A NOTE ON RADIO ANTIPODAL COLOURINGS OF PATHS

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Abstract. The radio antipodal number of a graph G is the smallest integer c such that there exists an assignment $f\colon V(G) \to \{1, 2, \ldots, c\}$ satisfying $|f(u) - f(v)| \geqslant D - d(u, v)$ for every two distinct vertices u and v of G, where D is the diameter of G. In this note we determine the exact value of the antipodal number of the path, thus answering the conjecture given in [G. Chartrand, D. Erwin and P. Zhang, Math. Bohem. 127 (2002), 57–69]. We also show the connections between this colouring and radio labelings.

Keywords: radio antipodal colouring, radio number, distance labeling

MSC 2000: 05C78, 05C12, 05C15

1. Introduction

Let G be a connected graph and let k be an integer, $k \geqslant 1$. The distance between two vertices u and v of G is denoted by $d(u, v)$ and the diameter of G by $D(G)$ or simply D. A *radio k-colouring* f of G is an assignment of positive integers to the vertices of G such that

$$|f(u) - f(v)| \geqslant 1 + k - d(u, v)$$

for every two distinct vertices u and v of G.

Following the notation of [1], [3], we define the *radio k-colouring number* $\mathrm{rc}_k(f)$ of a radio k-colouring f of G to be the maximum colour assigned to a vertex of G and the *radio k-chromatic number* $\mathrm{rc}_k(G)$ to be $\min\{\mathrm{rc}_k(f)\}$ taken over all radio k-colourings f of G.

Radio k-colourings generalize many graph colourings. For $k = 1$, $\mathrm{rc}_1(G) = \chi(G)$, the chromatic number of G. For $k = 2$, the radio 2-colouring problem corresponds to the well studied $L(2, 1)$-colouring problem and $\mathrm{rc}_2(G) = \lambda(G)$ (see [5] and references therein). For $k = D(G) - 1$, the radio $(D - 1)$-colouring is referred to as the *radio* *antipodal colouring*, because only antipodal vertices can have the same colour. In that case, $\mathrm{rc}_k(G)$ is called the *radio antipodal number*, also denoted by $\mathrm{ac}(G)$. Finally, for the case $k = D(G)$, $\mathrm{rc}_k(G)$ is called the *radio number* and is studied in [1], [6].

In [2] the antipodal number for cycles was discussed and bounds were given. In [3], Chartrand et al. gave general bounds for the antipodal number of a graph. The authors proved the following result for the radio antipodal number of the path:

Theorem 1 ([3]). *For every positive integer n,*

$$\mathrm{ac}(P_n) \leqslant \binom{n-1}{2} + 1.$$

Moreover, they conjectured that the above upper bound is the value of the antipodal number of the path. In [4], the authors found a sharper bound for the antipodal number of an odd path (thus showing that the conjecture was false):

Theorem 2 ([4]). *For the path P_n of odd order $n \geqslant 7$,*

$$\mathrm{ac}(P_n) \leqslant \binom{n-1}{2} - \frac{n-1}{2} + 4.$$

In this note we completely determine the antipodal number of the path:

Theorem 3. *For any $n \geqslant 5$,*

$$\mathrm{ac}(P_n) = \begin{cases} 2p^2 - 2p + 3 & \text{if } n = 2p + 1, \\ 2p^2 - 4p + 5 & \text{if } n = 2p. \end{cases}$$

Notice that for $n = 2p+1$ we have $\binom{n-1}{2} - \frac{n-1}{2} + 4 = p(2p-1) - p + 4 = 2p^2 - 2p + 4$, thus the bound of Theorem 2 is one from the optimal.

Examples of minimal antipodal colourings of P_7 and P_8 are given in Figure 1.

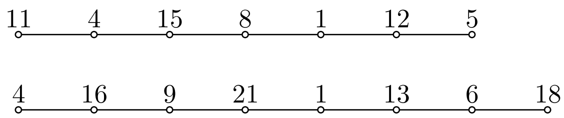

Figure 1. Antipodal colouring of P_7 and P_8.

In order to prove Theorem 3, we shall use a result of Liu and Zhu [6] about the radio number of the path. Notice that Liu and Zhu allow 0 to be used as a colour but we do not. Then, when presenting their result, we will make the necessary adjustment (adding "one") to be consistent with the rest of the paper.

Theorem 4 ([6])**.** *For any* $n \geqslant 3$

$$\mathrm{rc}_{n-1}(P_n) = \begin{cases} 2p^2+3 & \text{if } n = 2p+1,\\ 2p^2-2p+2 & \text{if } n = 2p.\end{cases}$$

2. Radio k-colourings

Lemma 1. *Let G be a graph of order n and let k be an integer. If f is a radio k-colouring of G then, for any integer $k' > k$, there exists a radio k'-colouring f' of G with $\mathrm{rc}_{k'}(f') \leqslant \mathrm{rc}_k(f) + (n-1)(k'-k)$.*

Proof. We construct a radio k'-colouring f' of G with $\mathrm{rc}_{k'}(f') = c+(n-1)(k'-k)$ from a radio k-colouring f with $\mathrm{rc}_k(f) = c$ in the following way: Let $x_1, x_2, \ldots, x_n$ be an ordering of the vertices of G such that $f(x_i) \leqslant f(x_{i+1})$, $1 \leqslant i \leqslant n-1$, and set

$$f'(x_i) = f(x_i) + (i-1)(k'-k).$$

For any two integers i and j, $1 \leqslant i < j \leqslant n$, we have $|f'(x_j) - f'(x_i)| = |f(x_j) - f(x_i)| + (j-i)(k'-k)$.

As $|f(x_j) - f(x_i)| \geqslant 1 + k - d(x_j, x_i)$ and $j - i \geqslant 1$, we obtain $|f'(x_j) - f'(x_i)| \geqslant 1 + k + (j-i)(k'-k) - d(x_j, x_i) \geqslant 1 + k' - d(x_j, x_i)$. Thus f' is a radio k'-colouring of G and $\mathrm{rc}_{k'}(f') = c + (n-1)(k'-k)$. $\square$

The above result can be strengthened a little in some cases:

Lemma 2. *Let G be a graph of order n and let k, k' be integers, $k' > k$. Given a radio k-colouring f of G, let $x_1, x_2, \ldots, x_n$ be an ordering of the vertices of G such that $f(x_i) \leqslant f(x_{i+1})$, $1 \leqslant i \leqslant n-1$ and let $\varepsilon_i = |f(x_i) - f(x_{i-1})| - (1+k-d(x_i, x_{i-1}))$, $2 \leqslant i \leqslant n$. Consider a set $I = \{i_1, i_2, \ldots, i_s\} \subset \{2, \ldots, n\}$, where $1 \leqslant s \leqslant n-1$, such that $i_{j+1} > i_j + 1$ for all $j, 1 \leqslant j \leqslant s-1$. Then there exists a radio k'-colouring f' of G with $\mathrm{rc}_{k'}(f') \leqslant \mathrm{rc}_k(f) + (n-1)(k'-k) - \sum\limits_{i \in I} \min(k'-k, \varepsilon_i)$.*

Proof. A radio k'-colouring f' of G is obtained simply by setting for all j with $1 \leqslant j \leqslant n-1$:

$$f'(x_j) = f(x_j) + (j-1)(k'-k) - \sum_{i \in I, i \leqslant j} \min(k'-k, \varepsilon_i).$$

The vertex x_n has the maximum colour: $f'(x_n) = f(x_n) + (n-1)(k'-k) - \sum\limits_{i \in I} \min(k'-k, \varepsilon_i) = \mathrm{rc}_k(f) + (n-1)(k'-k) - \sum\limits_{i \in I} \min(k'-k, \varepsilon_i)$.

Then, for any two integers j_1 and j_2, $1 \leqslant j_1 < j_2 \leqslant n$, let us show that the condition

$$|f'(x_{j_2}) - f'(x_{j_1})| \geqslant 1 + k' - d(x_{j_2}, x_{j_1})$$

is verified, i.e. that

$$|f(x_{j_2}) - f(x_{j_1})| + (j_2 - j_1)(k' - k) - \sum_{i \in I, j_1 < i \leqslant j_2} \min(k' - k, \varepsilon_i) \geqslant 1 + k' - d(x_{j_2}, x_{j_1}).$$

If $j_2 = j_1 + 1$, then $|f(x_{j_2}) - f(x_{j_1})| = 1 + k - d(x_{j_2}, x_{j_1}) + \varepsilon_{j_2}$. Thus $|f'(x_{j_2}) - f'(x_{j_1})| \geqslant 1 + k - d(x_{j_2}, x_{j_1}) + \varepsilon_{j_2} + (k' - k) - \min(k' - k, \varepsilon_{j_2}) \geqslant 1 + k' - d(x_{j_2}, x_{j_1})$.

If $j_2 > j_1 + 1$, then $\sum\limits_{i \in I, j_1 < i \leqslant j_2} \min(k' - k, \varepsilon_i) \leqslant (j_2 - j_1 - 1)(k' - k)$ since by the hypothesis there are no two consecutive integers in the set I. Thus $|f'(x_{j_2}) - f'(x_{j_1})| \geqslant 1 + k - d(x_{j_2}, x_{j_1}) + (j_2 - j_1)(k' - k) - (j_2 - j_1 - 1)(k' - k) = 1 + k' - d(x_{j_2}, x_{j_1})$.

Therefore, f' is a radio k'-colouring of G and $\mathrm{rc}_{k'}(f') = \mathrm{rc}_k(f) + (n-1)(k'-k) - \sum\limits_{i \in I} \min(k' - k, \varepsilon_i)$. □

3. Antipodal colourings of paths

Theorem 3 derives from the next two theorems.

Theorem 5. *For any $n \geqslant 5$,*

$$\mathrm{ac}(P_n) \leqslant \begin{cases} 2p^2 - 2p + 3 & \text{if } n = 2p + 1, \\ 2p^2 - 4p + 5 & \text{if } n = 2p. \end{cases}$$

Proof. The fact that $\mathrm{ac}(P_5) = 7$ is easily checked (see [3]). Thus take $n \geqslant 6$ and let $P_n = (u_1, u_2, \ldots, u_n)$. We consider two cases depending on whether n is even or odd.

Case 1. $n = 2p + 1$ is odd for an integer $p \geqslant 3$. Define a colouring f of P_{2p+1} by

$$\begin{cases} f(u_1) = 3p + 2, \\ f(u_2) = p + 1, \\ f(u_i) = i(2p - 1) - p + 3, & 3 \leqslant i \leqslant p, \\ f(u_{p+1}) = 2p + 2, \\ f(u_{p+2}) = 1, \\ f(u_{p+i}) = i(2p - 1) - 2p + 3, & 3 \leqslant i \leqslant p, \\ f(u_{2p+1}) = p + 2. \end{cases}$$

Then the vertex u_p has the maximum colour: $f(u_p) = p(2p-1)-p+3 = 2p^2-2p+3$. We only have to show that the distance condition is verified for two vertices u_i and u_{p+j}, $3 \leqslant i, j \leqslant p$ (the other cases can be easily checked). We want

$$\begin{aligned}
|f(u_{p+j}) - f(u_i)| &\geqslant 1 + (D-1) - d(u_{p+j}, u_i) \Leftrightarrow \\
|j(2p-1) - 2p + 3 - (i(2p-1) - p + 3)| &\geqslant 2p - (p + j - i) \Leftrightarrow \\
|(j-i)(2p-1) - p| &\geqslant p - j + i.
\end{aligned}$$

If $j-i \geqslant 1$ then $|(j-i)(2p-1)-p| = (j-i)(2p-1)-p \geqslant 2p-1-p = p-1 \geqslant p-j+i$.

If $j-i < 1$ then $|(j-i)(2p-1)-p| = -(j-i)(2p-1)+p = (i-j)(2p-1)+p \geqslant p-j+i$ for $p \geqslant 1$.

C a s e 2. $n = 2p$ is even for an integer $p \geqslant 3$. Define a colouring f of P_{2p} by

$$\begin{cases}
f(u_1) = p, & \\
f(u_i) = (p-i)(2p-1) + 2, & 2 \leqslant i \leqslant p-1, \\
f(u_p) = 2p^2 - 4p + 5, & \\
f(u_{p+i}) = 2p^2 - 4p + 6 - f(u_{p-i+1}), & 1 \leqslant i \leqslant p.
\end{cases}$$

Then the vertex u_p has the maximum colour: $f(u_p) = 2p^2 - 4p + 5$. We only have to show that the distance condition is verified for two vertices u_i and u_{p+j}, $2 \leqslant i \leqslant p-1$, $1 \leqslant j \leqslant p$ (the other cases can be easily checked). We want

$$\begin{aligned}
|f(u_{p+j}) - f(u_i)| &\geqslant 1 + (D-1) - d(u_{p+j}, u_i) \Leftrightarrow \\
|(p-j)(2p-1) + 3 - ((p-i)(2p-1) - p + 2)| &\geqslant 2p - 1 - (p + j - i) \Leftrightarrow \\
|(i-j)(2p-1) + p + 1| &\geqslant p - j + i - 1.
\end{aligned}$$

If $i-j \geqslant 0$ then $|(i-j)(2p-1)+p+1| = (i-j)(2p-1)+p+1 \geqslant p-j+i-1$ since $(i-j)(2p-2) \geqslant -1$ for $p \geqslant 1$.

If $i-j < 0$, i.e. if $j-i \geqslant 1$ then $|(i-j)(2p-1)+p+1| = (j-i)(2p-1)-p-1 \geqslant p-j+i-1$ since $2p(j-i) \geqslant 2p$. □

Theorem 6. *For any* $n \geqslant 5$,

$$\operatorname{ac}(P_n) \geqslant \begin{cases} 2p^2 - 2p + 3 & \text{if } n = 2p+1, \\ 2p^2 - 4p + 5 & \text{if } n = 2p. \end{cases}$$

P r o o f. For $n = 2p+1$, by Lemma 1 we have $\operatorname{rc}_{n-1}(P_n) \leqslant \operatorname{ac}(P_n) + (n-1)$. This together with Theorem 4 gives $\operatorname{ac}(P_{2p+1}) \geqslant 2p^2 + 3 - 2p$.

For $n = 2p$, let $D = D(P_{2p}) = 2p-1$. We will use Lemma 2 with the radio $(D-1)$-colouring f of P_{2p} described in the proof of Theorem 5 and with $k = D-1 = 2p-1$ and $k' = D = 2p$. Keeping the notation of Lemma 2, one can see that f is such that $x_1 = u_{p+1}$, $x_2 = u_1$, $x_3 = u_{2p-1}$, $x_4 = u_{p-1}$, …, $x_{2j+1} = u_{2p-j+1}$, $x_{2j} = u_{p-j+1}$, …, $x_{2p-1} = u_{2p}$, $x_{2p} = u_p$. Thus ε_3 verifies

$$\begin{aligned}\varepsilon_3 &= |f(x_3) - f(x_2)| - (1 + k - d(x_3, x_2))\\ &= |f(u_{2p-1}) - f(u_1)| - (1 + 2p - 2 - (2p-2))\\ &= |2p^2 - 4p + 6 - f(u_2) - f(u_1)| - 1\\ &= |2p^2 - 4p + 6 - (p-2)(2p-1) - 2 - p| - 1 = 1.\end{aligned}$$

A similar calculus gives $\varepsilon_{2p-1} = 1$ and $\varepsilon_i = 0$ for all other indices.

Thus, as $k' - k = 1$ and $p \geqslant 3$, applying Lemma 2 with $I = \{3, 2p-1\}$ gives

$$\mathrm{rc}_{2p-1}(P_{2p}) \leqslant \mathrm{ac}(P_{2p}) + (2p-1) - \varepsilon_3 - \varepsilon_{2p-1},$$

that is

$$\mathrm{ac}(P_{2p}) \geqslant \mathrm{rc}_{2p-1}(P_{2p}) - (2p-1) + \varepsilon_3 + \varepsilon_{2p-1}.$$

By virtue of Theorem 4 we obtain $\mathrm{ac}(P_{2p}) \geqslant 2p^2-2p+2-(2p-1)+1+1 = 2p^2-4p+5$. $\square$

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