Lower bounds for principal eigenvalues of elliptic operators

with applications to Friedrichs and trace constants

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Guaranteed error bounds

To compute (approximate) solution is not sufficient.
We should provide an information about the error.

Can we provide
a guranteed upper bound?
$\|u - u_h\| \leq \eta$

The sinking of the Sleipner A offshore platform.

Babuška, Verfürth, Ainsworth, Rannacher, Repin, ...

Complementary error estimates

Poisson problem:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Upper bound:

$$\|\nabla u - \nabla u_h\|_{L^2(\Omega)} \leq \|\mathbf{q} - \nabla u_h\|_{L^2(\Omega)} + C_{\mathrm{F}} \|f + \operatorname{div} \mathbf{q}\|_{L^2(\Omega)}$$

$$\forall u_h \in H_0^1(\Omega) \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Complementary error estimates

Poisson problem:

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$$\forall u_h \in H_0^1(\Omega) \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Proof. Let $v = u - u_h$.

$$\begin{aligned}(\nabla u - \nabla u_h, \nabla v) &= (f, v) - (\nabla u_h, \nabla v) + (\mathbf{q}, \nabla v) + (\operatorname{div}\mathbf{q}, v) \\ &= (\mathbf{q} - \nabla u_h, \nabla v) + (f + \operatorname{div}\mathbf{q}, v) \\ &\leq \|\mathbf{q} - \nabla u_h\|\,\|\nabla v\| + \|f + \operatorname{div}\mathbf{q}\|\,\|v\| \\ &\leq \|\mathbf{q} - \nabla u_h\|\,\|\nabla v\| + C_{\mathrm{F}}\|f + \operatorname{div}\mathbf{q}\|\,\|\nabla v\|\end{aligned}$$

Friedrichs inequality

$$\|v\|_{L^2(\Omega)} \leq C_{\mathrm{F}} \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$$

Optimal constant:

$$C_{\mathrm{F}} = \sup_{v \in H_0^1(\Omega)} \frac{\|v\|_{L^2(\Omega)}}{\|\nabla v\|_{L^2(\Omega)}} = \frac{1}{\sqrt{\inf_{v \in H_0^1(\Omega)} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}}}} = \frac{1}{\sqrt{\lambda_1}}$$

Laplace eigenvalue problem:

$$\text{Find } 0 \neq u_i \in H_0^1(\Omega) : \quad (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in H_0^1(\Omega)$$

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

Goal: $C_{\mathrm{F}} \leq C_{\mathrm{F}}^{\mathrm{up}} \quad \Leftrightarrow \quad \lambda_1^{\mathrm{low}} \leq \lambda_1$

Outline

1. Abstract theory
   - Hilbert space setting
   - Upper and lower bound on λ_1
2. Application to
   - Friedrichs inequality
   - trace inequality
3. Numerical examples
   - Friedrichs inequality
   - trace inequality

1. Abstract theory

Abstract setting

- V, H Hilbert spaces
- $\gamma : V \to H$ linear, continuous, compact

Eigenproblem: Find $\lambda_i \in \mathbb{R}$, $u_i \in V$, $u_i \neq 0$ such that

$$(u_i, v)_V = \lambda_i(\gamma u_i, \gamma v)_H \quad \forall v \in V$$

Properties:

- $\lambda_i > 0$ and $\gamma u_i \neq 0$

Abstract setting

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Properties:

- $(\gamma u_i, \gamma u_j)_H = \delta_{ij} \quad \forall i, j = 1, 2, \ldots$

Abstract setting

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Eigenproblem: Find $\lambda_i \in \mathbb{R}$, $u_i \in V$, $u_i \neq 0$ such that

$$(u_i, v)_V = \lambda_i (\gamma u_i, \gamma v)_H \quad \forall v \in V$$

Properties:

- $\{\lambda_i : \lambda_i \leq M\}$ is finite for all $M > 0$

Abstract setting

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Properties:

- $\lambda_1 = \inf_{v \in V, v \neq 0} \|v\|_V^2 / \|\gamma v\|_H^2$ is the smallest eigenvalue

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Properties:

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Theorem (abstract inequality):
There exists $C_\gamma > 0$ such that $\|\gamma v\|_H \leq C_\gamma \|v\|_V \quad \forall v \in V$.
Moreover, $C_\gamma = \lambda_1^{-1/2}$ is optimal.

Galerkin method – upper bound on λ_1

$V^h \subset V$

Discrete eigenproblem:
Find $\lambda_i^h \in \mathbb{R}$, $u_i^h \in V^h$, $u_i^h \neq 0$ such that

$$(u_i^h, v^h)_V = \lambda_i^h (\gamma u_i^h, \gamma v^h)_H \quad \forall v^h \in V^h$$

Theorem: $\lambda_1 \leq \lambda_1^h$
Proof:

$$\lambda_1 = \inf_{0 \neq v \in V} \frac{\|v\|_V^2}{\|v\|_H^2} \leq \inf_{0 \neq v^h \in V^h} \frac{\|v\|_V^2}{\|v\|_H^2} = \lambda_1^h$$

□

Lower bound on λ_1

Theorem

- $\lambda_* \in \mathbb{R}$, $u_* \in V$ arbitrary, $\|\gamma u_*\|_H = 1$
- $w \in V : \quad (w, v)_V = (u_*, v)_V - \lambda_*(\gamma u_*, \gamma v)_H \quad \forall v \in V$
- Relative closeness: $\left|\dfrac{\lambda_1 - \lambda_*}{\lambda_1}\right| \le \left|\dfrac{\lambda_i - \lambda_*}{\lambda_i}\right| \quad \forall i = 1, 2, \dots.$
- Abstract complementarity: $\|w\|_V \le A + C_\gamma B, \quad B < \lambda_*,\ C_\gamma = \dfrac{1}{\sqrt{\lambda_1}}$

Then

$$X_2^2 \le \lambda_1, \quad \text{where } X_2 = \frac{1}{2}\left(-A + \sqrt{A^2 + 4(\lambda_* - B)}\right).$$

Lower bound on λ_1

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Proof:

$$\min_i \left|\frac{\lambda_i - \lambda_*}{\lambda_i}\right| \leq \|\gamma w\|_H$$

[Kuttler, Sigillito, 1978]

Lower bound on λ_1

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Proof:

$$\frac{\lambda_* - \lambda_1}{\lambda_1} \leq \min_i \left|\frac{\lambda_i - \lambda_*}{\lambda_i}\right| \leq \|\gamma w\|_H$$

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Then

$$X_2^2 \le \lambda_1, \quad \text{where } X_2 = \frac{1}{2}\left(-A + \sqrt{A^2 + 4(\lambda_* - B)}\right).$$

Proof:

$$\frac{\lambda_* - \lambda_1}{\lambda_1} \le \min_i \left|\frac{\lambda_i - \lambda_*}{\lambda_i}\right| \le \|\gamma w\|_H \le C_\gamma \|w\|_V$$

Lower bound on λ_1

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Then

$$X_2^2 \le \lambda_1, \quad \text{where } X_2 = \frac{1}{2}\left(-A + \sqrt{A^2 + 4(\lambda_* - B)}\right).$$

Proof:

$$\frac{\lambda_* - \lambda_1}{\lambda_1} \le \min_i \left|\frac{\lambda_i - \lambda_*}{\lambda_i}\right| \le \|\gamma w\|_H \le C_\gamma \|w\|_V \le \frac{1}{\sqrt{\lambda_1}} A + \frac{1}{\lambda_1} B$$

Lower bound on λ_1

Theorem

- $\lambda_* \in \mathbb{R}$, $u_* \in V$ arbitrary, $\|\gamma u_*\|_H = 1$
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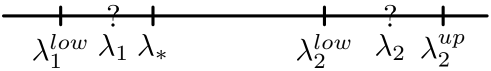
Proof:

$$\frac{\lambda_* - \lambda_1}{\lambda_1} \leq \min_i \left|\frac{\lambda_i - \lambda_*}{\lambda_i}\right| \leq \|\gamma w\|_H \leq C_\gamma \|w\|_V \leq \frac{1}{\sqrt{\lambda_1}} A + \frac{1}{\lambda_1} B$$

$$\Leftrightarrow \quad 0 \leq B - \lambda_* + \sqrt{\lambda_1} A + \lambda_1 \quad \Rightarrow \quad \sqrt{\lambda_1} \geq X_2 \qquad \square$$

Comment on relative closeness

It cannot be guaranteed unless lower bounds on λ_1 and λ_2 are known.



Observation:

$$D_1(\lambda_1^{\mathrm{low}}) = \frac{\lambda_* - \lambda_1^{\mathrm{low}}}{\lambda_1^{\mathrm{low}}} \qquad D_2(\lambda_2^{\mathrm{low}}) = \frac{\lambda_2^{\mathrm{low}} - \lambda_*}{\lambda_2^{\mathrm{up}}}$$

If $\lambda_1^{\mathrm{low}} \leq \lambda_1 \leq \lambda_* \leq \lambda_2^{\mathrm{low}} \leq \lambda_2 \leq \lambda_2^{\mathrm{up}}$

$\Rightarrow D_1(\lambda_1^{\mathrm{low}}) \leq D_2(\lambda_2^{\mathrm{low}}) \Rightarrow$ relative closeness

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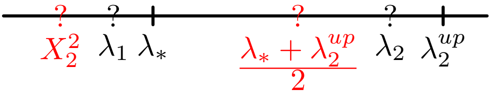

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If $\lambda_1^{\text{low}} \leq \lambda_1 \leq \lambda_* \leq \lambda_2^{\text{low}} \leq \lambda_2 \leq \lambda_2^{\text{up}}$

$\Rightarrow D_1(\lambda_1^{\text{low}}) \leq D_2(\lambda_2^{\text{low}}) \Rightarrow$ relative closeness

Diagnostics:

$$D_1 = D_1\left(X_2^2\right) \qquad D_2 = D_2\left(\frac{\lambda_* + \lambda_2^{\text{up}}}{2}\right)$$

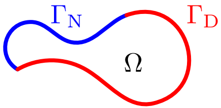
If $D_1 > D_2 \Rightarrow$ assumptions (probably) not satisfied

If $D_1 \ll D_2 \Rightarrow$ good confidence in relative closeness

2. Application of the abstract theory to specific cases.

Application to Friedrichs inequality

$$\|v\|_{L^2(\Omega)} \le C_{\mathrm{F}} \|v\|_a \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$$



Notation

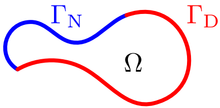
- $\|v\|_a^2 = a(v,v)$
- $a(u,v) = \int_\Omega (\nabla u)^T \mathcal{A} \nabla v \, \mathrm{d}x$
- $H^1_{\Gamma_{\mathrm{D}}}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_{\mathrm{D}}\}$

Assumptions

- $\mathcal{A} \in [L^\infty(\Omega)]^{d \times d}$ symmetric
- $\boldsymbol{\xi}^T \mathcal{A}(x) \boldsymbol{\xi} \ge \lambda_{\mathcal{A}} |\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $x \in \Omega$
- $a(\cdot,\cdot)$ scalar product in $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$

Application to Friedrichs inequality

$$\|v\|_{L^2(\Omega)} \le C_{\mathrm{F}} \|v\|_a \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$$



Setting

- $V = H^1_{\Gamma_{\mathrm{D}}}(\Omega), \quad (u,v)_V = a(u,v)$
- $H = L^2(\Omega), \quad (u,v)_H = (u,v)$
- $\gamma : H^1_{\Gamma_{\mathrm{D}}}(\Omega) \to L^2(\Omega)$ identity mapping,

  compact by Rellich theorem

Conclusions

- $C_{\mathrm{F}} = \lambda_1^{-1/2}$, where λ_1 is the smallest eigenvalue of

  $\lambda_i \in \mathbb{R},\ 0 \ne u_i \in H^1_{\Gamma_{\mathrm{D}}}(\Omega) : \quad a(u_i, v) = \lambda_i (u_i, v) \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$

$$\begin{aligned} -\operatorname{div}(\mathcal{A}\nabla u_i) &= \lambda_i u_i \quad \text{in } \Omega \\ u_i &= 0 \quad \text{on } \Gamma_{\mathrm{D}} \\ \partial u_i / \partial \mathbf{n} &= 0 \quad \text{on } \Gamma_{\mathrm{N}} \end{aligned}$$

Friedrichs inequality – complementarity

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Notation:

- $\mathbf{H}(\operatorname{div},\Omega) = \{\mathbf{q} \in [L^2(\Omega)]^d : \operatorname{div}\mathbf{q} \in L^2(\Omega)\}$
- $\|\mathbf{q}\|^2_{\mathcal{A}} = (\mathcal{A}\mathbf{q},\mathbf{q})$ a norm in $[L^2(\Omega)]^d$

Theorem: If

- $\lambda_* \in \mathbb{R}, \quad u_* \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$
- $w \in H^1_{\Gamma_{\mathrm{D}}}(\Omega): \quad a(w,v) = a(u_*,v) - \lambda_*(u_*,v) \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$

Then

$$\|w\|_a \le \underbrace{\|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}}}_{A} + C_{\mathrm{F}} \underbrace{\|\lambda_* u_* + \operatorname{div}\mathbf{q}\|_{L^2(\Omega)}}_{B} \quad \forall \mathbf{q} \in W_0,$$

where $W_0 = \{\mathbf{q} \in \mathbf{H}(\operatorname{div},\Omega) : \mathbf{q}\cdot\mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{N}}\}$

Choice of $\mathbf{q} \in W$

- $A = A(\mathbf{q}) = \|\nabla u_1^h - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}}$
  $B = B(\mathbf{q}) = \|\lambda_1^h u_1^h + \operatorname{div}\mathbf{q}\|_{L^2(\Omega)}$

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- Best choice: $\mathbf{q}^{\mathrm{best}} = \arg\min\limits_{\mathbf{q}\in W_0}\left\{A(\mathbf{q}) + C_{\mathrm{F}} B(\mathbf{q})\right\}$

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- Best choice: $\mathbf{q}^{\mathrm{best}} = \arg\min_{\mathbf{q}\in W_0}\left\{A(\mathbf{q}) + C_{\mathrm{F}} B(\mathbf{q})\right\}$
- Practical: $\mathbf{q}^h = \arg\min_{\mathbf{q}\in W_0^h}\left\{(1+\varrho^{-1})A^2(\mathbf{q}) + (1+\varrho)(\lambda_1^h)^{-1}B^2(\mathbf{q})\right\}$
- $W_0^h \subset W_0$ Raviart-Thomas finite element space

Choice of $\mathbf{q} \in W$

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- $A = A(\mathbf{q}) = \|\nabla u_1^h - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}}$
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- $W_0^h \subset W_0$ Raviart-Thomas finite element space
- Equivalent to

$$\mathbf{q}^h \in W_0^h : \quad \mathcal{B}(\mathbf{q}^h, \mathbf{w}^h) = \mathcal{F}(\mathbf{w}^h) \quad \forall \mathbf{w}^h \in W_0^h$$

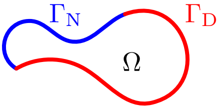
where

$$\mathcal{B}(\mathbf{q}, \mathbf{w}) = (\operatorname{div}\mathbf{q}, \operatorname{div}\mathbf{w}) + \frac{\lambda_1^h}{\varrho}(\mathcal{A}^{-1}\mathbf{q}, \mathbf{w}),$$

$$\mathcal{F}(\mathbf{w}) = \frac{\lambda_1^h}{\varrho}(\nabla u_1^h, \mathbf{w}) - (\lambda_1^h u_1^h, \operatorname{div}\mathbf{w})$$

Application to trace inequality

$$\|v\|_{L^2(\Gamma_\mathrm{N})} \le C_\mathrm{T} \|v\|_a \quad \forall v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$$



$\mathrm{meas}_{d-1}\,\Gamma_\mathrm{N} > 0$

Setting

- $V = H^1_{\Gamma_\mathrm{D}}(\Omega), \quad (u,v)_V = a(u,v)$
- $H = L^2(\Gamma_\mathrm{N}), \quad (u,v)_H = (u,v)_{\Gamma_\mathrm{N}}$
- $\gamma : H^1_{\Gamma_\mathrm{D}}(\Omega) \to L^2(\Gamma_\mathrm{N})$ trace operator, compact, see e.g. [Kufner, John, Fučík, 1977]

Conclusions

- $C_\mathrm{T} = \lambda_1^{-1/2}$, where λ_1 is the smallest eigenvalue of

$$\lambda_i \in \mathbb{R},\ 0 \neq u_i \in H^1_{\Gamma_\mathrm{D}}(\Omega) : \quad a(u_i, v) = \lambda_i (u_i, v)_{\Gamma_\mathrm{N}} \quad \forall v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$$

$$\begin{aligned} -\mathrm{div}(\mathcal{A}\nabla u_i) &= 0 \quad \text{in } \Omega \\ u_i &= 0 \quad \text{on } \Gamma_\mathrm{D} \\ \partial u_i/\partial \mathbf{n} &= \lambda_i u_i \quad \text{on } \Gamma_\mathrm{N} \end{aligned}$$

Trace inequality – complementarity

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$$\|w\|_a \le A(\mathbf{q}) + C_{\mathrm{T}} B(\mathbf{q}) \quad \forall \mathbf{q} \in \mathbf{H}(\mathrm{div},\Omega)$$

- $A(\mathbf{q}) = \|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}} + C_{\mathrm{F}}\|\operatorname{div}\mathbf{q}\|_{L^2(\Omega)}$
  $B(\mathbf{q}) = \|\lambda_* u_* - \mathbf{q}\cdot\mathbf{n}\|_{L^2(\Gamma_{\mathrm{N}})}$

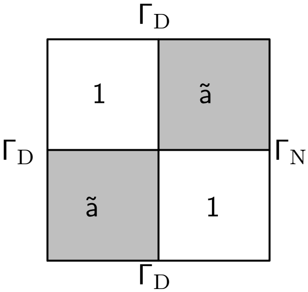
Variants

- $A(\mathbf{q}) = \|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}}, \quad \forall \mathbf{q} \in \mathbf{H}(\mathrm{div},\Omega),\ \operatorname{div}\mathbf{q} = 0$
- $A(\mathbf{q}) = \|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}} + C_{\mathrm{P}}\|\operatorname{div}\mathbf{q}\|_{L^2(\Omega)}$
  $\forall \mathbf{q} \in \mathbf{H}(\mathrm{div},\Omega),\ \int_\Omega \operatorname{div}\mathbf{q}\,\mathrm{d}x = 0$
- $A(\mathbf{q}) = \|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}} + \dfrac{h}{\lambda_{\mathcal{A}}\pi}\|\operatorname{div}\mathbf{q}\|_{L^2(\Omega)}$
  $\forall \mathbf{q} \in \mathbf{H}(\mathrm{div},\Omega) : \int_K \operatorname{div}\mathbf{q}\,\mathrm{d}x = 0\ \forall K \in \mathcal{T}_h$

3. Numerical examples

Example 1: Friedrichs inequality

$$\|v\|_{L^2(\Omega)} \leq C_{\mathrm{F}} \|\mathcal{A}^{1/2} \nabla v\|_{L^2(\Omega)} \qquad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$$



- $\mathcal{A}(x_1, x_2) = \begin{cases} 1 & \text{for } x_1 x_2 \leq 0 \\ \tilde{a} & \text{for } x_1 x_2 > 0 \end{cases}$
- $u_1^h \in V^h = \{v^h \in H^1_{\Gamma_{\mathrm{D}}}(\Omega) : v^h|_K \in P^1(K), \ \forall K \in \mathcal{T}_h\}$
- $\mathbf{q}^h \in W_0^h = \{\mathbf{w}_h \in W_0 : \mathbf{w}_h \in [P^2(K)]^2, \ \forall K \in \mathcal{T}_h\}$
- Adaptive algorithm

Example 1: Adaptive algorithm

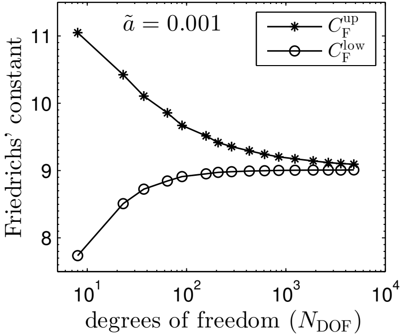
1. Initial mesh $\mathcal{T}_h$
2. Galerkin approximation: $\lambda_1^h \in \mathbb{R}$, $u_1^h \in V^h \Rightarrow C_{\mathrm{F}}^{\mathrm{low}} = (\lambda_1^h)^{-1/2}$
3. Find: $\mathbf{q}^h \in W_0^h$
4. Upper bounds: $C_{\mathrm{F}}^{\mathrm{up}} = 1/X_2$
5. Stop if $E_{\mathrm{REL}} = \frac{C_{\mathrm{F}}^{\mathrm{up}} - C_{\mathrm{F}}^{\mathrm{low}}}{C_{\mathrm{F}}^{\mathrm{avg}}} \leq E_{\mathrm{TOL}}$, where $C_{\mathrm{F}}^{\mathrm{avg}} = (C_{\mathrm{F}}^{\mathrm{up}} + C_{\mathrm{F}}^{\mathrm{low}})/2$
6. Error indicators:
   $$\eta_K^2 = \frac{1+\varrho}{\varrho} \|\nabla u_1^h - \mathcal{A}^{-1}\mathbf{q}^h\|_{\mathcal{A},K}^2 + \frac{1+\varrho}{\lambda_1^h} \|\lambda_1^h u_1^h + \operatorname{div} \mathbf{q}^h\|_{L^2(K)}^2$$
7. Identify elements to refine (bulk criterion)
8. Refine $\mathcal{T}_h$
9. Go to 2.

Example 1: results

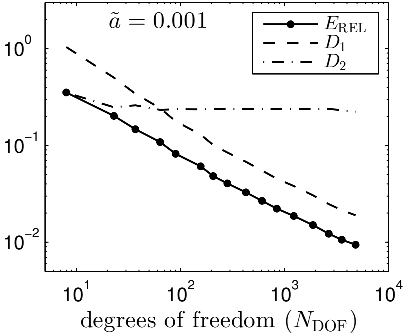
$\tilde{a}$	C_F^{low}	C_F^{up}	E_{REL}	N_{DOF}
0.001	9.0086	9.0939	0.94 %	4 832
0.01	2.8697	2.8971	0.95 %	5 003
0.1	1.0035	1.0124	0.88 %	7 866
1	0.5693	0.5743	0.86 %	4 802
10	0.3173	0.3201	0.88 %	7 866
100	0.2870	0.2897	0.95 %	5 003
1000	0.2849	0.2876	0.94 %	4 832

Note: $C_F = 4/(\pi\sqrt{5}) \approx 0.5694$ for $\tilde{a} = 1$.

Example 1: Convergence plot $\tilde{a} = 0.001$



Lower and upper bound.



Relative error and heuristic indicators.

Example 1: Adapted meshes $\tilde{a} = 0.001$

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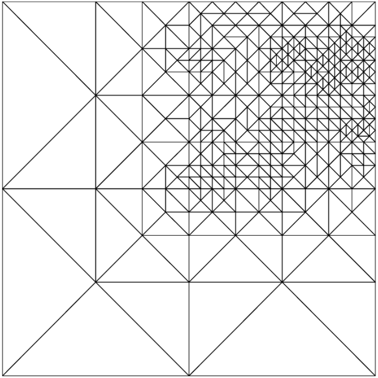
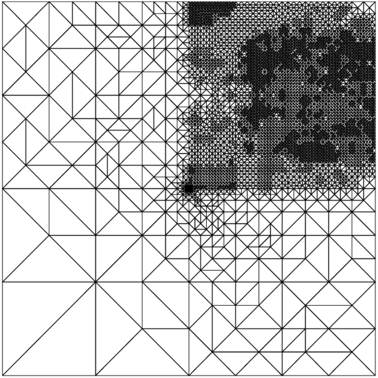


Adaptive step 8.



Adaptive step 16 (final).

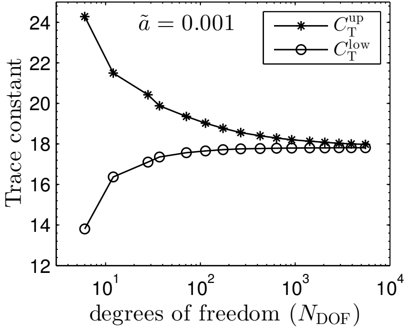
Example 2: Trace inequality

$$\|v\|_{L^2(\Gamma_{\mathrm{N}})} \le C_{\mathrm{T}} \|\mathcal{A}^{1/2}\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$$

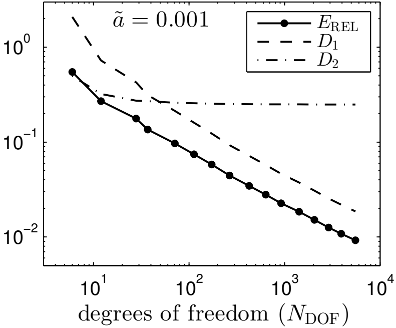
$\tilde{a}$	$C_{\mathrm{T}}^{\mathrm{low}}$	$C_{\mathrm{T}}^{\mathrm{up}}$	E_{REL}	N_{DOF}
0.001	17.8110	17.9760	0.92 %	5 523
0.01	5.6490	5.7047	0.98 %	5 418
0.1	1.8433	1.8593	0.86 %	7 775
1	0.7963	0.8033	0.88 %	5 499
10	0.5829	0.5880	0.86 %	7 775
100	0.5649	0.5705	0.98 %	5 421
1000	0.5632	0.5685	0.92 %	5 523

Note: $C_{\mathrm{T}} = \sqrt{2/(\pi \coth \pi)} \approx 0.7964$ for $\tilde{a} = 1$

Example 2: Convergence plot $\tilde{a} = 0.001$

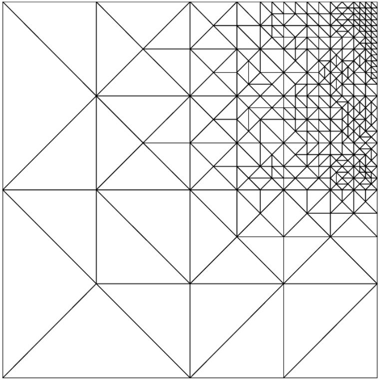


Lower and upper bound.

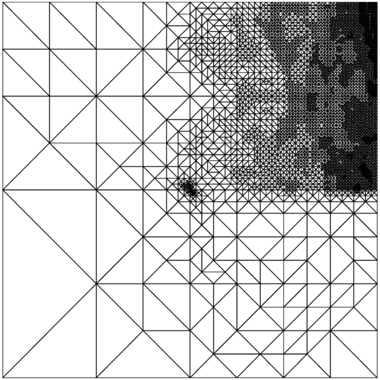


Relative error and heuristic indicators.

Example 2: Adapted meshes $\tilde{a} = 0.001$



Adaptive step 8.



Adaptive step 15 (final).

Other approaches

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Guaranteed lower bounds on eigenvalues:

- A priori-a posteriori inequalities [Kuttler, Sigillito, 1978]
- Analytic estimates – special cases [Mikhlin, 1986]
- Nonconforming elements – asymptotic bounds [Lin, Luo, Xie, Yang, Zhang, Andreev,. . .]
- Nonconforming elements and postprocessing [Carstensen, Gedicke 2013]
- Method of intermediate problems [Fox, Rheinboldt 1996, . . .]
- Inclusion theorems [Goerisch, He, Zimmermann, . . .]
- Maxwell operator [Barrenechea, Boulton, Boussaid 2013]

Conclusions

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- General method for two-sided bounds of principal eigenvalues
- Straightforward applications
- Guaranteed *lower* bound if
  - no round-off errors
  - all integrals evaluated exactly
  - domain Ω represented exactly
  - relative closeness (crucial)
- Guaranteed *upper* bound if
  - exact solution of matrix eigenproblem

Outlook

- Local construction of $\mathbf{q}$
- Convergence of the adaptive algorithm
- Nonlinear and nonsymmetric problems

Acknowledgement

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Thank you for your attention

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