constants

Abstract eigenvalue problem: Find $\lambda_n \in \mathbb{R}$, $u_n \in V \setminus \{0\}$:

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Theorem

$$|v|_b \leq \lambda_1^{-1/2} \|v\|_a \quad \forall v \in V, \quad \text{with equality for } v = u_1.$$

Proof

Let $v \in V$, $|v|_b \neq 0$.

$$\lambda_1 = \min_{w \in V, |w|_b \neq 0} \frac{\|w\|_a^2}{|w|_b^2} \leq \frac{\|v\|_a^2}{|v|_b^2} \quad \Leftrightarrow \quad |v|_b^2 \leq \lambda_1^{-1} \|v\|_a^2$$

□

Optimal constants

Abstract eigenvalue problem: Find $\lambda_n \in \mathbb{R}$, $u_n \in V \setminus \{0\}$:

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Theorem

$$|v|_b \le \lambda_1^{-1/2} \|v\|_a \quad \forall v \in V, \quad \text{with equality for } v = u_1.$$

Example 1: Dirichlet Laplacian.

$V = H_0^1(\Omega)$, $\quad \|v\|_a = \|\nabla v\|_{L^2(\Omega)} \quad |v|_b = \|v\|_{L^2(\Omega)}$

Corollary 1. The optimal constant in Friedrichs inequality

$$\|v\|_{L^2(\Omega)} \le C_{\mathrm{F}} \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega) \quad \text{is} \quad C_{\mathrm{F}} = \lambda_1^{-1/2},$$

where λ_1 is the principal eigenvalue of the Dirichlet Laplacian.

- $\Omega = (0, L) \quad \Rightarrow \quad C_{\mathrm{F}} = \frac{L}{\pi}$
- $\Omega = (0, L_1) \times (0, L_2) \quad \Rightarrow \quad C_{\mathrm{F}} = \frac{1}{\pi} \left(\frac{1}{L_1^2} + \frac{1}{L_2^2} \right)^{-1/2}$

Optimal constants

Abstract eigenvalue problem: Find $\lambda_n \in \mathbb{R}$, $u_n \in V \setminus \{0\}$:

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Theorem

$$|v|_b \leq \lambda_1^{-1/2} \|v\|_a \quad \forall v \in V, \quad \text{with equality for } v = u_1.$$

Example 2: Neumann Laplacian.

$V = \{v \in H^1(\Omega) : \int_\Omega v \, \mathrm{d}x = 0\}$, $\quad \|v\|_a = \|\nabla v\|_{L^2(\Omega)}$, $\quad |v|_b = \|v\|_{L^2(\Omega)}$

Corollary 2. The optimal constant in Poincaré inequality

$$\|v\|_{L^2(\Omega)} \leq C_{\mathrm{P}} \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1(\Omega), \int_\Omega v \, \mathrm{d}x = 0, \quad \text{is} \quad C_{\mathrm{P}} = \lambda_1^{-1/2},$$

where λ_1 is the principal eigenvalue of the Neumann Laplacian.

- $\Omega = (0, L_1) \times (0, L_2) \quad \Rightarrow \quad C_{\mathrm{P}} = \dfrac{\max\{L_1, L_2\}}{\pi}$

Optimal constants

Abstract eigenvalue problem: Find $\lambda_n \in \mathbb{R}$, $u_n \in V \setminus \{0\}$:

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Theorem

$$|v|_b \leq \lambda_1^{-1/2} \|v\|_a \quad \forall v \in V, \quad \text{with equality for } v = u_1.$$

Example 3: Steklov eigenvalue problem.

$$V = H^1(\Omega), \quad \|v\|_a^2 = \|\nabla v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2, \quad |v|_b = \|v\|_{L^2(\partial\Omega)}$$

Corollary 3. The optimal constant in trace inequality

$$\|v\|_{L^2(\partial\Omega)} \leq C_{\mathrm{T}} \|v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega) \quad \text{is} \quad C_{\mathrm{T}} = \lambda_1^{-1/2},$$

where λ_1 is the principal eigenvalue of the Steklov problem.

- $\Omega = (-L, L)^2 \quad \Rightarrow \quad C_{\mathrm{T}} = \left(\sqrt{2} \coth(\sqrt{2} L/2)\right)^{1/2}$

3. Rayleigh–Ritz (Galerkin) method

Rayleigh–Ritz (Galerkin) method

Eigenvalue problem: Find $\lambda_n \in \mathbb{R}$, $u_n \in V \setminus \{0\}$:

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Finite dimensional subspace: $V_h \subset V$, $\dim V_h = N < \infty$.

Discrete eigenvalue problem: Find $\lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h \setminus \{0\}$:

$$a(u_{h,n}, v_h) = \lambda_{h,n} b(u_{h,n}, v_h) \quad \forall v_h \in V_h$$

Generalized eigenvalue problem for matrices:

$$A\boldsymbol{x}_n = \lambda_{h,n} B\boldsymbol{x}_n,$$

where $u_{h,n} = \sum_{j=1}^{N} x_j \varphi_j, \quad A_{ij} = a(\varphi_j, \varphi_i), \quad B_{ij} = b(\varphi_j, \varphi_i)$

Properties

Discrete eigenvalue problem: Find $\lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h \setminus \{0\}$:

$$a(u_{h,n}, v_h) = \lambda_{h,n} b(u_{h,n}, v_h) \quad \forall v_h \in V_h$$

- $0 < \lambda_{h,1} \leq \lambda_{h,2} \leq \cdots \leq \lambda_{h,N}$
- $\dfrac{1}{\lambda_{h,i}} a(u_{h,i}, u_{h,j}) = b(u_{h,i}, u_{h,j}) = \delta_{ij} \quad \forall i,j = 1,2,\ldots,N.$
- Minimum principle:

$$\lambda_{h,1} = \min_{v_h \in V_h,\ |v_h|_b \neq 0} R(v_h) \qquad u_{h,1} = \operatorname*{arg\,min}_{v_h \in V_h,\ |v_h|_b \neq 0} R(v_h),$$

$$\lambda_{h,n} = \min_{v_h \in \mathcal{M}^{\perp}_{h,n-1}} R(v_h) \qquad u_{h,n} = \operatorname*{arg\,min}_{v_h \in \mathcal{M}^{\perp}_{h,n-1}} R(v_h),$$

where $\mathcal{M}^{\perp}_{h,n-1} = \{v_h \in V_h : |v_h|_b \neq 0 \text{ and } b(v_h, u_{h,i}) = 0 \quad \forall i = 1,2,\ldots,n-1\}.$

Properties

Discrete eigenvalue problem: Find $\lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h \setminus \{0\}$:

$$a(u_{h,n}, v_h) = \lambda_{h,n} b(u_{h,n}, v_h) \quad \forall v_h \in V_h$$

- $0 < \lambda_{h,1} \leq \lambda_{h,2} \leq \cdots \leq \lambda_{h,N}$
- $\dfrac{1}{\lambda_{h,i}} a(u_{h,i}, u_{h,j}) = b(u_{h,i}, u_{h,j}) = \delta_{ij} \quad \forall i, j = 1, 2, \ldots, N.$
- Min-max principle:

$$\lambda_{h,n} = \min_{E_h \in \mathcal{V}_h^{(n)}} \max_{v_h \in E_h} R(v_h)$$

  where $\mathcal{V}_h^{(n)}$ is the set of all n-dimensional subspaces of V_h.
- Theorem.

$$\lambda_n \leq \lambda_{h,n}, \quad n = 1, 2, \ldots, N$$

  Proof.

$$\mathcal{V}_h^{(n)} \subset \mathcal{V}^{(n)} \quad \Rightarrow \quad \lambda_n = \min_{E \in \mathcal{V}^{(n)}} \max_{v \in E} R(v) \leq \lambda_{h,n} \qquad \square$$

4. Lower bounds on eigenvalues

4.1 Weinstein's bound

Lower bounds – history

Standard (conforming) approach:
Temple (1928), Weinstein (1937), Kato (1949), Lehmann (1949), Goerisch (1985), . . .

Nonconforming FEM:
Carstensen, Gedicke, Gallistl (2014), Xuefeng LIU (2015), . . .

Many results: M.G. Armentano, G. Barrenechea, H. Behnke, R.G. Duran, L. Grubišić, Jun Hu, J.R. Kuttler, Y.A. Kuznetsov, Fubiao Lin, Qun Lin, M. Plum, S.I. Repin, V.G. Sigillito, M. Vohralík, Hehu Xie, Yidu Yang, Zhimin Zhang, . . . *many others*

Recall

Find $\lambda_n \in \mathbb{R}$ and $u_n \in V \setminus \{0\}$ such that

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

- V is a Hilbert space.
- $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are two bilinear forms on V.
- $V = \mathcal{K} \oplus \mathcal{M}$
- $\mathcal{K} = \{v \in V : |v|_b = 0\}$
- $\mathcal{M} = \text{span}\{u_1, u_2, \dots\}$
- $v = v^{\mathcal{K}} + v^{\mathcal{M}}$
- $v^{\mathcal{M}} = \sum_{n=1}^{\infty} c_n u_n, \quad c_n = b(v^{\mathcal{M}}, u_n) = b(v, u_n)$
- $|v|_b^2 = \sum_{n=1}^{\infty} |b(v, u_n)|^2$
- $\|v\|_a^2 = \|v^{\mathcal{K}}\|_a^2 + \|v^{\mathcal{M}}\|_a^2 \quad$ with $\|v^{\mathcal{M}}\|_a^2 = \sum_{n=1}^{\infty} \lambda_n |b(v, u_n)|^2$

Weinstein's bound

Theorem

Let $\lambda_* \in \mathbb{R}$ and $u_* \in V$ with $|u_*|_b \neq 0$ be arbitrary and $w \in V$ be given by

$$a(w, v) = a(u_*, v) - \lambda_* b(u_*, v) \quad \forall v \in V.$$

Then

$$\min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \leq \frac{\|w\|_a^2}{|u_*|_b^2}.$$

Proof: $w = w^{\mathcal{K}} + w^{\mathcal{M}}$

$$\|w^{\mathcal{M}}\|_a^2 = \sum_{j=1}^{\infty} \lambda_j |b(w, u_j)|^2 = \sum_{j=1}^{\infty} \frac{|a(w, u_j)|^2}{\lambda_j}$$

$$= \sum_{j=1}^{\infty} \frac{|a(u_*, u_j) - \lambda_* b(u_*, u_j)|^2}{\lambda_j} = \sum_{j=1}^{\infty} \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} |b(u_*, u_j)|^2$$

Thus,

$$\|w\|_a^2 \geq \|w^{\mathcal{M}}\|_a^2 \geq \min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \sum_{j=1}^{\infty} |b(u_*, u_j)|^2 \qquad \square$$

Weinstein's bound

Corollary: Let λ_n has multiplicity m, i.e.,
$\lambda_{n-1} \neq \lambda_n = \cdots = \lambda_{n+m-1} \neq \lambda_{n+m}$. If

$$\sqrt{\lambda_{n-1}\lambda_n} \leq \lambda_* \leq \sqrt{\lambda_n \lambda_{n+m}} \qquad \text{(closeness)}$$

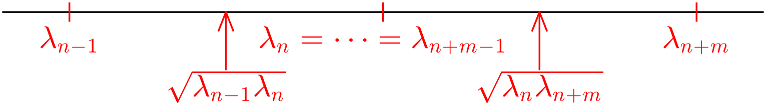
and

$$\|w\|_a \leq \eta$$

then

$$\ell_n \leq \lambda_n,$$

where $\ell_n = \dfrac{1}{4|u_*|_b^2}\left(-\eta + \sqrt{\eta^2 + 4\lambda_* |u_*|_b^2}\right)^2.$



Weinstein's bound

Corollary: Let λ_n has multiplicity m, i.e., $\lambda_{n-1} \neq \lambda_n = \cdots = \lambda_{n+m-1} \neq \lambda_{n+m}$. If

$$\sqrt{\lambda_{n-1}\lambda_n} \leq \lambda_* \leq \sqrt{\lambda_n \lambda_{n+m}} \qquad \text{(closeness)}$$

and

$$\|w\|_a \leq \eta$$

then

$$\ell_n \leq \lambda_n,$$

where $\ell_n = \dfrac{1}{4|u_*|_b^2}\left(-\eta + \sqrt{\eta^2 + 4\lambda_*|u_*|_b^2}\right)^2$.

Proof: Clearly,

$$\frac{(\lambda_n - \lambda_*)^2}{\lambda_n} = \min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \leq \frac{\|w\|_a^2}{|u_*|_b^2} \leq \frac{\eta^2}{|u_*|_b^2}$$

and solve for λ_n.

Complementary upper bound on the residual

Laplace eigenvalue problem: Find λ_n and $u_n \in H_0^1(\Omega) \setminus \{0\}$:

$$(\nabla u_n, \nabla v) = \lambda_n (u_n, v) \quad \forall v \in H_0^1(\Omega)$$

Definition. Flux $\mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega)$ is equilibrated if $-\operatorname{div} \mathbf{q} = \lambda_* u_*$.

Theorem. If $\mathbf{q}$ is an equilibrated flux then

$$\|\nabla w\|_0 \leq \eta = \|\nabla u_* - \mathbf{q}\|_0.$$

Proof: Let $v \in H_0^1(\Omega)$, then

$$\begin{aligned}(\nabla w, \nabla v) &= (\nabla u_*, \nabla v) - \lambda_* (u_*, v) - (\operatorname{div} \mathbf{q}, v) - (\mathbf{q}, \nabla v) \\ &= (\nabla u_* - \mathbf{q}, \nabla v) - (\lambda_* u_* + \operatorname{div} \mathbf{q}, v) \\ &\leq \|\nabla u_* - \mathbf{q}\|_0 \|\nabla v\|_0 \qquad \square\end{aligned}$$

[Neittaanmäki, Repin 2004], [Repin 2008], [Braess, Schöberl, 2008], [Ainsworth, Vejchodský 2011,2014], [Vohralík at al.]

Avoiding equilibration

Shifted eigenvalue problem:

$$\underbrace{(\nabla u_n, \nabla v) + \gamma(u_n, v)}_{a_\gamma(u_n, v)} = (\lambda_n + \gamma)(u_n, v) \quad \forall v \in H_0^1(\Omega)$$

Theorem. Let $\mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega)$ and $\gamma > 0$. Then

$$\|\nabla w\|_0 \le \|w\|_{a_\gamma} \le \eta, \quad \eta^2 = \|\nabla u_* - \mathbf{q}\|_0^2 + \frac{1}{\gamma}\|\lambda_* u_* + \operatorname{div}\mathbf{q}\|_0^2$$

Proof:

$$\begin{aligned} a_\gamma(w, v) &= (\nabla u_*, \nabla v) - \lambda_*(u_*, v) - (\operatorname{div}\mathbf{q}, v) - (\mathbf{q}, \nabla v) \\ &= (\nabla u_* - \mathbf{q}, \nabla v) - (\lambda_* u_* + \operatorname{div}\mathbf{q}, v) \\ &\le \|\nabla u_* - \mathbf{q}\|_0 \|\nabla v\|_0 + \gamma^{-1/2}\|\lambda_* u_* + \operatorname{div}\mathbf{q}\|_0 \; \gamma^{1/2}\|v\|_0 \\ &\le \left(\|\nabla u_* - \mathbf{q}\|_0^2 + \gamma^{-1}\|\lambda_* u_* + \operatorname{div}\mathbf{q}\|_0^2\right)^{1/2} \left(\|\nabla v\|_0^2 + \gamma\|v\|_0^2\right)^{1/2} \end{aligned}$$

Thus, $\|w\|_{a_\gamma}^2 \le \|\nabla u_* - \mathbf{q}\|_0^2 + \gamma^{-1}\|\lambda_* u_* + \operatorname{div}\mathbf{q}\|_0^2$ □

How to compute $\mathbf{q}$?

Global flux reconstruction: Find $\mathbf{q}_h \in \mathbf{W}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$ minimizing

$$\eta^2 = \|\nabla u_* - \mathbf{q}_h\|_0^2 + \frac{1}{\gamma}\|\lambda_* u_* + \mathrm{div}\, \mathbf{q}_h\|_0^2$$

FEM space:

$V_h = \{v_h \in V : v_h|_K \in \mathbb{P}^1(K)\ \forall K \in \mathcal{T}_h\}$

FEM approximation:

$u_* = u_{h,n} \in V_h,\ \lambda_* = \lambda_{h,n}$

Raviart–Thomas space:

$\mathbf{RT}_1(K) = [\mathbb{P}^1(K)]^2 \oplus \boldsymbol{x}\mathbb{P}^1(K)$ (local)

$\boldsymbol{W}_h = \{\mathbf{q}_h \in \mathbf{H}(\mathrm{div}, \Omega) : \mathbf{q}_h|_K \in \mathbf{RT}_1(K) \quad \forall K \in \mathcal{T}_h\}$ (global)

How to compute $\mathbf{q}$?

Global flux reconstruction: Find $\mathbf{q}_h \in \mathbf{W}_h \subset \mathbf{H}(\mathrm{div},\Omega)$ minimizing

$$\eta^2 = \|\nabla u_* - \mathbf{q}_h\|_0^2 + \frac{1}{\gamma}\|\lambda_* u_* + \mathrm{div}\,\mathbf{q}_h\|_0^2$$

Euler–Lagrange equations:

$$(\mathbf{q}_h, \mathbf{w}_h) + \frac{1}{\gamma}(\mathrm{div}\,\mathbf{q}_h, \mathrm{div}\,\mathbf{w}_h) = (\nabla u_*, \mathbf{w}_h) - \frac{1}{\gamma}(\lambda_* u_*, \mathrm{div}\,\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \boldsymbol{W}_h$$

Equivalent to linear system:

$$M\mathbf{y} = F,$$

where $\mathbf{q}_h = \sum_j y_j \boldsymbol{\psi}_j$,

$$M_{ij} = (\boldsymbol{\psi}_j, \boldsymbol{\psi}_i) + \frac{1}{\gamma}(\mathrm{div}\,\boldsymbol{\psi}_j, \mathrm{div}\,\boldsymbol{\psi}_i),$$

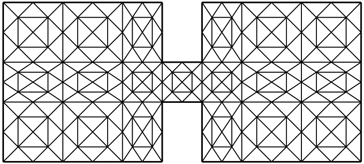
$$F_i = (\nabla u_*, \boldsymbol{\psi}_i) - \frac{1}{\gamma}(\lambda_* u_*, \mathrm{div}\,\boldsymbol{\psi}_i)$$

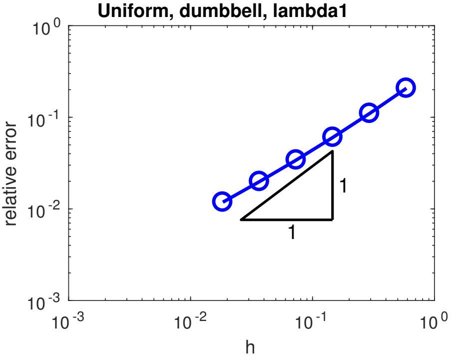
Example: dumbbell

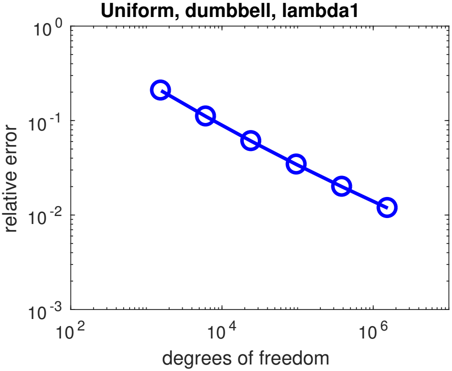
$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = \text{dumbbell}$$
$$u_n = 0 \quad \text{on } \partial\Omega$$

Rel. error: $\dfrac{|\lambda_n - \lambda_{h,n}|}{\lambda_n} \leq \dfrac{\lambda_{h,n} - \ell_n}{\ell_n}$

$\gamma = 10^{-6}$

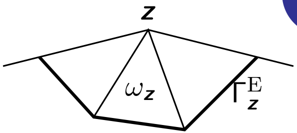






Local flux reconstruction

Flux reconstruction:



$$\mathbf{q}_h = \sum_{z \in \mathcal{N}_h} \mathbf{q}_z$$

Local problems: Find $\mathbf{q}_z \in \mathbf{W}_z$ minimizing

$$\|\varphi_{\boldsymbol{z}} \nabla u_* - \mathbf{q}_{\boldsymbol{z}}\|^2_{L^2(\omega_{\boldsymbol{z}})} + \frac{1}{\gamma} \|\lambda_* \varphi_{\boldsymbol{z}} u_* + \operatorname{div} \mathbf{q}_{\boldsymbol{z}}\|^2_{L^2(\omega_{\boldsymbol{z}})}$$

Euler–Lagrange equations:

$$(\mathbf{q}_{\boldsymbol{z}}, \mathbf{w}_h)_{\omega_{\boldsymbol{z}}} + \frac{1}{\gamma} (\operatorname{div} \mathbf{q}_{\boldsymbol{z}}, \operatorname{div} \mathbf{w}_h)_{\omega_{\boldsymbol{z}}} = (\varphi_{\boldsymbol{z}} \nabla u_*, \mathbf{w}_h)_{\omega_{\boldsymbol{z}}} - \frac{1}{\gamma} (\lambda_* \varphi_{\boldsymbol{z}} u_*, \operatorname{div} \mathbf{w}_h)_{\omega_{\boldsymbol{z}}}$$

$$\forall \mathbf{w}_h \in \boldsymbol{W}_{\boldsymbol{z}}$$

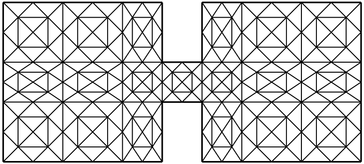
Patch of elements: $\omega_{\boldsymbol{z}} = \bigcup \{K \in \mathcal{T}_h : \boldsymbol{z} \in K\}$

Partition of unity: $\sum_{\boldsymbol{z} \in \mathcal{N}_h} \varphi_{\boldsymbol{z}} = 1$

$\boldsymbol{W}_{\boldsymbol{z}} = \{\mathbf{q} \in \mathbf{H}(\operatorname{div}, \omega_{\boldsymbol{z}}) : \mathbf{q}|_K \in \mathbf{RT}_1(K)\ \forall K \subset \omega_{\boldsymbol{z}},\ \mathbf{q} \cdot \boldsymbol{n}_{\boldsymbol{z}} = 0 \text{ on } \Gamma_{\boldsymbol{z}}^{\mathrm{E}}\}$

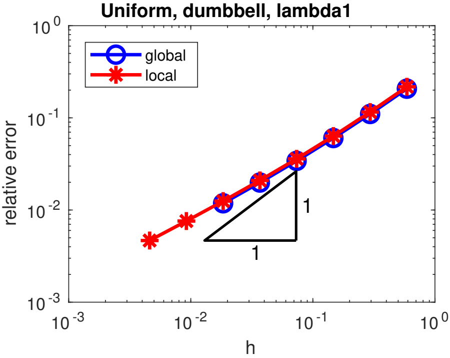
Example: dumbbell

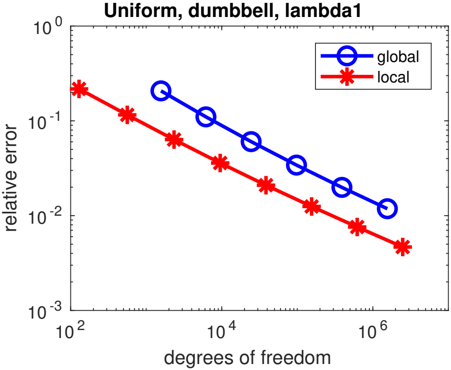
$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = \text{dumbbell}$$
$$u_n = 0 \quad \text{on } \partial\Omega$$



Rel. error: $\dfrac{|\lambda_n - \lambda_{h,n}|}{\lambda_n} \le \dfrac{\lambda_{h,n} - \ell_n}{\ell_n}$

$\gamma = 10^{-6}$

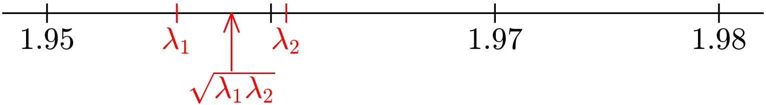




Closeness assumption for dumbbell

$$\sqrt{\lambda_{n-1}\lambda_n} \le \lambda_* \le \sqrt{\lambda_n \lambda_{n+m}} \quad \Rightarrow \quad \ell_n \le \lambda_n$$

Exact eigenvalues: $\lambda_1 = 1.955793794588$, $\lambda_2 = 1.960683031595$



h	ℓ_1	$\lambda_{h,1}$	closeness
$h_1 = 1.1781$	1.6618	2.0228	no

Closeness assumption for dumbbell

$$\sqrt{\lambda_{n-1}\lambda_n} \le \lambda_* \le \sqrt{\lambda_n \lambda_{n+m}} \quad \Rightarrow \quad \ell_n \le \lambda_n$$

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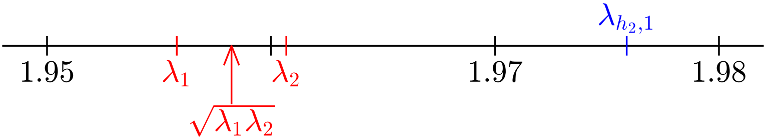

h	ℓ_1	$\lambda_{h,1}$	closeness
$h_1 = 1.1781$	1.6618	2.0228	no
$h_2 = 0.5890$	1.7711	1.9759	no

Closeness assumption for dumbbell

$$\sqrt{\lambda_{n-1}\lambda_n} \leq \lambda_* \leq \sqrt{\lambda_n \lambda_{n+m}} \quad \Rightarrow \quad \ell_n \leq \lambda_n$$

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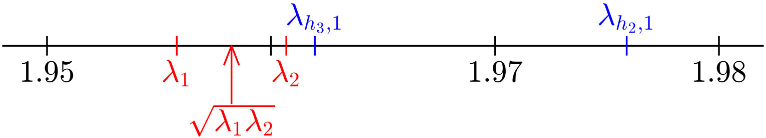

h	ℓ_1	$\lambda_{h,1}$	closeness
$h_1 = 1.1781$	1.6618	2.0228	no
$h_2 = 0.5890$	1.7711	1.9759	no
$h_3 = 0.2945$	1.8449	1.9620	no

Closeness assumption for dumbbell

$$\sqrt{\lambda_{n-1}\lambda_n} \le \lambda_* \le \sqrt{\lambda_n \lambda_{n+m}} \quad \Rightarrow \quad \ell_n \le \lambda_n$$

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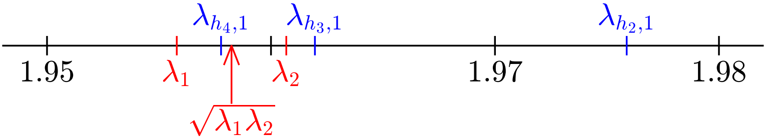
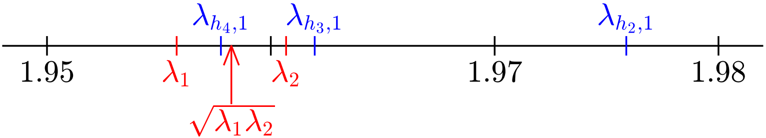


h	ℓ_1	$\lambda_{h,1}$	closeness
$h_1 = 1.1781$	1.6618	2.0228	no
$h_2 = 0.5890$	1.7711	1.9759	no
$h_3 = 0.2945$	1.8449	1.9620	no
$h_4 = 0.1473$	1.8899	1.9578	yes

Closeness assumption for dumbbell

$$\sqrt{\lambda_{n-1}\lambda_n} \leq \lambda_* \leq \sqrt{\lambda_n \lambda_{n+m}} \quad \Rightarrow \quad \ell_n \leq \lambda_n$$

Exact eigenvalues: $\lambda_1 = 1.955793794588$, $\lambda_2 = 1.960683031595$



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$h_1 = 1.1781$	1.6618	2.0228	no
$h_2 = 0.5890$	1.7711	1.9759	no
$h_3 = 0.2945$	1.8449	1.9620	no
$h_4 = 0.1473$	1.8899	1.9578	yes
$h_5 = 0.0736$	1.9163	1.9565	yes
$h_6 = 0.0368$	1.9319	1.9560	yes
$h_7 = 0.0184$	1.9411	1.9559	yes

Weinstein's bound – summary

- easy to use
- it is a generalization of Bauer–Fike estimates for matrices
- good for general symmetric elliptic problems
- sub-optimal speed of convergence
- a priori information on spectrum needed for guaranteed lower bounds

4. Lower bounds on eigenvalues

4.2 Lehmann–Goerisch method

Lehmann–Goerisch method

General setting:

Find $\lambda_n \in \mathbb{R}$ and $u_n \in V \setminus \{0\}$ such that

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Lehmann method

Theorem

Let $\tilde{\lambda}_N < \rho \le \lambda_{N+1}$

- $\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_N \in V$ be linearly independent
- $A_{0,ij} = a(\tilde{u}_i, \tilde{u}_j)$
- $A_{1,ij} = b(\tilde{u}_i, \tilde{u}_j)$
- $w_i \in V: \quad a(w_i, v) = b(\tilde{u}_i, v) \quad \forall v \in V$
  $A_{2,ij} = a(w_i, w_j)$

- $\mu_1 \le \mu_2 \le \dots \le \mu_N: \quad (\rho A_1 - A_0)\boldsymbol{x} = \mu(A_0 - 2\rho A_1 + \rho^2 A_2)\boldsymbol{x}$

Then $0 < \mu_1$ and

$$\rho - \frac{\rho}{1 + \mu_n} \le \lambda_n, \quad n = 1, 2, \dots, N.$$

[Lehmann 1949, 1950]

Lehmann–Goerisch method

Theorem

Let $\tilde{\lambda}_N < \rho \le \lambda_{N+1}$

- $\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_N \in V$ be linearly independent
- $A_{0,ij} = a(\tilde{u}_i, \tilde{u}_j)$
- $A_{1,ij} = b(\tilde{u}_i, \tilde{u}_j)$
- X ... vector space
  $\mathcal{B}$ ... positive semidefinite symmetric bilinear form on X
  $T : V \to X$ ... linear operator:
  (a) $\mathcal{B}(Tu, Tv) = a(u, v) \quad \forall u, v \in V$
  (b) $\hat{\mathbf{w}}_i \in X : \quad \mathcal{B}(\hat{\mathbf{w}}_i, Tv) = b(\tilde{u}_i, v) \quad \forall v \in V$
  (c) $\hat{A}_{2,ij} = \mathcal{B}(\hat{\mathbf{w}}_i, \hat{\mathbf{w}}_j)$
- $\hat{\mu}_1 \le \hat{\mu}_2 \le \dots \le \hat{\mu}_N : \quad (\rho A_1 - A_0)\hat{\mathbf{x}} = \hat{\mu}(A_0 - 2\rho A_1 + \rho^2 \hat{A}_2)\hat{\mathbf{x}}$

Then $0 < \hat{\mu}_1$ and

$$\rho - \frac{\rho}{1 + \hat{\mu}_n} \le \lambda_n, \quad n = 1, 2, \dots, N.$$

[Lehmann 1949, 1950], [Goerisch, Haunhorst 1985]

Proof: Lehmann $\Rightarrow$ Goerisch

It suffices to show that $\hat{A}_2 - A_2$ is positive semidefinite, because

$\Rightarrow$ $0 < \hat{\mu}_i \leq \mu_i$ for all $i = 1, 2, \ldots, N$,

$\Rightarrow$ $\rho - \dfrac{\rho}{1 + \hat{\mu}_n} \leq \rho - \dfrac{\rho}{1 + \mu_n} \leq \lambda_n.$

To show that $\hat{A}_2 - A_2$ is positive semidefinite:
Let $\boldsymbol{x} \in \mathbb{R}^N$, $\tilde{u} = \sum_{i=1}^{N} x_i \tilde{u}_i$, $w = \sum_{i=1}^{N} x_i w_i$, $\hat{\mathbf{w}} = \sum_{i=1}^{N} x_i \hat{\mathbf{w}}_i$, and

$$0 \leq \mathcal{B}(\hat{\mathbf{w}} - Tw, \hat{\mathbf{w}} - Tw) = \mathcal{B}(\hat{\mathbf{w}}, \hat{\mathbf{w}}) - 2\,\underbrace{\mathcal{B}(\hat{\mathbf{w}}, Tw)}_{\substack{\overset{(b)}{=} b(\tilde{u}, w) \\ = a(w,w)}} + \underbrace{\mathcal{B}(Tw, Tw)}_{\overset{(a)}{=} a(w,w)}.$$

Thus,

$$0 \leq \mathcal{B}(\hat{\mathbf{w}}, \hat{\mathbf{w}}) - a(w, w) \overset{(c)}{=} \boldsymbol{x}^T (\hat{A}_2 - A_2) \boldsymbol{x}.$$

□

Application to Laplace eigenvalue problem

$$(\nabla u_i, \nabla v) + \gamma(u_i, v) = (\lambda_i + \gamma)(u_i, v) \quad \forall v \in H_0^1(\Omega),\ \Omega \subset \mathbb{R}^2$$

Setting

- $V = H_0^1(\Omega),\ a(u,v) = (\nabla u, \nabla v) + \gamma(u,v),\ b(u,v) = (u,v)$
- $X = \left[L^2(\Omega)\right]^3$
- $\mathcal{B}(\hat{\boldsymbol{u}}, \hat{\mathbf{v}}) = (\hat{u}_1, \hat{v}_1) + (\hat{u}_2, \hat{v}_2) + \gamma(\hat{u}_3, \hat{v}_3)$
- $Tu = \begin{pmatrix} \nabla u \\ u \end{pmatrix}$

Facts

(a) $\mathcal{B}(Tu, Tv) = a(u,v)$

Application to Laplace eigenvalue problem

$$(\nabla u_i, \nabla v) + \gamma(u_i, v) = (\lambda_i + \gamma)(u_i, v) \quad \forall v \in H_0^1(\Omega),\ \Omega \subset \mathbb{R}^2$$

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- $Tu = \begin{pmatrix} \nabla u \\ u \end{pmatrix}$

Facts

(a) $\mathcal{B}(Tu, Tv) = a(u,v)$

(b) $\mathcal{B}(\hat{\mathbf{w}}_i, Tv) = b(\tilde{u}_i, v) \Leftarrow \hat{\mathbf{w}}_i = \begin{pmatrix} \boldsymbol{\sigma}_i \\ \hat{w}_{i,3} \end{pmatrix} \quad \boldsymbol{\sigma}_i \in \mathbf{H}(\mathrm{div}, \Omega)$

$$(\boldsymbol{\sigma}_i, \nabla v) + \gamma(\hat{w}_{i,3}, v) = (\tilde{u}_i, v) \quad \forall v \in V$$

$$-(\mathrm{div}\, \boldsymbol{\sigma}_i, v) + \gamma(\hat{w}_{i,3}, v) = (\tilde{u}_i, v) \quad \forall v \in V$$

$$\hat{w}_{i,3} = \frac{1}{\gamma}(\tilde{u}_i + \mathrm{div}\, \boldsymbol{\sigma}_i)$$

Application to Laplace eigenvalue problem

$$(\nabla u_i, \nabla v) + \gamma(u_i, v) = (\lambda_i + \gamma)(u_i, v) \quad \forall v \in H_0^1(\Omega),\ \Omega \subset \mathbb{R}^2$$

Setting

- $V = H_0^1(\Omega),\ a(u,v) = (\nabla u, \nabla v) + \gamma(u,v),\ b(u,v) = (u,v)$
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- $\mathcal{B}(\hat{\boldsymbol{u}}, \hat{\mathbf{v}}) = (\hat{u}_1, \hat{v}_1) + (\hat{u}_2, \hat{v}_2) + \gamma(\hat{u}_3, \hat{v}_3)$
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(a) $\mathcal{B}(Tu, Tv) = a(u,v)$

(b) $\mathcal{B}(\hat{\mathbf{w}}_i, Tv) = b(\tilde{u}_i, v) \Leftarrow \hat{\mathbf{w}}_i = \begin{pmatrix} \boldsymbol{\sigma}_i \\ \frac{1}{\gamma}(\tilde{u}_i + \operatorname{div} \boldsymbol{\sigma}_i) \end{pmatrix} \quad \boldsymbol{\sigma}_i \in \mathbf{H}(\operatorname{div}, \Omega)$

Application to Laplace eigenvalue problem

$$(\nabla u_i, \nabla v) + \gamma(u_i, v) = (\lambda_i + \gamma)(u_i, v) \quad \forall v \in H_0^1(\Omega),\ \Omega \subset \mathbb{R}^2$$

Setting

- $V = H_0^1(\Omega),\ a(u,v) = (\nabla u, \nabla v) + \gamma(u,v),\ b(u,v) = (u,v)$
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Facts

(a) $\mathcal{B}(Tu, Tv) = a(u,v)$

(b) $\mathcal{B}(\hat{\mathbf{w}}_i, Tv) = b(\tilde{u}_i, v) \Leftarrow \hat{\mathbf{w}}_i = \begin{pmatrix} \boldsymbol{\sigma}_i \\ \frac{1}{\gamma}(\tilde{u}_i + \operatorname{div} \boldsymbol{\sigma}_i) \end{pmatrix} \quad \boldsymbol{\sigma}_i \in \mathbf{H}(\operatorname{div}, \Omega)$

(c) $\hat{A}_{2,ij} = \mathcal{B}(\hat{\mathbf{w}}_i, \hat{\mathbf{w}}_j) \ \Leftrightarrow\ \hat{A}_{2,ij} = (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) + \frac{1}{\gamma}(\tilde{u}_i + \operatorname{div} \boldsymbol{\sigma}_i, \tilde{u}_j + \operatorname{div} \boldsymbol{\sigma}_j)$

Application to Laplace eigenvalue problem

Theorem (Lehmann–Goerisch)

Let $\tilde{\lambda}_N + \gamma < \rho \le \lambda_{N+1} + \gamma$, $\gamma > 0$

- $\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_N \in V$ be linearly independent
- $A_{0,ij} = (\nabla \tilde{u}_i, \nabla \tilde{u}_j) + \gamma(\tilde{u}_i, \tilde{u}_j)$
- $A_{1,ij} = (\tilde{u}_i, \tilde{u}_j)$
- $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \dots, \boldsymbol{\sigma}_N \in \mathbf{H}(\operatorname{div}, \Omega)$ be arbitrary
  $\hat{A}_{2,ij} = (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) + \frac{1}{\gamma}(\tilde{u}_i + \operatorname{div} \boldsymbol{\sigma}_i, \tilde{u}_j + \operatorname{div} \boldsymbol{\sigma}_j)$
- $\hat{\mu}_1 \le \hat{\mu}_2 \le \dots \le \hat{\mu}_N : \quad (\rho A_1 - A_0)\hat{\boldsymbol{x}} = \hat{\mu}(A_0 - 2\rho A_1 + \rho^2 \hat{A}_2)\hat{\boldsymbol{x}}$

Then $0 < \hat{\mu}_1$ and

$$\ell_n = \rho - \gamma - \frac{\rho}{1 + \hat{\mu}_n} \le \lambda_n, \quad n = 1, 2, \dots, N$$

[Behnke, Mertins, Plum, Wieners 2000]

How to find good $\hat{\mathbf{w}}_i$?

Observation: Let $\tilde{u}_i \approx u_i$ and $\tilde{\lambda}_i \approx \lambda_i$.

$$\Rightarrow \quad a(w_i, v) = b(\tilde{u}_i, v) \approx \frac{1}{\tilde{\lambda}_i} a(\tilde{u}_i, v) \quad \forall v \in V$$

$$\Rightarrow \quad w_i \approx \frac{1}{\tilde{\lambda}_i} \tilde{u}_i$$

Need

$$\Rightarrow \quad \hat{A}_2 \approx A_2$$

$$\Rightarrow \quad \mathcal{B}(\hat{\mathbf{w}}_i, \hat{\mathbf{w}}_j) \approx a(w_i, w_j) \overset{(a)}{=} \mathcal{B}(Tw_i, Tw_j)$$

$$\Rightarrow \quad \hat{\mathbf{w}}_i \approx Tw_i \approx \frac{1}{\tilde{\lambda}_i} T\tilde{u}_i$$

Natural idea

make $|\frac{1}{\tilde{\lambda}_i} T\tilde{u}_i - \hat{\mathbf{w}}_i|^2_{\mathcal{B}}$ small

For Laplacian: Find $\boldsymbol{\sigma}_{h,i} \in \mathbf{H}(\mathrm{div}, \Omega)$ that

makes $\left\| \frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma} - \boldsymbol{\sigma}_{h,i} \right\|_0^2 + \frac{1}{\gamma} \left\| \frac{\lambda_{h,i} u_{h,i}}{\lambda_{h,i} + \gamma} + \mathrm{div}\, \boldsymbol{\sigma}_{h,i} \right\|_0^2$ small

Choice of $\boldsymbol{\sigma}_i$ – global

Global minimization:

Find $\boldsymbol{\sigma}_{h,i} \in \boldsymbol{W}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$, $i = 1, 2, \dots, N$, that minimizes

$$\left\| \frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma} - \boldsymbol{\sigma}_{h,i} \right\|_0^2 + \frac{1}{\gamma} \left\| \frac{\lambda_{h,i} u_{h,i}}{\lambda_{h,i} + \gamma} + \mathrm{div}\, \boldsymbol{\sigma}_{h,i} \right\|_0^2$$

Euler–Lagrange equations:

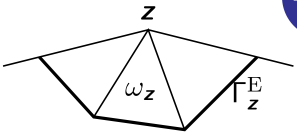
$$(\boldsymbol{\sigma}_{h,i}, \mathbf{w}_h) + \frac{1}{\gamma} (\mathrm{div}\, \boldsymbol{\sigma}_{h,i}, \mathrm{div}\, \mathbf{w}_h) = \left(\frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma}, \mathbf{w}_h \right) - \frac{1}{\gamma} \left(\frac{\lambda_{h,i} u_{h,i}}{\lambda_{h,i} + \gamma}, \mathrm{div}\, \mathbf{w}_h \right)$$

$$\forall \mathbf{w}_h \in \boldsymbol{W}_h$$

$$\boldsymbol{W}_h = \{ \boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega) : \boldsymbol{\sigma}_h|_K \in \mathbf{RT}_1(K) \quad \forall K \in \mathcal{T}_h \}$$

Choice of $\boldsymbol{\sigma}_i$ – local

Flux reconstruction:



$$\boldsymbol{\sigma}_{h,i} = \sum_{\boldsymbol{z}\in\mathcal{N}_h} \boldsymbol{\sigma}_{\boldsymbol{z},i}$$

Local problems: Find $\boldsymbol{\sigma}_{\boldsymbol{z},i} \in \boldsymbol{W}_{\boldsymbol{z}}$, $i = 1, 2, \ldots, N$ minimizing

$$\left\| \varphi_{\boldsymbol{z}} \frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma} - \boldsymbol{\sigma}_{\boldsymbol{z},i} \right\|_{0,\omega_{\boldsymbol{z}}}^2 + \frac{1}{\gamma} \left\| \frac{\lambda_{h,i}\varphi_{\boldsymbol{z}} u_{h,i}}{\lambda_{h,i} + \gamma} + \operatorname{div} \boldsymbol{\sigma}_{\boldsymbol{z},i} \right\|_{0,\omega_{\boldsymbol{z}}}^2$$

Euler–Lagrange equations:

$$(\boldsymbol{\sigma}_{\boldsymbol{z},i}, \mathbf{w}_h)_{\omega_{\boldsymbol{z}}} + \frac{1}{\gamma} (\operatorname{div} \boldsymbol{\sigma}_{\boldsymbol{z},i}, \operatorname{div} \mathbf{w}_h)_{\omega_{\boldsymbol{z}}}$$

$$= \left(\varphi_{\boldsymbol{z}} \frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma}, \mathbf{w}_h \right)_{\omega_{\boldsymbol{z}}} - \frac{1}{\gamma} \left(\frac{\varphi_{\boldsymbol{z}} \lambda_{h,i} u_{h,i}}{\lambda_{h,i} + \gamma}, \operatorname{div} \mathbf{w}_h \right)_{\omega_{\boldsymbol{z}}} \quad \forall \mathbf{w}_h \in \boldsymbol{W}_{\boldsymbol{z}}$$

Patch of elements: $\omega_{\boldsymbol{z}} = \bigcup \{ K \in \mathcal{T}_h : \boldsymbol{z} \in K \}$

Partition of unity: $\sum_{\boldsymbol{z}\in\mathcal{N}_h} \varphi_{\boldsymbol{z}} = 1$

$\boldsymbol{W}_{\boldsymbol{z}} = \{ \boldsymbol{\sigma} \in \mathbf{H}(\operatorname{div}, \omega_{\boldsymbol{z}}) : \boldsymbol{\sigma}|_K \in \mathbf{RT}_1(K)\ \forall K \subset \omega_{\boldsymbol{z}},\ \boldsymbol{\sigma} \cdot \boldsymbol{n}_{\boldsymbol{z}} = 0 \text{ on } \Gamma_{\boldsymbol{z}}^{\mathrm{E}} \}$

Comparison of flux reconstructions

Weinstein: Find $\mathbf{q}_{h,i} \in \mathbf{W}_h$ minimizing

$$\|\nabla u_{h,i} - \mathbf{q}_{h,i}\|_0^2 + \frac{1}{\gamma}\|\lambda_{h,i} u_{h,i} + \operatorname{div} \mathbf{q}_{h,i}\|_0^2$$

Lehmann–Goerisch: Find $\boldsymbol{\sigma}_{h,i} \in \boldsymbol{W}_h$ minimizing

$$\left\| \frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma} - \boldsymbol{\sigma}_{h,i} \right\|_0^2 + \frac{1}{\gamma} \left\| \frac{\lambda_{h,i} u_{h,i}}{\lambda_{h,i} + \gamma} + \operatorname{div} \boldsymbol{\sigma}_{h,i} \right\|_0^2$$

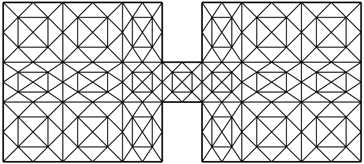
Thus,

$$\boldsymbol{\sigma}_{h,i} = \frac{\mathbf{q}_{h,i}}{\lambda_{h,i} + \gamma}$$

[Vejchodský 2018]

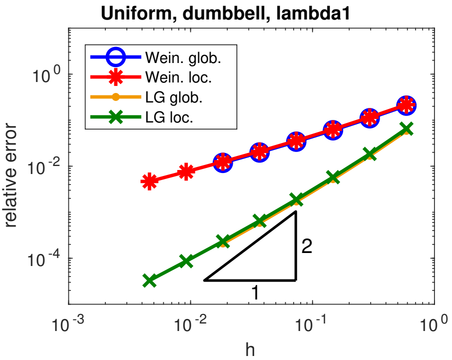
Example: dumbbell

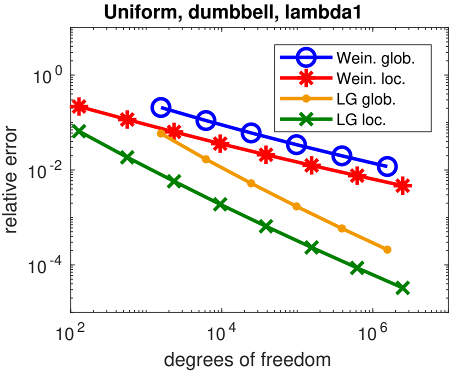
$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = \text{dumbbell}$$
$$u_n = 0 \quad \text{on } \partial\Omega$$



Rel. error: $\dfrac{|\lambda_n - \lambda_{h,n}|}{\lambda_n} \leq \dfrac{\lambda_{h,n} - \ell_n}{\ell_n}$

$\gamma = 10^{-6}$





How to get the a priori lower bound ρ?

Monotonicity principle: If $V \subset \widetilde{V}$ then $\mathcal{V}^{(n)} \subset \widetilde{\mathcal{V}}^{(n)}$ and

$$\tilde{\lambda}_n = \min_{E \in \widetilde{\mathcal{V}}^{(n)}} \max_{v \in E} R(v) \le \min_{E \in \mathcal{V}^{(n)}} \max_{v \in E} R(v) = \lambda_n$$

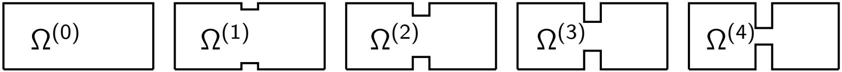
Example 1.

$$\Omega \subset \widetilde{\Omega} \quad \Rightarrow \quad H_0^1(\Omega) \subset H_0^1(\widetilde{\Omega}) \quad \Rightarrow \quad \tilde{\lambda}_n \le \lambda_n$$

Example 2.

$$H_0^1(\Omega) \subset H^1(\Omega) \quad \Rightarrow \quad \lambda_n^{\mathrm{Neumann}} \le \lambda_n^{\mathrm{Dirichlet}}$$

Homotopy



Analytically:	$\rho = 12.16$	$\rho = 11.39$	$\rho = 10.77$	$\rho = 9.988$
$12.16 \le \lambda_{17}^{(0)}$	$\ell_{15} \doteq 11.39$	$\ell_{13} \doteq 10.77$	$\ell_{11} \doteq 9.988$	

[Plum 1990, 1991]

Adaptive mesh refinement

Recall the residual

$$w \in V : \quad (\nabla w, \nabla v) = (\nabla u_{h,i}, \nabla v) - \lambda_{h,i}(u_{h,i}, v) \quad \forall v \in V$$

Recall theorem:

$$\|\nabla w\|_0 \le \eta, \quad \text{where } \eta^2 = \|\nabla u_{h,i} - \mathbf{q}_{h,i}\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\|\lambda_{h,i} u_{h,i} + \operatorname{div} \mathbf{q}_{h,i}\|^2_{L^2(\Omega)}$$

Local error indicators for mesh refinement:

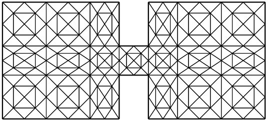
$$\eta_K^2 = \|\nabla u_{h,i} - \mathbf{q}_{h,i}\|^2_{L^2(K)} + \frac{1}{\gamma}\|\lambda_{h,i} u_{h,i} + \operatorname{div} \mathbf{q}_{h,i}\|^2_{L^2(K)}$$

Note: Good for both Weinstein and Lehmann–Goerisch method:

$$\boldsymbol{\sigma}_{h,i} = \frac{\mathbf{q}_{h,i}}{\lambda_{h,i} + \gamma}$$

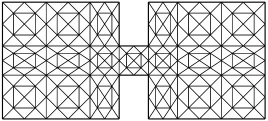
Example: dumbbell

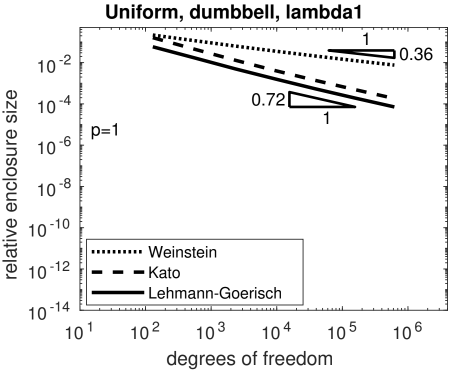
$$
\begin{aligned}
-\Delta u_i &= \lambda_i u_i \quad && \text{in } \Omega \\
u_i &= 0 && \text{on } \partial\Omega
\end{aligned}
$$



Example: dumbbell

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

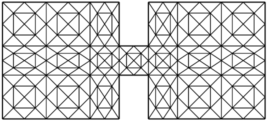


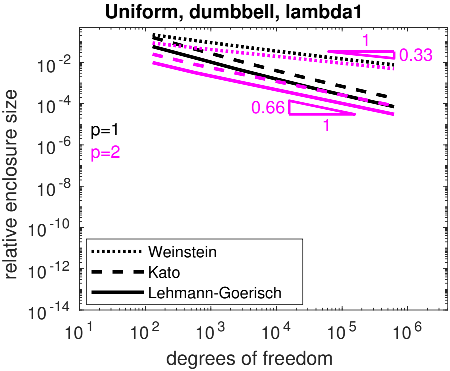


- relative enclosure size: $(\lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}, \quad \rho := \ell_{11}^{\mathrm{Wein}} \approx 10.0017 \le \lambda_{11}$

Example: dumbbell

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$





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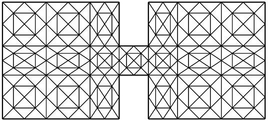

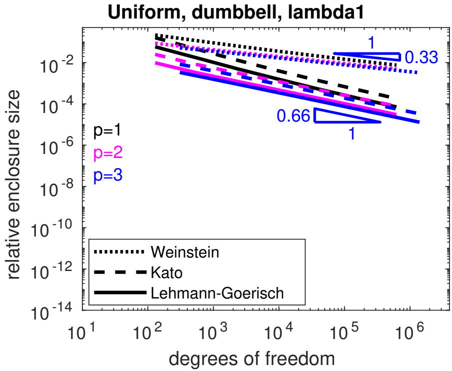

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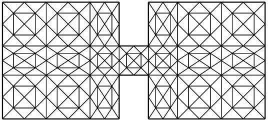

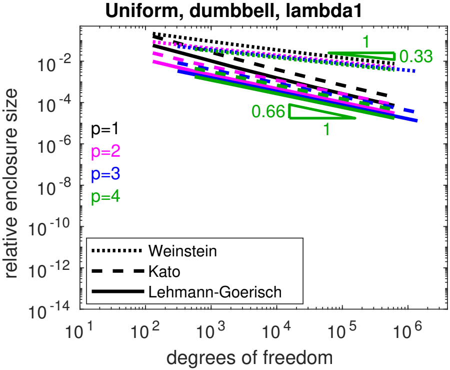

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Example: dumbbell

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\begin{aligned}
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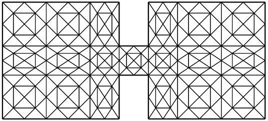
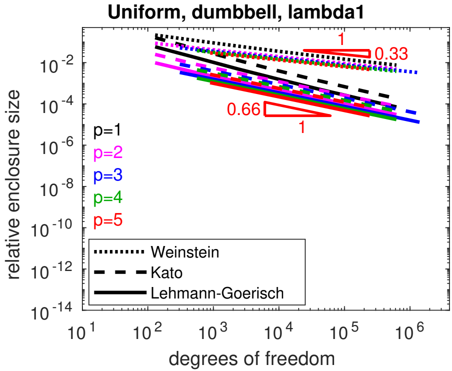
$$





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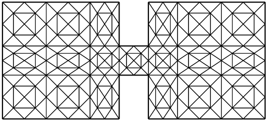

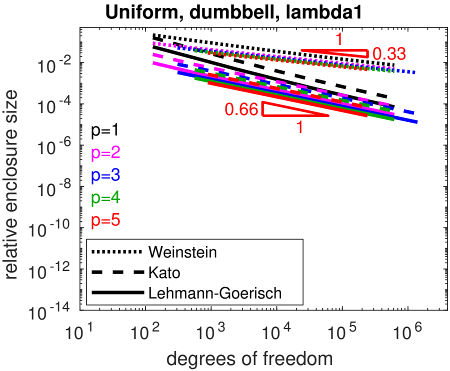

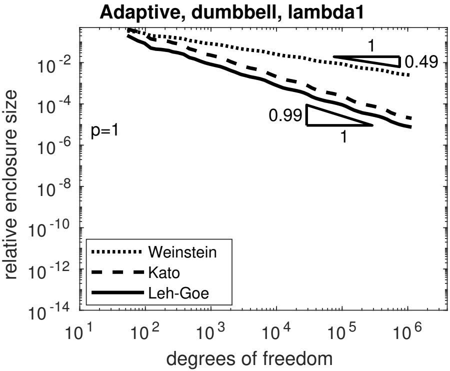

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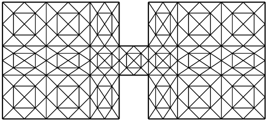

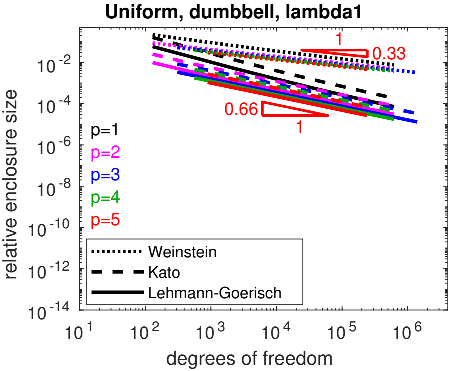

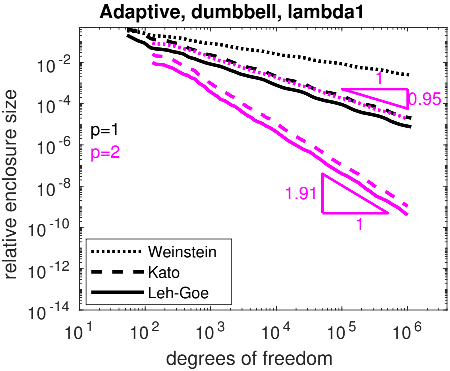

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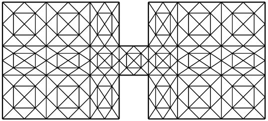

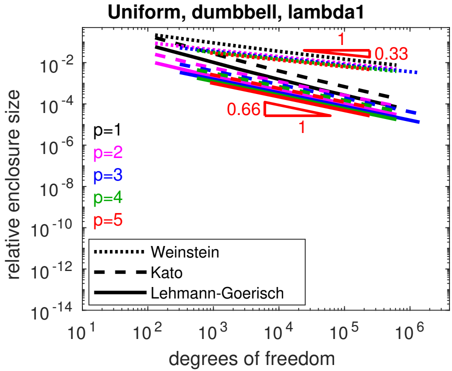

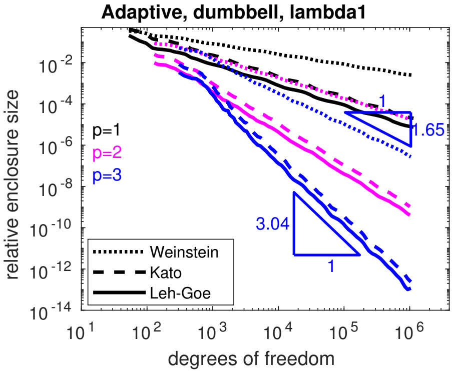

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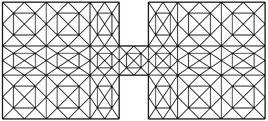

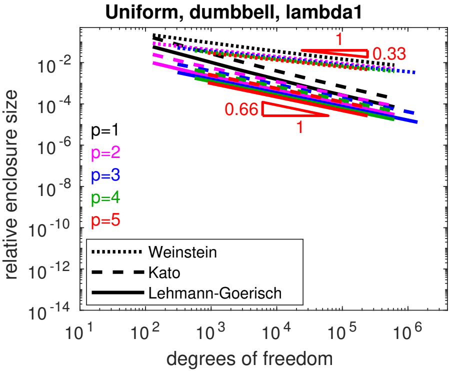

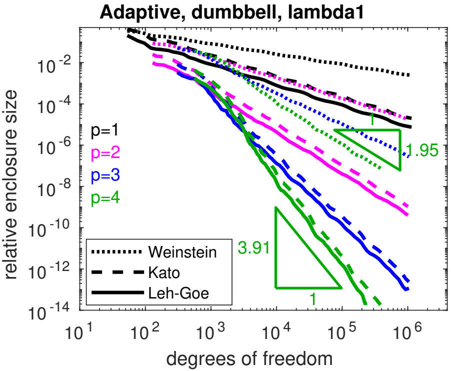

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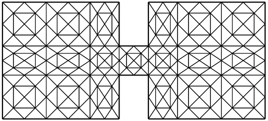

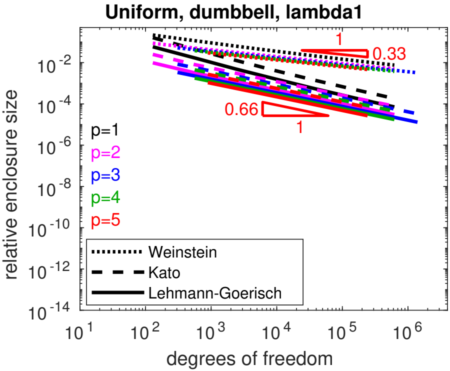

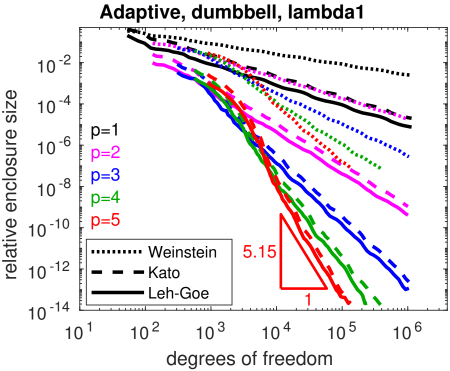

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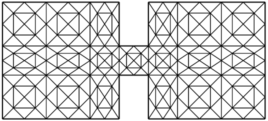

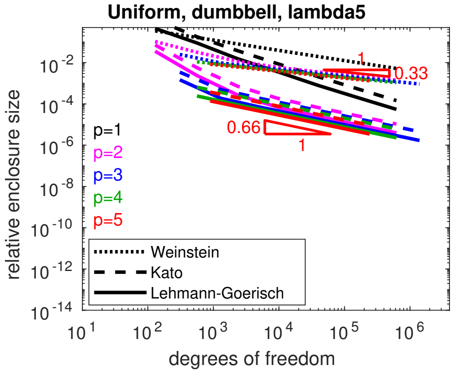

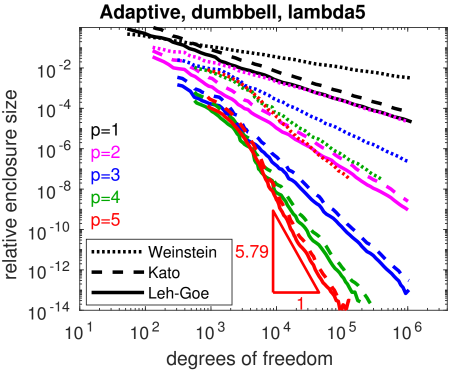

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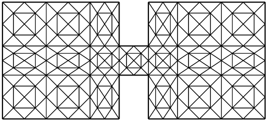
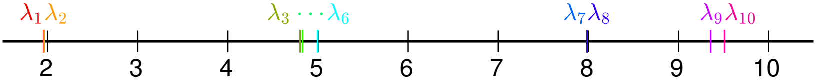




Computed bounds ($p = 5$, adaptive):

$$1.9557937945883 \leq \lambda_1 \leq 1.9557937945884$$
$$1.9606830315950 \leq \lambda_2 \leq 1.9606830315951$$
$$4.8007611240339 \leq \lambda_3 \leq 4.8007611240345$$
$$4.8298952545005 \leq \lambda_4 \leq 4.8298952545010$$
$$4.9968370972489 \leq \lambda_5 \leq 4.9968370972490$$
$$4.9968509041015 \leq \lambda_6 \leq 4.9968509041016$$
$$7.9869672921028 \leq \lambda_7 \leq 7.9869672921038$$
$$7.9870343068216 \leq \lambda_8 \leq 7.9870343068227$$

Lehmann–Goerisch method – summary

- optimal speed of convergence
- implementation based on standard FEM
- adaptivity for free
- naturally generalize to higher orders
- good for a wide class of problems
- an a priori lower bound on some eigenvalue is needed

4. Lower bounds on eigenvalues

4.3 Interpolation constant based methods

[Carstensen, Gallistl, Gedicke 2014], [Liu 2015]

Nonconforming approximation

Eigenvalue problem: Find λ_n and $u_n \in V \setminus \{0\}$ such that

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Finite dimensional space: $\dim V_h = N < \infty$, but it can be $V_h \not\subset V$.
Discrete eigenvalue problem: Find $\lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h \setminus \{0\}$:

$$a(u_{h,n}, v_h) = \lambda_{h,n} b(u_{h,n}, v_h) \quad \forall v_h \in V_h$$

Definition:
$V(h) = V + V_h = \{v + v_h : v \in V,\ v_h \in V_h\}$

Extensions of bilinear forms:
$a_h, b_h : V(h) \times V(h) \to \mathbb{R}$
$a_h(u, v) = a(u, v) \quad \text{and} \quad b_h(u, v) = b(u, v) \quad \forall u, v \in V$
$a_h(\cdot, \cdot)$ is symmetric and $V(h)$-elliptic
$b_h(\cdot, \cdot)$ is symmetric and positive semidefinite on $V(h)$
Notation: $a = a_h$ and $b = b_h$

Lemmas

Lemma 1 (Discrete Friedrichs inequality).

$$|v_h|_b \leq \lambda_{h,1}^{-1/2} \|v_h\|_a \quad \forall v_h \in V_h$$

Proof. $\lambda_{h,1} = \min_{w_h \in V_h} \frac{\|w_h\|_a^2}{|w_h|_b^2} \leq \frac{\|v_h\|_a^2}{|v_h|_b^2}$ $\square$

Elliptic projection: $P_h : V(h) \to V_h$

$$a(u - P_h u, v_h) = 0 \quad \forall v_h \in V_h$$

Lemma 2.

$$\|u\|_a^2 = \|P_h u\|_a^2 + \|u - P_h u\|_a^2$$

Proof.

$$\|u - P_h u\|_a^2 = \|u\|_a^2 - 2a(u, P_h u) + \|P_h u\|_a^2$$

$$a(u, P_h u) = a(P_h u, P_h u) = \|P_h u\|_a^2$$ $\square$

Lower bound

Theorem. Let $|u - P_h u|_b \le C_h \|u - P_h u\|_a$ for all $u \in V$. Then

$$\frac{\lambda_{h,n}}{1 + \lambda_{h,n} C_h^2} \le \lambda_n, \quad n = 1, 2, \ldots, N.$$

Proof (for λ_1 only). Let $v \in V$.

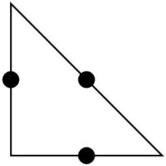
$$\begin{aligned}
|v|_b &\le |P_h v|_b + |v - P_h v|_b \\
&\le \lambda_{h,1}^{-1/2} \|P_h v\|_a + C_h \|v - P_h v\|_a \\
&\le \left(\lambda_{h,1}^{-1} + C_h^2\right)^{1/2} \left(\|P_h v\|_a^2 + \|v - P_h v\|_a^2\right)^{1/2} \\
&= \left(\frac{1 + \lambda_{h,1} C_h^2}{\lambda_{h,1}}\right)^{1/2} \|v\|_a
\end{aligned}$$

$$\lambda_1 = \min_{v \in V} \frac{\|v\|_a^2}{|v|_b^2} \ge \frac{\lambda_{h,1}}{1 + \lambda_{h,1} C_h^2}$$

Crouzeix–Raviart (CR) elements

Laplace eigenvalue problem: Find $\lambda_n \in \mathbb{R}$, $u_n \in H_0^1(\Omega) \setminus \{0\}$:

$$(\nabla u_n, \nabla v) = \lambda_n (u_n, v) \quad \forall v \in H_0^1(\Omega)$$



CR space: $v_h \in V_h^{\mathrm{CR}}$ if

- $v_h|_K \in \mathbb{P}^1(K)$
- v_h is continuous at midpoints of interior edges
- $v_h = 0$ at midpoints of boundary edges

CR eigenvalue problem: Find $\lambda_{h,i}^{\mathrm{CR}} \in \mathbb{R}$, $u_{h,i}^{\mathrm{CR}} \in V_h^{\mathrm{CR}} \setminus \{0\}$:

$$(\nabla u_{h,i}^{\mathrm{CR}}, \nabla v_h) = \lambda_{h,i}^{\mathrm{CR}} (u_{h,i}^{\mathrm{CR}}, v_h) \quad \forall v_h \in V_h^{\mathrm{CR}}.$$

Crouzeix–Raviart interpolation

Let e_i, $i = 1, 2, 3$, be edges of triangle K.
Definition: $\Pi_h : H^1(K) \to \mathbb{P}^1(K)$ such that

$$\int_{e_i} u - \Pi_h u \,\mathrm{d}s = 0 \quad \forall i = 1, 2, 3.$$

Note: If m_i is a midpoint of e_i then $\Pi_h u(m_i) = \frac{1}{|e_i|} \int_{e_i} u \,\mathrm{d}s.$

Lemma. $\Pi_h = P_h$
Proof.
Let $u \in H^1(\Omega) \oplus V_h^{\mathrm{CR}}$ and $v_h \in V_h^{\mathrm{CR}}$.

$$a(u - \Pi_h u, v_h) = \sum_{K \in \mathcal{T}_h} \int_K \nabla(u - \Pi_h u) \cdot \nabla v_h$$

$$= \sum_{K \in \mathcal{T}_h} \left(\sum_{i=1}^{3} \int_{e_i} (u - \Pi_h u) \underbrace{\frac{\partial v_h}{\partial \boldsymbol{n}}}_{=\text{const.}} \,\mathrm{d}s - \int_K (u - \Pi_h u) \underbrace{\Delta v_h}_{=0} \,\mathrm{d}x \right) = 0$$

The value of C_h

Interpolation error estimate:

$$\|u - \Pi_h u\|_{L^2(\Omega)} \leq C_h \|\nabla u - \nabla \Pi_h u\|_{L^2(\Omega)}$$

Local interpolation error estimate:

$$\|u - \Pi_h u\|_{L^2(K)} \leq C_h(K) \|\nabla u - \nabla \Pi_h u\|_{L^2(K)}$$

Lemma.

$$C_h \leq \max_{K \in \mathcal{T}_h} C_h(K)$$

Proof.

$$\|u - \Pi_h u\|^2_{L^2(\Omega)} = \sum_{K \in \mathcal{T}_h} \|u - \Pi_h u\|^2_{L^2(K)} \leq \sum_{K \in \mathcal{T}_h} C_h^2(K) \|\nabla u - \nabla \Pi_h u\|^2_{L^2(K)}$$

$$\leq \max_{K \in \mathcal{T}_h} C_h^2(K) \|\nabla u - \nabla \Pi_h u\|^2_{L^2(\Omega)}$$

□

Explicit estimates of C_h

Interval

- $C_h = h/\pi$

Triangle

- $C_h = 0.4396h$ [Carstensen, Gedicke 2014]
- $C_h = 0.2983h$ [Carstensen, Gallistl 2014]
- $C_h = 0.1893h$ [Liu 2015]

Tetrahedron

- $C_h = 0.3804h$ [Liu 2015]

Explicit estimate of C_h for an interval

Setting: $\Omega = (\alpha, \beta), \quad V = H_0^1(\alpha, \beta),$

$a(u, v) = \int_\alpha^\beta u'v' \,\mathrm{d}x, \; b(u, v) = \int_\alpha^\beta uv \,\mathrm{d}x$

Partition: $\alpha = z_0 < z_1 < \cdots < z_N = \beta$

Elements: $K_i = [z_{i-1}, z_i], \; i = 1, 2, \dots, N,$

$h_i = z_i - z_{i-1}, \; h = \max_{i=1,\dots,N} h_i$

CR space: $V_h = \{v \in H_0^1(\alpha, \beta) : v|_{K_i} \in \mathbb{P}^1(K_i), \;\; i = 1, 2, \dots, N\}$

Interpolation: $\Pi_h : H_0^1(\alpha, \beta) \to V_h$

$(\Pi_h u)(x_i) = u(x_i), \; i = 0, \dots, N$

Lemma.

$$\|u - \Pi_h u\|_{L^2(\Omega)} \le \frac{h}{\pi} \|u' - (\Pi_h u)'\|_{L^2(\Omega)}$$

Proof.

$$\min_{v \in H^1(K_i)} R(v - \Pi_h v) = \min_{w \in H_0^1(K_i)} R(w) = R\left(\sin \frac{\pi(x - z_i)}{h_i}\right) = \pi^2 / h_i^2$$

Upper bound

Interpolation to continuous functions: $\mathcal{I} : V_h^{\mathrm{CR}} \to \widetilde{V}_h \subset H^1(\Omega)$

Examples:

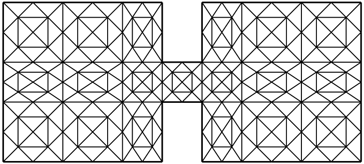
- Oswald quasi-interpolation [Oswald 1994]
- Interpolation to refined mesh [Carstensen, Merdon 2013]

Upper bound

- $\mathcal{T}_h^*$ is the red refinement of $\mathcal{T}_h$
- $u_{h,i}^* = \mathcal{I}_{\mathrm{CM}} \tilde{u}_{h,i}^{\mathrm{CR}}$ for $i = 1, 2, \dots, m$
- $\boldsymbol{S}, \boldsymbol{Q} \in \mathbb{R}^{m \times m}$ with entries $\boldsymbol{S}_{j,k} = (\nabla u_{h,j}^*, \nabla u_{h,k}^*)$ and $\boldsymbol{Q}_{j,k} = (u_{h,j}^*, u_{h,k}^*)$
- $\boldsymbol{S}\mathbf{y}_i = \Lambda_i^* \boldsymbol{Q}\mathbf{y}_i, \quad i = 1, 2, \dots, m$
- $\Lambda_1^* \leq \Lambda_2^* \leq \cdots \leq \Lambda_m^*$
- $\lambda_i \leq \Lambda_i^* \quad$ for $i = 1, 2, \dots, m$

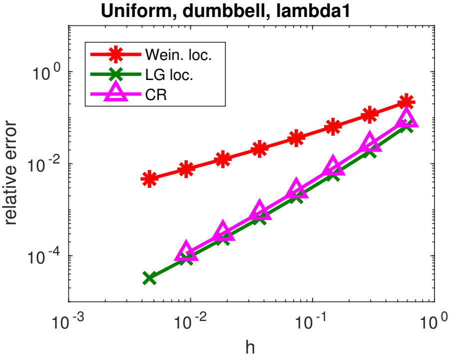
Example: dumbbell

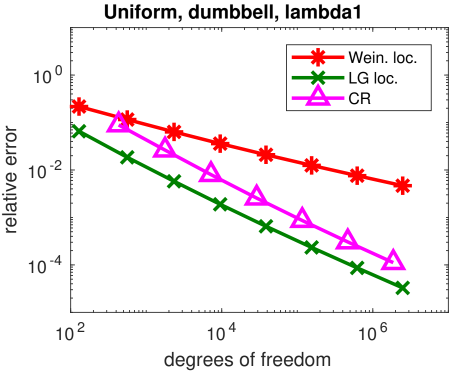
$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = \text{dumbbell}$$
$$u_n = 0 \quad \text{on } \partial\Omega$$



Rel. error: $\dfrac{|\lambda_n - \lambda_{h,n}|}{\lambda_n} \le \dfrac{\lambda_{h,n} - \ell_n}{\ell_n}$

$\gamma = 10^{-6}$





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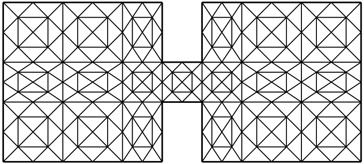
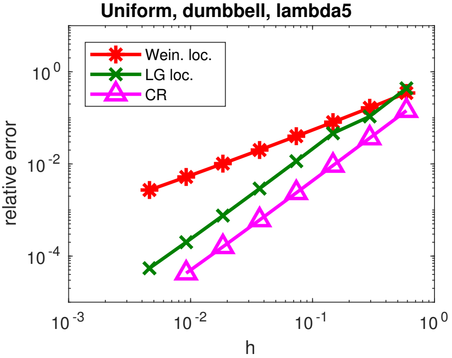
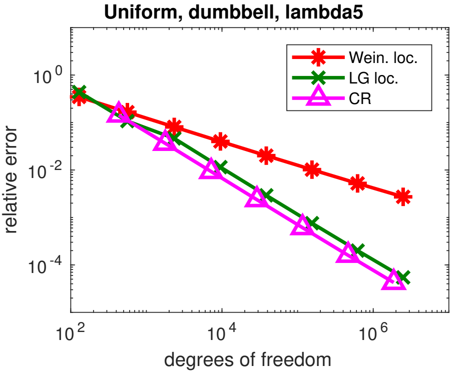
$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = \text{dumbbell}$$
$$u_n = 0 \quad \text{on } \partial\Omega$$



Rel. error: $\dfrac{|\lambda_n - \lambda_{h,n}|}{\lambda_n} \le \dfrac{\lambda_{h,n} - \ell_n}{\ell_n}$

$\gamma = 10^{-6}$





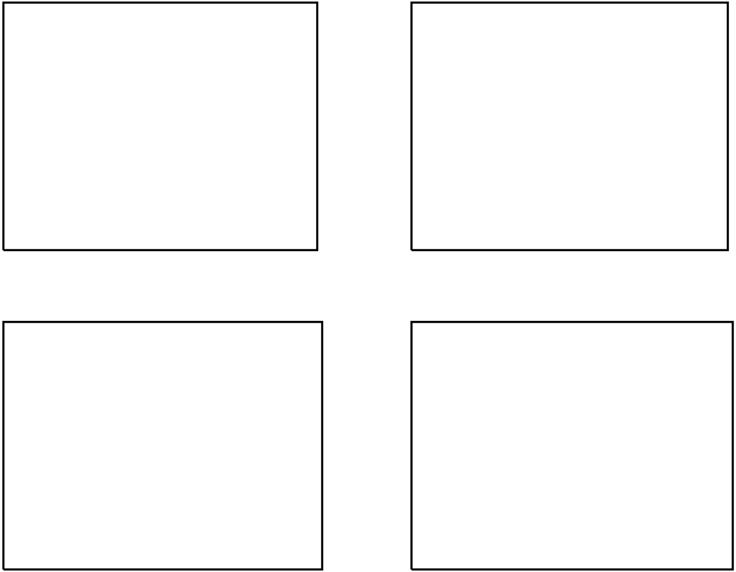
Interpolation constant based method – summary

- no a priori information needed
- optimal speed of convergence
- easy to implement
- interpolation constant known in special cases only
- adaptivity is not for free
- higher order variant is not available

5. Guaranteed bounds on eigenfunctions

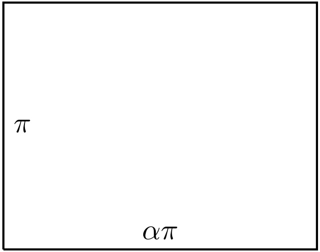
[work in progress, collaboration with X. Liu]

Laplace eigenvalue problem in a rectangle

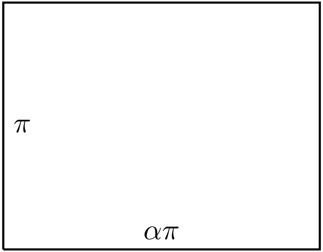


Laplace eigenvalue problem in a rectangle

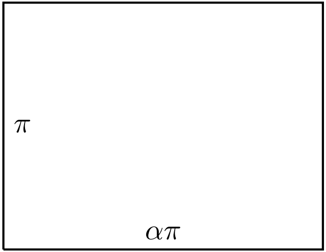
$\alpha = 1.27$



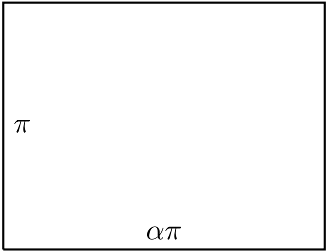
$\alpha = 1.28$



$\alpha = 1.29$

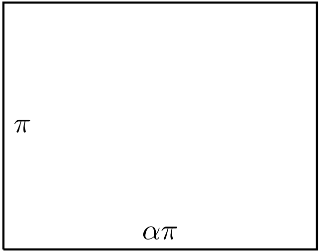


$\alpha = 1.30$

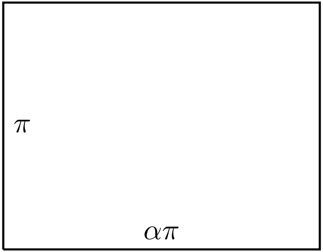


Laplace eigenvalue problem in a rectangle

$\alpha = 1.27, \ \lambda_4 = 6.4800$



$\alpha = 1.28$



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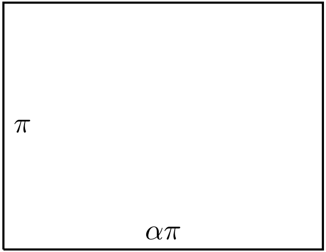
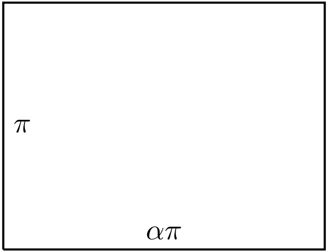
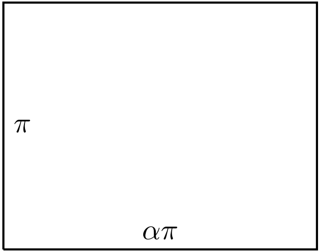


$\alpha = 1.30$

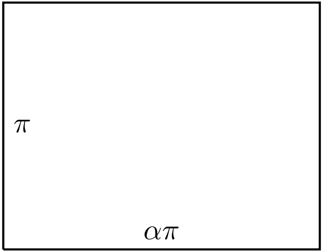


Laplace eigenvalue problem in a rectangle

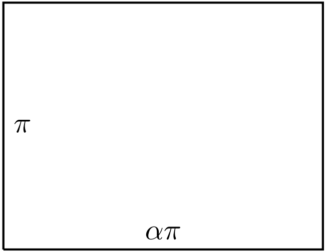
$\alpha = 1.27,\ \lambda_4 = 6.4800$



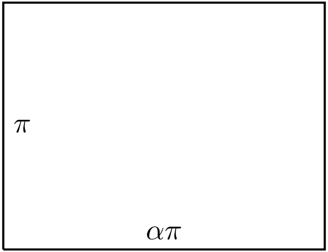
$\alpha = 1.28,\ \lambda_4 = 6.4414$



$\alpha = 1.29$



$\alpha = 1.30$



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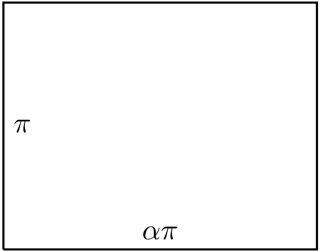

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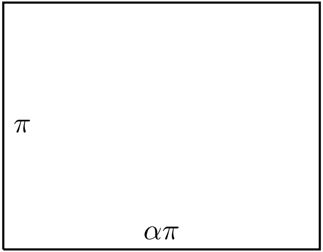
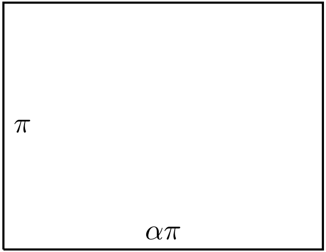


$\alpha = 1.29,\ \lambda_4 = 6.4037$



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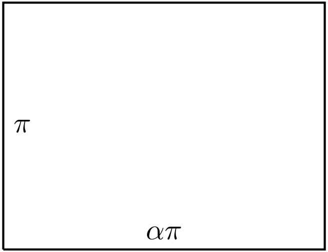

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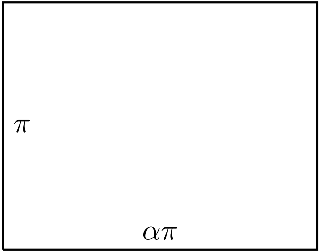

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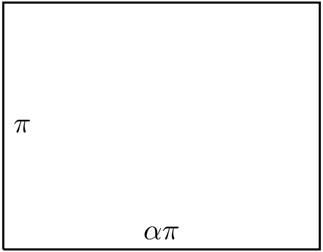

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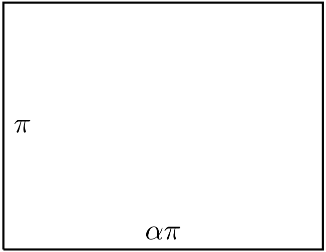
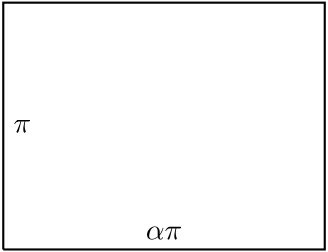


$\alpha = 1.30,\ \lambda_4 = 6.3254$



Laplace eigenvalue problem in a rectangle

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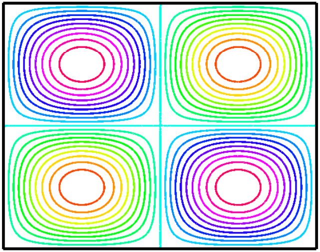

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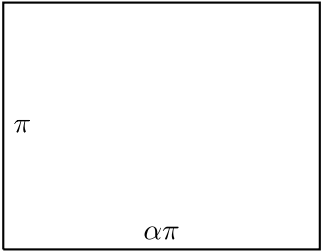
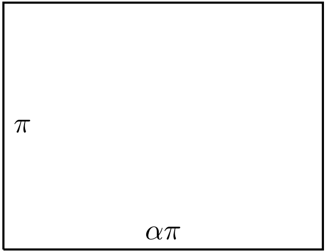


$\alpha = 1.29,\ \lambda_4 = 6.4037$



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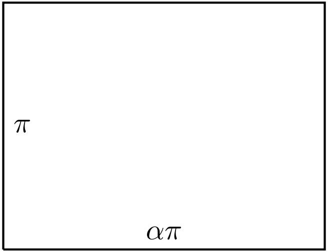

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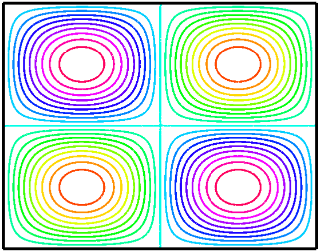

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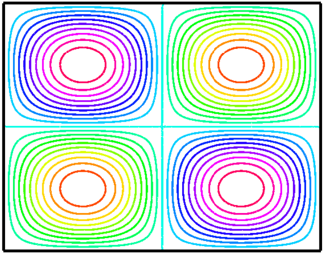

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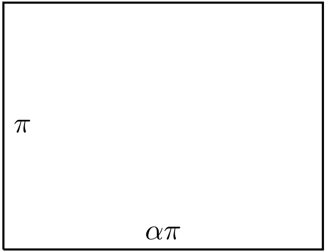

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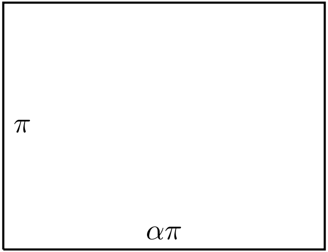
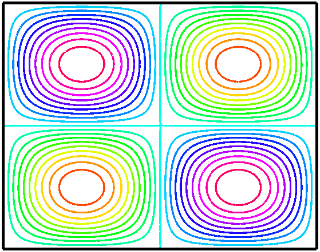


Laplace eigenvalue problem in a rectangle

$\alpha = 1.27,\ \lambda_4 = 6.4800$



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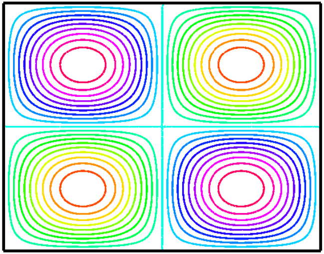

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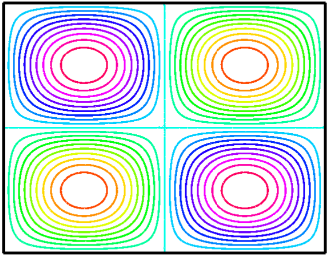

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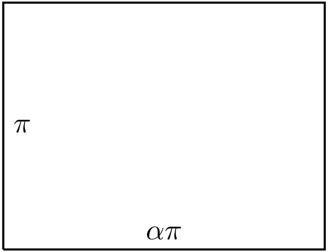

Laplace eigenvalue problem in a rectangle

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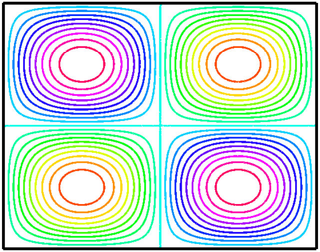
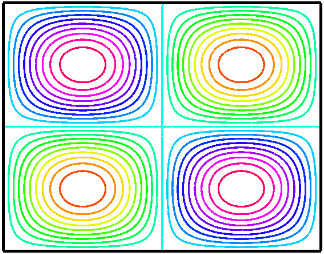
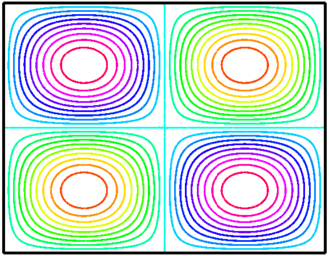


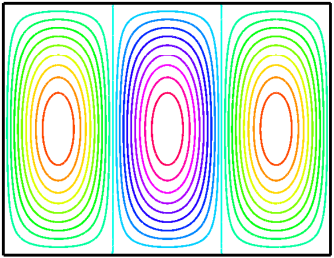
$\alpha = 1.28,\ \lambda_4 = 6.4414$



$\alpha = 1.29,\ \lambda_4 = 6.4037$



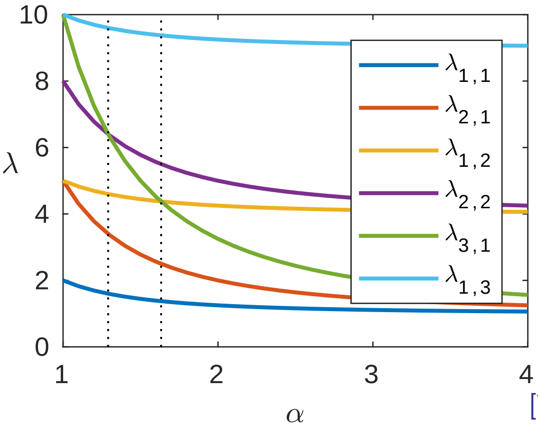
$\alpha = 1.30,\ \lambda_4 = 6.3254$



Laplace eigenvalue problem in a rectangle

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = (0, \alpha\pi) \times (0, \pi)$$
$$u_n = 0 \quad \text{on } \partial\Omega$$

Dependence of eigenvalues on α



$$\lambda_{k,m} = \frac{k^2}{\alpha^2} + m^2$$

$$u_{k,m} = \sin\frac{kx}{\alpha}\sin(my)$$

$$\alpha^* = \sqrt{5/3} \approx 1.2910$$

[Trefethen, Betcke 2006]

Error bounds on eigenfunctions

Problem

- Eigenfunctions may be ill-posed $\Rightarrow$ spaces of eigenfunctions
- Directed distance of spaces $\delta(E, E_h)$ [Meyer 2000]

Assume

- $\lambda_n, \lambda_{n+1}, \ldots, \lambda_N$ (cluster)
- $E = \operatorname{span}\{u_n, u_{n+1}, \ldots, u_N\}$ (space of eigenfunctions)
- $E_h = \operatorname{span}\{u_{h,n}, u_{h,n+1}, \ldots, u_{h,N}\}$ (its approximation)
- $\ell_i \le \lambda_i \le \lambda_{h,i}$ (two sided bounds on eigenvalues)

$\Rightarrow$

Compute an upper bound on $\delta(E, E_h)$

Directed distance of spaces

Definition

Let E and E_h be two subspaces of a Hilbert space V then

$$\delta(E, E_h) = \max_{\substack{v \in E \\ \|v\|=1}} \min_{v_h \in E_h} \|v - v_h\|$$

Properties

- if $\dim E = \dim E_h$ then $\delta(E, E_h) = \delta(E_h, E)$
- $\delta^2(E, E_h) = 1 - \min_{\substack{v \in E \\ \|v\|=1}} \max_{\substack{v_h \in E_h \\ \|v_h\|=1}} |(v, v_h)|^2$

Example

Let $E = \mathrm{span}\{u\}$ and $E_h = \mathrm{span}\{u_h\}$ then

$$\delta^2(E, E_h) = 1 - \frac{|(u, u_h)|^2}{\|u\|^2 \|u_h\|^2} = 1 - \cos^2\alpha = \sin^2\alpha$$

$$\|u - u_h\|^2 = \|u\|^2 + \|u_h\|^2 - 2\|u\|\|u_h\|\sqrt{1 - \delta^2(E, E_h)}$$

Lehmann-like estimate of eigenfunctions

Eigenvalue problem:
Find $\lambda_n > 0$ and $u_n \in V \setminus \{0\}$ such that

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V.$$

Consider

- $E = \operatorname{span}\{u_n, u_{n+1}, \dots, u_N\}, \quad b(u_i, u_j) = \delta_{ij}, \quad a(u_i, u_j) = \lambda_i \delta_{ij}$
- $E_h = \operatorname{span}\{u_{h,n}, u_{h,n+1}, \dots, u_{h,N}\}$

Goal

Upper bound on $\delta(E, E_h) = \max_{\substack{v \in E \\ \|v\|_a = 1}} \min_{v_h \in E_h} \|v - v_h\|_a$

Lehmann-like estimate of eigenfunctions

Theorem

Let $\lambda_{n-1} \le \xi < \lambda_n, \quad \lambda_N < \rho \le \lambda_{N+1}, \quad \theta \ge \max_{i=n,\dots,N} \left(\frac{\xi+\rho}{\lambda_i} - \frac{\xi\rho}{\lambda_i^2} \right)$

- $u_{h,n}, u_{h,n+1}, \dots, u_{h,N} \in V$ be linearly independent
- $A_{0,ij} = a(u_{h,i}, u_{h,j})$
- $A_{1,ij} = b(u_{h,i}, u_{h,j})$
- $w_i \in V : \quad a(w_i, v) = b(u_{h,i}, v) \quad \forall v \in V$
  $A_{2,ij} = a(w_i, w_j)$
- $\mu_{\min}$ be the smallest eigenvalue of $[(\xi+\rho)A_1 - \xi\rho A_2]\,\boldsymbol{x} = \mu A_0 \boldsymbol{x}$

Then

$$\delta^2(E, E_h) \le \frac{\theta - \mu_{\min}}{\theta - 1}$$

Lehmann–Goerisch-like estimate of eigenfunctions

Theorem

Let $\lambda_{n-1} \le \xi < \lambda_n, \quad \lambda_N < \rho \le \lambda_{N+1}, \quad \theta \ge \max_{i=n,\dots,N} \left(\frac{\xi+\rho}{\lambda_i} - \frac{\xi\rho}{\lambda_i^2} \right)$

- $u_{h,n}, u_{h,n+1}, \dots, u_{h,N} \in V$ be linearly independent
- $A_{0,ij} = a(u_{h,i}, u_{h,j})$
- $A_{1,ij} = b(u_{h,i}, u_{h,j})$
- X ... vector space
  $\mathcal{B}$ ... positive semidefinite symmetric bilinear form on X
  $T : V \to X$ ... linear operator:
  (a) $\mathcal{B}(Tu, Tv) = a(u,v) \quad \forall u,v \in V$
  (b) $\hat{\mathbf{w}}_i \in X : \quad \mathcal{B}(\hat{\mathbf{w}}_i, Tv) = b(\tilde{u}_i, v) \quad \forall v \in V$
  (c) $\hat{A}_{2,ij} = \mathcal{B}(\hat{\mathbf{w}}_i, \hat{\mathbf{w}}_j)$
- $\hat{\mu}_{\min}$ be the smallest eigenvalue of $\left[(\xi+\rho)A_1 - \xi\rho\hat{A}_2\right]\boldsymbol{x} = \hat{\mu}A_0\boldsymbol{x}$

Then

$$\delta^2(E, E_h) \le \frac{\theta - \hat{\mu}_{\min}}{\theta - 1}$$

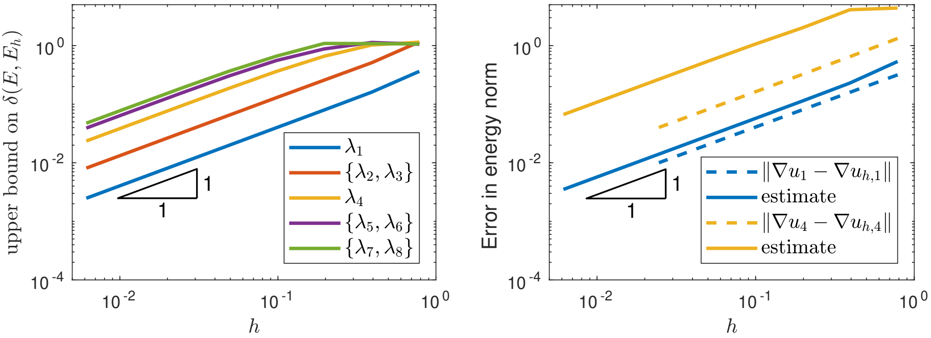
Example

Laplace eigenvalue problem in a square

$$
\begin{aligned}
-\Delta u_n &= \lambda_n u_n && \text{in } \Omega = (0,\pi)^2 \\
u_n &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Exact eigenvalues

$\lambda_1 = 2, \quad \lambda_2 = \lambda_3 = 5, \quad \lambda_4 = 8, \quad \lambda_5 = \lambda_6 = 10, \quad \lambda_7 = \lambda_8 = 13$



6. Literature

Literature (very incomplete)

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- S. Brenner, R. Scott, *The mathematical theory of finite element methods*, Springer 1994, 3 editions.
- T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1976.

Literature (very incomplete)

Papers on conforming approaches

- G. Temple, *The theory of Rayleighs principle as applied to continuous systems*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 119 (2) (1928) 276–293.
- A. Weinstein, *Étude des Spectres des quations aux Dérivées Partielles de la Théorie des Plaques élastiques*, in: Mem. Sci. Math., vol. 88, Gauthier-Villars, Paris, 1937, p. 63.
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- F. Goerisch, H. Haunhorst, *Eigenwertschranken für Eigenwertaufgaben mit partiellen Differentialgleichungen*, ZAMM Z. Angew. Math. Mech. 65 (3) (1985) 129–135.

Literature (very incomplete)

Papers on interpolation constant based method:

- C. Carstensen, J. Gedicke, *Guaranteed lower bounds for eigenvalues*, Math. Comp. 83 (290) (2014) 2605–2629.
- C. Carstensen, D. Gallistl, *Guaranteed lower eigenvalue bounds for the biharmonic equation*, Numer. Math. 126 (1) (2014) 33–51.
- X. Liu, S. Oishi, *Verified eigenvalue evaluation for the Laplacian over polygonal domains of arbitrary shape*, SIAM J. Numer. Anal. 51 (3) (2013) 1634–1654.
- X. Liu, *A framework of verified eigenvalue bounds for self-adjoint differential operators*, Appl. Math. Comput. 267 (2015) 341–355.

Literature (very incomplete)

My contributions

- I. Šebestová, T. Vejchodský, *Two-sided bounds for eigenvalues of differential operators with applications to Friedrichs, Poincaré, trace, and similar constants*, SIAM J. Numer. Anal. 52 (2014), no. 1, 308–329.
- T. Vejchodský, *Flux reconstructions in the Lehmann–Goerisch method for lower bounds on eigenvalues*, J. Comput. Appl. Math. 340 (2018) 676–690.
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Acknowledgement

My research of eigenvalue problems is supported by the Neuron Impuls project no. 24/2016

Neuron BENEVOLENT FUND FOR SUPPORT OF SCIENCE

Appendix

Guaranteed computations

Weinstein bound:

- λ_*, u_*, $\mathbf{q}$ can be arbitrary
- $\eta^2 = \|\nabla u_* - \mathbf{q}\|_0^2 + \frac{1}{\gamma}\|\lambda_* u_* + \operatorname{div}\mathbf{q}\|_0^2$

  must be evaluated exactly $(*)$

Lehmann–Goerisch method:

- $\tilde{u}_i$, $\boldsymbol{\sigma}_i$ can be arbitrary
- $(A_0 - \rho A_1)\hat{\boldsymbol{x}} = \hat{\mu}(A_0 - 2\rho A_1 + \rho^2 \hat{A}_2)\hat{\boldsymbol{x}}$

  must be solved exactly $(*)$

Interpolation constant based method:

- $\lambda_{h,i}^{\mathrm{CR}}$ must be computed exactly $(*)$

$(*)$ or bounded by interval arithmetic!

Thank you for your attention

Tomáš Vejchodský (vejchod@math.cas.cz)

Institute of Mathematics
Czech Academy of Sciences

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