Finite difference MAC scheme for the compressible Navier-Stokes equations

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July 11, 2018 Tongji University

Compressible barotropic Navier-Stokes

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0. \tag{1a}$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \nabla \cdot \mathbb{S} \tag{1b}$$

ρ : density

$\mathbf{u}$: velocity

p : pressure, $p = a\rho^\gamma$

$\mathbb{S}$: viscous stress, $\mathbb{S} = \mu \nabla \mathbf{u}, \mu > 0$

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Boundary condition for $\mathbf{u}$

$$\mathbf{u}|_{\partial\Omega} = 0 \quad \text{or} \quad \text{periodic} \tag{1c}$$

Initial values

$$\rho(\mathbf{x}, 0) = \rho_0 > 0 \tag{1d}$$

Finite Volume-Finite Element by T. Karper, 2013, $\gamma > 3$

- E. Feireisl, R. Hošek, D. Maltese, A. Novotný, 2017
  bounded numerical solution
- E. Feireisl, M. Lukáčová-Medviďová, 2017
  dissipative measure-valued solution, $\gamma \in (1, 2)$

Motivation

Finite Volume-Finite Element by T. Karper, 2013, $\gamma > 3$

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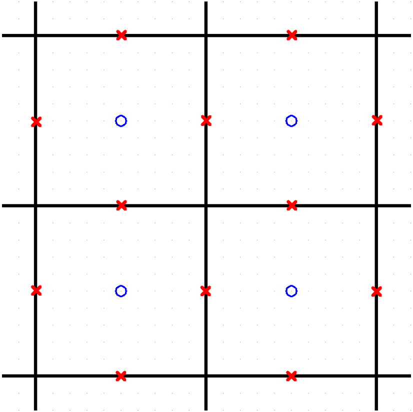
Our interests: Finite Difference, stability and convergence

Notations I

- Elements: $\Omega_h = \cup K$
- Faces: $\mathcal{E}$
- Exterior faces: $\mathcal{E}_{ext} = \partial\Omega \cup \mathcal{E}$.
- Interior faces: $\mathcal{E}_{int} = \mathcal{E} \setminus \mathcal{E}_{ext}$
- Interior faces of K: $\mathcal{E}_{int}(K)$
- Interior neighbours of K: $\mathcal{N}(K)$
- $\sigma = \overrightarrow{K|L}$ if $x_L = x_K + \frac{h}{2}\mathbf{e}_s$
  $\sigma_{K,s+}$
- Primary grid $\circ$: ρ, p
- Dual grid $\times$: $\mathbf{u}$



Between grids

$$\{f\}_\sigma = \frac{1}{2}(f_K + f_L), \quad \forall \sigma = K|L \in \mathcal{E}_{int}$$

$$\bar{\mathbf{g}}_K = \frac{1}{2}\begin{pmatrix} g^1_{\sigma_{K,1+}} + g^1_{\sigma_{K,1-}} \\ g^2_{\sigma_{K,2+}} + g^2_{\sigma_{K,2-}} \\ g^3_{\sigma_{K,3+}} + g^3_{\sigma_{K,3-}} \end{pmatrix}$$

Notations II

Between grids

$$\{f\}_\sigma = \frac{1}{2}(f_K + f_L), \quad \forall \sigma = K|L \in \mathcal{E}_{int}$$

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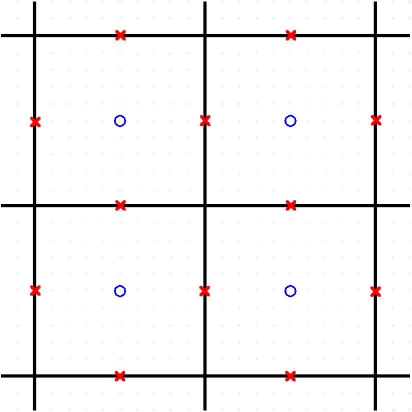
Functional spaces

$X(\Omega_h)$: P0 on primary grid Ω_h

$X(\mathcal{E}_{int})^d$: P0 on dual grid $\mathcal{E}$, and $\mathbf{g}|_{\mathcal{E}_{ext}} = \mathbf{0}$



Time

$$\partial_h^t \phi^n = \frac{\phi^n - \phi^{n-1}}{\Delta t}$$

Space

Let $f \in X(\Omega_h)$, $\mathbf{g} \in X(\mathcal{E}_{int})^d$

$$(\partial_h^s f)_\sigma = \frac{f_L - f_K}{h}, \; \sigma = \overrightarrow{K|L}$$

$$(\Delta_h f)_K = \frac{1}{h^2} \sum_{L \in \mathcal{N}(K)} (f_L - f_K)$$

$$(\Delta_h g)_\sigma = \frac{1}{h^2} \sum_{s=1}^{d} (g_{\sigma - \mathbf{e}_s} - 2 g_\sigma + g_{\sigma + \mathbf{e}_s}).$$

Upwind flux

$$\mathrm{Up}[f, \mathbf{u}]_\sigma = f_K (u_\sigma^s)^+ + f_L (u_\sigma^s)^-$$
$$f^+ = \max\{0, f\}, \quad f^- = \min\{0, f\}$$

Discrete differential operators II

Upwind flux

$$\mathrm{Up}[f,\mathbf{u}]_\sigma = f_K (u_\sigma^s)^+ + f_L (u_\sigma^s)^-$$
$$f^+ = \max\{0, f\}, \quad f^- = \min\{0, f\}$$

Upwind discrete derivative and upwind divergence

$$\partial_s^{\mathrm{Up}}[f,\mathbf{u}]_K = \frac{\mathrm{Up}[f,\mathbf{u}]_{\sigma_{K,s+}} - \mathrm{Up}[f,\mathbf{u}]_{\sigma_{K,s-}}}{h}$$
$$\mathrm{div}_{\mathrm{Up}}[g,\mathbf{u}]_K = \sum_{s=1}^{d} \partial_s^{\mathrm{Up}}[f,\mathbf{u}]_K$$

Upwind flux

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Upwind discrete derivative and upwind divergence

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$$\text{div}_{\text{Up}}[g,\mathbf{u}]_K = \sum_{s=1}^{d} \partial_s^{\text{Up}}[f,\mathbf{u}]_K$$

Let $f \in X(\Omega_h), \mathbf{v} = [v^1, \cdots, v^d] \in X(\mathcal{E}_{int})^d$, then $\sum_{K \in \Omega_h} \int_K \text{div}_{\text{Up}}[f,\mathbf{v}]_K = 0.$

Numerical Scheme

$$\partial_h^t \rho_K^n + \mathrm{div}_{\mathrm{Up}}[\rho^n, \mathbf{u}^n]_K - h^\alpha (\Delta_h \rho^n)_K = 0, \tag{2a}$$

$$\{\partial_h^t (\rho \bar{\mathbf{u}})^n\}_\sigma + \{\mathrm{div}_{\mathrm{Up}}[\rho^n \bar{\mathbf{u}}^n, \mathbf{u}^n]\}_\sigma + \left(\partial_h^s p(\rho^n)\right)_\sigma \mathbf{e}_s$$
$$- \mu\left(\Delta_h \mathbf{u}^n\right)_\sigma - h^\alpha \sum_{r=1}^{d} \left\{\partial_h^r\left(\{\hat{\mathbf{u}}^n\} \partial_h^r \rho^n\right)\right\}_\sigma = 0, \tag{2b}$$

for all $K \in \Omega_h, \sigma \in \mathcal{E}_{int}$ and $n = \{1, \ldots, N\}$, with boundary conditions.

Renormalized continuity equation

$$\boxed{\sum_{K\in\Omega_h}\int_K \partial_h^t B(\rho_K^n) + \left(B'(\rho_K^n)\rho_K^n - B(\rho_K^n)\right)(\mathrm{div}_h \mathbf{u}^n)_K + \mathcal{P}_K\right) = 0}$$

$$\mathcal{P}_K = \frac{\Delta t}{2} B''(\overline{\rho_K^{n-1,n}})|\partial_h^t \rho_K^n|^2 + \frac{1}{2}\sum_{\sigma\in\mathcal{E}(K)}\left(h|\mathbf{u}_\sigma| + h^\alpha\right)B''(\rho_\sigma^\star)|(\partial_h \rho)_\sigma|^2$$

$\mathcal{P}_K \geq 0$ provided B is convex.

Existence of num sol

Lemma 1

Let $\rho_h^{n-1} \in X(\Omega_h)$, $\mathbf{u}_h^{n-1} \in X(\mathcal{E}_{int})^d$ *be given;* $\rho_K^{n-1} > 0$ *for all* $K \in \Omega_h$*. Then the numerical scheme (2) admits a solution*

$$\rho_h^n \in X(\Omega_h), \rho_K^n > 0 \text{ for all } K \in \Omega_h, \mathbf{u}_h^n \in X(\mathcal{E}_{int})^d.$$

Moreover, it satisfies the discrete conservation of mass

$$\sum_{K\in\Omega_h} \int_K \rho_K^n = \sum_{K\in\Omega_h} \int_K \rho_K^{n-1}.$$

Mass conservation

$$\partial_h^t \rho_K^n + \mathrm{div}_{\mathrm{Up}}[\rho^n, \mathbf{u}^n]_K - h^{\alpha}(\Delta_h \rho^n)_K = 0$$

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$$\boxed{\sum_{K \in \Omega_h} \int_K \rho_K^n = \sum_{K \in \Omega_h} \int_K \rho_K^{n-1}}$$

Positivity–nonnegativity

Recall the renormalized continuity equation

$$\sum_{K \in \Omega_h} \left(\partial_h^t B(\rho_K^n) + \Big(B'(\rho_K^n)\rho_K^n - B(\rho_K^n) \Big) (\mathrm{div}_h \mathbf{u}^n)_K + \mathcal{P}_K \right) = 0,$$

with test function

$$B(z) = \begin{cases} -z & \text{for } z < 0, \\ 0 & \text{for } z \geq 0. \end{cases}$$

Positivity–nonnegativity

Recall the renormalized continuity equation

$$\sum_{K\in\Omega_h}\left(\partial_h^t B(\rho_K^n) + \Big(B'(\rho_K^n)\rho_K^n - B(\rho_K^n)\Big)(\mathrm{div}_h \mathbf{u}^n)_K + \mathcal{P}_K\right) = 0,$$

with test function

$$B(z) = \begin{cases} -z & \text{for } z < 0, \\ 0 & \text{for } z \geq 0. \end{cases}$$

$$B(z) \geq 0$$

$$B'(z)z - B(z) = 0$$

$$\sum_{K\in\Omega_h}\int_K B(\rho_K^n) = \sum_{K\in\Omega_h}\int_K \left(B(\rho_K^{n-1}) - P_K\right) \leq 0$$

$$\boxed{\rho_K^n \geq 0}$$

Positivity–strict positivity

Let $K \in \Omega_h$ satisfy $\rho_K^n \leq \rho_L^n$ for all $L \in \Omega_h$. Then we have

$$
\begin{aligned}
\rho_K^n - \rho_K^{n-1} &= -\Delta t \mathrm{div}_{\mathrm{Up}}[\rho^n, \mathbf{u}^n]_K + \Delta t h^\alpha (\Delta_h \rho^n) \\
&\geq -\frac{\Delta t}{h} \sum_{s=1}^{d} \left(\rho_K^n u_{\sigma_{K,s+}}^s - \rho_K^n u_{\sigma_{K,s-}}^s + (\rho_{K+h\mathbf{e}_s}^n - \rho_K^n) u_{\sigma_{K,s+}}^{s\,-} + (\rho_K^n - \rho_{K-h\mathbf{e}_s}^n) u_{\sigma_{K,s-}}^{s\,+} \right) \\
&\geq -\Delta t \rho_K^n |(\mathrm{div}_h \mathbf{u}^n)_K|
\end{aligned}
$$

Let $K \in \Omega_h$ satisfy $\rho_K^n \leq \rho_L^n$ for all $L \in \Omega_h$. Then we have

$$\rho_K^n - \rho_K^{n-1} = -\Delta t \mathrm{div}_{\mathrm{Up}}[\rho^n, \mathbf{u}^n]_K + \Delta t h^{\alpha}(\Delta_h \rho^n)$$

$$\geq -\frac{\Delta t}{h} \sum_{s=1}^{d} \Big(\rho_K^n u^s_{\sigma_{K,s+}} - \rho_K^n u^s_{\sigma_{K,s-}} + (\rho^n_{K+h\mathbf{e}_s} - \rho_K^n) u^{s\,-}_{\sigma_{K,s+}} + (\rho_K^n - \rho^n_{K-h\mathbf{e}_s}) u^{s\,+}_{\sigma_{K,s-}}$$

$$\geq -\Delta t \rho_K^n \left| (\mathrm{div}_h \mathbf{u}^n)_K \right|$$

$$\rho_L^n \geq \rho_K^n \geq \frac{1}{1 + \Delta t \left| (\mathrm{div}_h \mathbf{u}^n)_K \right|} \rho_K^{n-1} > 0, \qquad \text{for any } L \in \Omega_h$$

Lemma 2

Let $(\rho_h, \mathbf{u}_h)$ be the numerical solution obtained by the scheme (2). For any $m = 1, \cdots, N$ the following estimate holds,

$$E^m + \Delta t \mu \sum_{n=1}^{m} \sum_{K \in \Omega_h} \int_K \sum_{r=1}^{3} \sum_{s=1}^{3} |(\partial_h^r (u^s)^n)_K|^2 + \sum_{j=1}^{4} \mathcal{N}_j \leq E^0.$$

$$E^m = \sum_{K \in \Omega_h} \int_K \left(\rho_K^m \frac{|\bar{\mathbf{u}}_K^m|^2}{2} + \frac{1}{\gamma - 1} p(\rho_K^m) \right)$$

$$\mathcal{N}_1 = \Delta t \sum_{n=1}^{m} \sum_{K \in \Omega_h} \int_K \sum_{s=1}^{d} \frac{1}{2} \left((h^\alpha + h^2 (u_{\sigma, s\mp}^{s,n})^{\pm}) p''(\rho_{\sigma, s\mp}^{n,\star}) |(\partial_h^s \rho^n)_{\sigma, s\mp}|^2 \right),$$

$$\mathcal{N}_2 = \Delta t^2 \sum_{n=1}^{m} \sum_{K \in \Omega_h} \int_K \frac{p''(\rho_K^\eta)}{2} |\partial_t^h \rho_K^n|^2, \quad \mathcal{N}_3 = \Delta t^2 \sum_{n=1}^{m} \sum_{K \in \Omega_h} \int_K \frac{\rho_K^{n-1}}{2} |\partial_t^h \bar{\mathbf{u}}_K^n|^2,$$

$$\mathcal{N}_4 = \Delta t \frac{h}{4} \sum_{n=1}^{m} \sum_{\Gamma \in \mathcal{E}_{int}} \int_\Gamma |Up[\rho^n, \mathbf{u}^n]_\sigma| |(\partial_h^s \bar{\mathbf{u}}^n)_\sigma|^2.$$

Uniform bounds

Lemma 3

Let $(\rho_h, \mathbf{u}_h)$ *be a numerical solution obtained by the scheme (2).*
Suppose $1 < \gamma < 2, \; 1 < \alpha < 2\gamma - 1$.
Then we have

$$\|\rho_h\|_{L^\infty(L^\gamma(\Omega))} \lesssim 1$$

$$\|p(\rho_h)\|_{L^\infty(L^1(\Omega))} \lesssim 1$$

$$\|\nabla_h \mathbf{u}_h\|_{L^2(L^2(\Omega))} \lesssim 1$$

$$\|\mathbf{u}_h\|_{L^2(L^6(\Omega))} \lesssim 1$$

$$\|\sqrt{\rho_h}\bar{\mathbf{u}}_h\|_{L^\infty(L^2(\Omega))} \lesssim 1$$

$$h\|\sqrt{\rho_h}\|_{L^2(L^\infty(\Omega))} \lesssim h^\theta, \; \theta = 1 - \frac{\alpha + 1}{2\gamma} > 0.$$

Consistency

Lemma 4

Let $\rho_h^n, \mathbf{u}_h^n$ be the solution to the numerical scheme (2). Then

$$\int_\Omega \partial_h^t \rho_h^n \phi dx - \int_\Omega \rho_h^n \mathbf{u}_h^n \cdot \nabla_x \phi dx = \mathcal{O}(h^{\beta_1}), \beta_1 > 0.$$

$$\int_\Omega \partial_h^t (\rho_h \bar{\mathbf{u}}_h)^n \cdot \mathbf{v} dx - \int_\Omega \rho_h^n \bar{\mathbf{u}}_h^n \otimes \bar{\mathbf{u}}_h^n : \nabla_x \mathbf{v} dx - \int_\Omega p(\rho_h^n) div_x \mathbf{v} dx$$
$$+ \mu \int_\Omega (\nabla_h \mathbf{u}_h^n) : \nabla_x \mathbf{v} dx = \mathcal{O}(h^{\beta_2}), \beta_2 > 0.$$

Dissipative measure-valued solution

Definition 5

We say that a parameterized measure $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$,

$$\nu \in L^{\infty}_{weak}\left((0,T)\times\Omega; \mathcal{P}\left([0,\infty)\times R^N\right)\right)$$

is a dissipative measure-valued solution of the Navier-Stokes system in $(0,T)\times\Omega$, if the following holds for a.a. $\tau\in(0,T)$, for any $\psi\in C^1\left((0,T)\times\Omega; R^d\right)$

$$\left[\int_{\Omega}\langle\nu_{\tau,x};\rho\rangle\psi(\tau,\cdot)dx\right]_{t=0}^{t=\tau} = \int_0^{\tau}\int_{\Omega}[\langle\nu_{t,x};\rho\rangle\partial_t\psi + \langle\nu_{t,x};\rho\mathbf{u}\rangle\cdot\nabla_x\psi]dxdt$$

$$\left[\int_{\Omega}\langle\nu_{\tau,x};\rho\mathbf{u}\rangle\psi(\tau,\cdot)dx\right]_{t=0}^{t=\tau} = \int_0^{\tau}\int_{\Omega}[\langle\nu_{t,x};\rho\mathbf{u}\rangle\partial_t\psi + \langle\nu_{t,x};\rho\mathbf{u}\otimes\mathbf{u}\rangle:\nabla_x\psi + \langle\nu_{t,x};p(\rho)\rangle]dxdt$$

$$-\int_0^{\tau}\int_{\Omega}\mathcal{S}(\nabla\mathbf{u}):\nabla_x\psi dxdt, + \int_0^{\tau}\int_{\Omega}\mathcal{R};\nabla_x\psi dxdt$$

$$\left[\int_{\Omega}\langle\nu_{\tau,x};E\rangle\psi(\tau,\cdot)dx\right]_{t=0}^{t=\tau} + \int_0^{\tau}\int_{\Omega}\mathcal{S}(\nabla\mathbf{u}):\nabla_x\psi dxdt + \mathcal{D}(\tau)\leq 0,$$

where

$$\int_0^{\tau}\|\mathcal{R}\|_{\mathcal{M}(\Omega)}dt \leq \int_0^{\tau}\mathcal{D}(\tau)dt$$

Theorem 6

Let $1 < \gamma < 2, \Delta t \approx h, \ 1 < \alpha < 2\gamma - 1$ *and the initial data satisfy*

$$\rho_0 \in L^\infty(R^d), \ \rho_0 \geq \underline{\rho} > 0 \ a.a. \ in \ R^d, \ \mathbf{u}_0 \in L^2(R^d).$$

Then any Young measure $\nu_{t,x}$ *generated by the numerical sol of scheme* (2) *represents a dissipative measure-valued solution of NS (1).*

[1]Feireisl et.al. Dissipative measure-valued solutions to the compressible Navier–Stokes system. Calc. Vari. Partial Differ. Equ. 2016

Convergence

Theorem 6

Let $1 < \gamma < 2, \Delta t \approx h,\ 1 < \alpha < 2\gamma - 1$ *and the initial data satisfy*

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Then any Young measure $\nu_{t,x}$ *generated by the numerical sol of scheme* (2) *represents a dissipative measure-valued solution of NS (1).*

Applying the weak-strong uniqueness [1] we conclude

Theorem 7

In addition to the hypotheses of Theorem 6, suppose the NS (2) endowed with the periodic boundary condition admits a regular solution.
Then

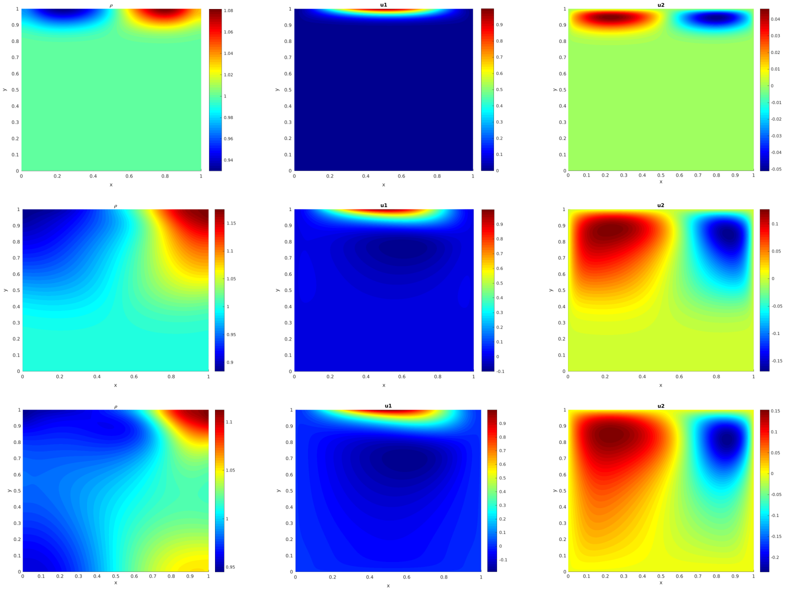
$$\rho_h \to \rho \text{ (strongly) in } L^\gamma\left((0,T)\times K\right),$$

$$\mathbf{u}_h \to \mathbf{u} \text{ (strongly) in } L^2\left((0,T)\times K; R^d\right)$$

for any compact $K \subset \Omega$.

[1] Feireisl et.al. Dissipative measure-valued solutions to the compressible Navier–Stokes system. Calc. Vari. Partial Differ. Equ. 2016

Test-1 Dirichlet boundary



Test-1 Dirichlet boundary

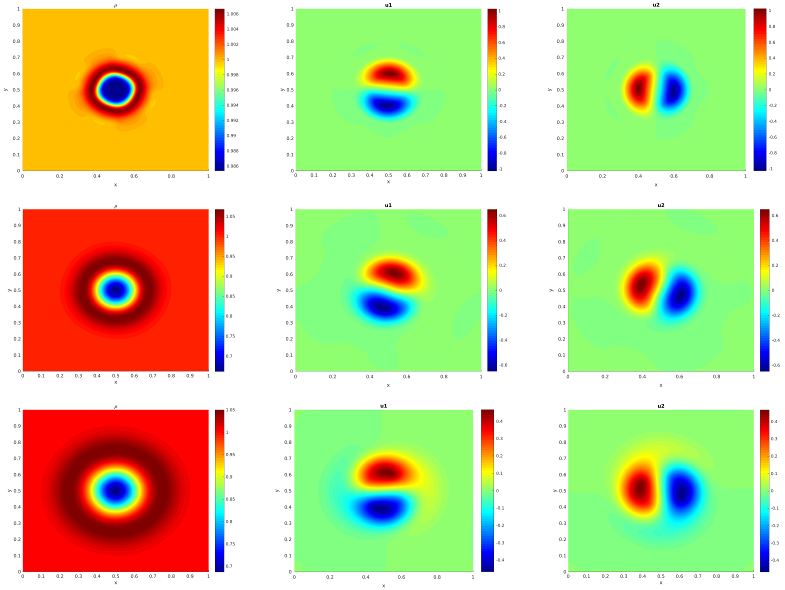
$$\Omega = [0,1]^2, \mu = 0.01, \ a = 1.0, \gamma = 1.4, \alpha = 0.83.$$

Cavity flow, upper boundary $\mathbf{u} = (16x^2(1-x)^2, 0)^T$.

Table : Convergence results

h	$\|\nabla \mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\rho\|_{L^1(L^1)}$	EOC	$\|\rho\|_{L^\infty(L^\gamma)}$	EOC
1/16	6.17e-01	–	4.65e-02	–	7.74e-03	–	4.94e-02	–
1/32	3.08e-01	1.00	2.32e-02	1.00	4.23e-03	0.87	3.19e-02	0.63
1/64	1.51e-01	1.03	1.12e-02	1.05	2.15e-03	0.97	1.96e-02	0.70
1/128	6.60e-02	1.19	4.75e-03	1.23	8.45e-04	1.35	9.97e-03	0.97

Test-2 Periodic boundary



Test-2 Periodic boundary

$$U(0,x,y) = u_r(r) * (y-0.5)/r,$$
$$V(0,x,y) = u_r(r) * (0.5-x)/r.$$

where $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}$ and

$$u_r(r) = \sqrt{\gamma}\begin{cases} 2r/R & \text{if } 0 \le r < R/2, \\ 2(1-r/R) & \text{if } R/2 \le r < R, \\ 0 & \text{if } r \ge R, \end{cases}$$

Table : Convergence results of Gresho vortex test

h	$\|\nabla \mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\rho\|_{L^1(L^1)}$	EOC	$\|\rho\|_{L^\infty(L^\gamma)}$	EOC
1/16	2.23e-01	–	7.84e-03	–	3.19e-06	–	6.66e-03	–
1/32	1.19e-01	0.91	4.09e-03	0.94	1.63e-06	0.97	4.27e-03	0.64
1/64	6.04e-02	0.97	2.01e-03	1.03	5.92e-07	1.46	2.27e-03	0.91
1/128	2.66e-02	1.18	8.98e-03	1.16	2.24e-07	1.40	1.17e-03	0.96

Thank you for your attention!