THE PDE DESCRIBING CONSTANT MEAN CURVATURE SURFACES

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Dedicated to Professor Dr. Jindřich Nečas on the occasion of his 70th birthday

Abstract. We give an expository account of a Weierstrass type representation of the non-zero constant mean curvature surfaces in space and discuss the meaning of the representation from the point of view of partial differential equations.

Keywords: constant mean curvature surfaces, nonlinear partial differential equation, dressing action, Weierstrass type representation

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In these notes, we would like to share with the reader some recent developments concerning surfaces having non-zero constant mean curvature. Since we hope to get the main ideas acrossed while keeping our exposition short, parts of our context will not be very rigorous and our list of references will be fairly incomplete.

We will start with the definitions of Gaussian curvature and mean curvature of a surface and recall briefly the history of surfaces having constant Gaussian curvature or constant mean curvature. Then, the PDE describing the (non-zero) constant mean curvature surfaces will be introduced. After that, we will present a Weierstrass type representation of the constant mean curvature surfaces discovered by Dorfmeister, Pedit and the author using techniques from soliton theory, and explain the meaning of the representation from the point of view of PDE's.

The reader is urged to check into the gallery of constant mean curvature surfaces at `http://www.gang.umass.edu`, for the graphs there can greatly help the reader understanding these notes.

1. Definitions

Let $M \subset \mathbb{R}^3$ be an immersed surface, $p \in M$ a point, $\boldsymbol{n}$ a unit normal vector of M at p, and $\boldsymbol{t}$ a unit tangent vector to M at p. Then, near p, the intersection of M and the plane through $\boldsymbol{n}$ and $\boldsymbol{t}$ is a plane curve, whose curvature at p relative to $\boldsymbol{n}$ will be denoted by $\kappa(p, \boldsymbol{t})$. As a function of $\boldsymbol{t}$, $\kappa(p, \boldsymbol{t})$ has a maximum and a minimum, to be denoted by $\kappa_1(p)$ and $\kappa_2(p)$, respectively. Then, locally on the surface,

$$\text{Gaussian curvature } K = \kappa_1 \kappa_2, \tag{1.1}$$

$$\text{mean curvature } H = \frac{\kappa_1 + \kappa_2}{2}, \tag{1.2}$$

$$\text{mean curvature vector } \boldsymbol{H} = H\boldsymbol{n}. \tag{1.3}$$

Remark 1.4. The Gaussian curvature K and the mean curvature vector $\boldsymbol{H}$ do *not* change when $\boldsymbol{n}$ is replaced by $-\boldsymbol{n}$, even though H gets a negative sign. Therefore, K and $\boldsymbol{H}$ are defined on the *whole* surface. Actually, K only depends on the induced metric on the surface, by Gauss's Theorema Egregium.

Remark 1.5. Rescaling $\mathbb{R}^3$ is equivalent to multiplying K and $\boldsymbol{H}$ by a positive constant.

Remark 1.6. A point p is called an *umbilic point* if $\kappa_1(p) = \kappa_2(p)$. At such a point, the tangent directions can not be distinguished geometrically.

Surfaces with constant K or constant length $\boldsymbol{H}$ will be called *basic surfaces*. The following classification results are well-known (see, for example, [8]).

Theorem 1.7. *Any complete surface whose Gaussian curvature K always equals a positive constant is a sphere.*

Theorem 1.8. *Any complete surface whose Gaussian curvature K always equals zero is a general cylinder.*

Here, by a *general cylinder* we mean the surface obtained by moving a line along a curve without changing the direction of the line (which is, of course, assumed to be never tangent to the curve).

Theorem 1.9. *There is no complete surface whose Gaussian curvature K always equals a negative constant.*

The usual proof of this result of Hilbert uses the sine-Gordon equation

$$u_{xt} = \sin u \tag{1.10}$$

and the fact that it does not have any solution u on $\mathbb{R}^2$ satisfying $0 < u < \pi$. Such a solution can be interpreted as the angle between the so-called asymptotic directions on the surface.

If $\boldsymbol{H}$ has constant length, then H can also be defined on the whole surface and is either always zero or always equal to a non-zero constant. When $H \equiv 0$, the surface is called a *minimal surface*. For such surfaces, we have the following famous result of Weierstrass [8].

Theorem 1.11. *Each minimal surface is given by the integrals of two holomorphic functions on the surface.*

When H is a non-zero constant, the surface is orientable. By using the other unit normal vector field on the surface and rescaling $\mathbb{R}^3$ if necessary, we can always assume that $H \equiv 1/2$. We will call these surfaces *CMC surfaces*. These surfaces show up as, for example, interface surfaces in nature and have been mentioned in Professor Finn's talk.

In summary, CMC surfaces are basic surfaces and they can be regarded as interface surfaces in nature.

2. Examples

The spheres and round cylinders are the first few examples of CMC surfaces. In 1841, Delaunay classified all the revolutional CMC surfaces, to be called *Delaunay surfaces*. In addition to the spheres and cylinders, Delaunay surfaces are generated by the trajectory of a focus of an ellipse or hyperbola when the ellipse or hyperbola is rolled on the axes of the surfaces. For an excellent account of these surfaces, see [9].

For some time, no new CMC surfaces were found. In his seminar lectures [13] in the 1940's, H. Hopf even asked: are there any *compact* CMC surfaces other than the spheres? In 1986, using classical existence methods in PDE, Wente [21] answered the above question affirmatively and provided many new examples of CMC surfaces (see also [22] and [23]).

Theorem 2.1. *There exist infinitely many constant mean curvature tori.*

Wente's tori can be graphed on a computer, see [22]. And [21] inspired a considerable amount of work on CMC tori, including [1], [18], [2], [7], [14] and [4].

In [16] and [17], using existence methods by the Inverse Function Theorem in infinite dimensions, Kapouleas proved the following results.

Theorem 2.2. *For each integer $g \geqslant 2$, there is a compact constant mean curvature surface of genus g.*

Here we would like to remark that this is an existence result. So far, we have not seen any graph of the compact CMC surfaces that are shown to exist by the theorem. On the other hand, [12] provides several graphs of compact CMC surfaces; however, to the best of our knowledge, an existence proof corresponding to these graphs is still missing.

In addition to these compact CMC surfaces, many new non-compact surfaces were also found. See, for example, [19], [15], [10], [20] and [11]. We also would like to remark that we have not seen any graph of some of the surfaces proved to exist in one of these references.

In summary, there are many CMC surfaces. A unified treatment and a general procedure for graphing them are needed.

3. PDE describing CMC surfaces

Let $\widetilde{M}$ be the universal covering of M, then the lift $\varphi\colon \widetilde{M} \longrightarrow \mathbb{R}^3$ of the CMC immersion from M into $\mathbb{R}^3$ is also an immersion with CMC $H \equiv 1/2$ with respect to a unit normal vector field $\boldsymbol{n}$ along $\varphi(\widetilde{M})$. By changing the coordinate system on $\widetilde{M}$ if necessary, we can assume that φ is conformal, i.e., the induced metric on $\varphi(\widetilde{M})$ is given by

$$(\mathrm{d}s)^2 = 4\mathrm{e}^{2u}[(\mathrm{d}x)^2 + (\mathrm{d}y)^2] \tag{3.1}$$

for some function $u\colon \widetilde{M} \longrightarrow \mathbb{R}$. Elements of $\mathbb{R}^3$ will be regarded as column vectors. Let $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ be the unit tangent vector fields in the directions of $\frac{\partial}{\partial x}\varphi$ and $\frac{\partial}{\partial y}\varphi$, respectively, then we have a moving frame

$$F = (\, \boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{n}\,) \tag{3.2}$$

along $\varphi(\widetilde{M})$. By interchanging x and y if necessary, we assume that

$$F \in \mathrm{SO}(3). \tag{3.3}$$

In terms of the complex coordinate $z = x + \mathrm{i}y$ (with the metric on $\mathbb{R}^3$ being $\mathbb{C}$-linearly extended to $\mathbb{C}^3$), the Frenet equations on F can be written as follows (see, for example, Section 1 of [18]).

$$\partial_z F = F(J + U_z), \qquad \partial_{\bar{z}} F = F(-U_{\bar{z}} + \overline{J}), \tag{3.4}$$

where

$$
(3.5)\quad U = \begin{pmatrix} 0 & \mathrm{i}u & 0 \\ -\mathrm{i}u & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad J = \frac{1}{2}\begin{pmatrix} 0 & 0 & -E\mathrm{e}^{-u} - \mathrm{e}^{u} \\ 0 & 0 & -\mathrm{i}E\mathrm{e}^{-u} + \mathrm{i}\mathrm{e}^{u} \\ E\mathrm{e}^{-u} + \mathrm{e}^{u} & \mathrm{i}E\mathrm{e}^{-u} - \mathrm{i}\mathrm{e}^{u} & 0 \end{pmatrix}
$$

with $E = \langle \varphi_{zz}, \boldsymbol{n} \rangle$. The integrability conditions for the system (3.4), i.e., the Gauss equation and Codazzi equations for φ, are

$$
(3.6)\qquad 4u_{z\bar{z}} - E\overline{E}\mathrm{e}^{-2u} + \mathrm{e}^{2u} = 0
$$

and $E_{\bar{z}} = 0$. So, E is a holomorphic function on $\widetilde{M}$. Actually, $E(z)(\mathrm{d}z)^2$ is the so-called *Hopf differential* of φ. The zeros of $E(z)(\mathrm{d}z)^2$ are exactly the umbilic points of the CMC surface. So, the umbilic points are isolated, except on the spheres.

Conversely, given a holomorphic function E on a simply connected domain in $\mathbb{C}$, and a real solution u to (3.6) on the domain, one can solve (3.4) for F with an arbitrary initial condition (i.e., a rotation of $\mathbb{R}^3$), then recover φ from $\frac{\partial}{\partial x}\varphi$ and $\frac{\partial}{\partial y}\varphi$.

Note that for the spheres, $E \equiv 0$; while for the round cylinders, $u \equiv 0$ and E is a non-zero constant.

In summary, a simply connected CMC immersion is equivalent to a holomorphic function E and a real solution u to the nonlinear PDE (3.6) on a simply connected domain in $\mathbb{C}$.

4. Dressing action

Observation: if E and u give a CMC surface, then for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, λE and u also give a CMC surface, i.e., each CMC surface belongs to a 1-parameter family of CMC surfaces, to be called the *associated family*. From the PDE point of view, here we use a certain symmetry of (3.6).

For each $\lambda = \mathrm{e}^{\mathrm{i}\theta} \in \mathrm{S}^1$, where $\theta \in [0, 2\pi)$, let

$$
(4.1)\qquad (\, \boldsymbol{e}_1(z,\bar{z},\lambda) \quad \boldsymbol{e}_2(z,\bar{z},\lambda) \quad \boldsymbol{n}(z,\bar{z},\lambda) \,), \qquad z \in \widetilde{M}
$$

be a frame produced as above along the CMC surface corresponding to u and $\lambda^2 E$ and set

$$
(4.2)\qquad F_\lambda(z,\bar{z}) = (\, \boldsymbol{e}_1(z,\bar{z},\lambda) \quad \boldsymbol{e}_2(z,\bar{z},\lambda) \quad \boldsymbol{n}(z,\bar{z},\lambda) \,) \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

for $z \in \widetilde{M}$. Then, by (3.4),

$$\partial_z F_\lambda = F_\lambda(J\lambda^{-1} + U_z), \qquad \partial_{\bar z} F_\lambda = F_\lambda(-U_{\bar z} + \overline{J}\lambda). \tag{4.3}$$

Here we see that the role of the parameter λ in (4.3) is to separate the intrinsic and extrinsic parts of the coefficient matrices.

The system (4.3) is equivalent to the system

$$\begin{aligned} \partial_z \Phi &= \Phi \frac{1}{2}\begin{pmatrix} u_z & e^u\lambda^{-1} \\ Ee^{-u}\lambda^{-1} & -u_z \end{pmatrix} = \Phi(\alpha_{-1}\lambda^{-1} + \alpha_0^-), \\ \partial_{\bar z} \Phi &= \Phi \frac{1}{2}\begin{pmatrix} -u_{\bar z} & -\overline{E}e^{-u}\lambda \\ -e^u\lambda & u_{\bar z} \end{pmatrix} = \Phi(\alpha_0^+ + \alpha_1\lambda) \end{aligned} \tag{4.4}$$

on $\Phi\colon \widetilde{M} \longrightarrow \mathrm{SU}(2)$. Actually, Φ is a lift of F_λ under the adjoint representation $\mathrm{Ad}\colon \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$. So, we have shown the following result.

Lemma 4.5. *A constant mean curvature surface* $\Longleftrightarrow F_\lambda(z, \bar z) = \Phi(z, \bar z, \lambda)$.

Now comes the magic. Take a complex matrix

$$H_+(\lambda) = \begin{pmatrix} a_0 & 0 \\ 0 & d_0 \end{pmatrix} + \begin{pmatrix} 0 & b_1 \\ c_1 & 0 \end{pmatrix}\lambda + \begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & b_3 \\ c_3 & 0 \end{pmatrix}\lambda^3 + \dots \tag{4.6}$$

with determinant 1, then

$$\begin{aligned} H_+(\lambda)\Phi(z, \bar z, \lambda) &= \Psi(z, \bar z, \lambda) G_+(z, \bar z, \lambda) \\ &= \Psi(z, \bar z, \lambda)(G_0(z, \bar z) + G_1(z, \bar z)\lambda + G_2(z, \bar z)\lambda^2 + \dots), \end{aligned} \tag{4.7}$$

$$\begin{aligned} \Psi^{-1}\partial_z\Psi &= G_+(\alpha_{-1}\lambda^{-1} + \alpha_0^-)G_+^{-1} - \partial_z G_+ \cdot G_+^{-1} \\ &= \beta_{-1}\lambda^{-1} + \beta_0^- + \dots, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \Psi^{-1}\partial_{\bar z}\Psi &= G_+(\alpha_0^+ + \alpha_1\lambda)G_+^{-1} - \partial_{\bar z} G_+ \cdot G_+^{-1} \\ &= \beta_0^+ + \beta_1\lambda + \dots. \end{aligned} \tag{4.9}$$

Thus,

$$\Psi^{-1}\partial_z\Psi = \beta_{-1}\lambda^{-1} + \beta_0^-, \qquad \Psi^{-1}\partial_{\bar z}\Psi = \beta_0^+ + \beta_1\lambda, \tag{4.10}$$

and hence Ψ gives another CMC surface, to be called the *dressing action* by H_+ of the old CMC surface.

From the PDE point of view, the dressing action yields new solutions from old ones. In particular, one gets infinitely many non-trivial solutions from the trivial one. A nature question then is: how powerful is dressing? The following is a result from [7].

Theorem 4.11. *Any constant mean curvature torus can be obtained from the round cylinder by dressing.*

It is easy to see that the Hopf differential does not change under dressing. The next question is: is there anything else invariant under dressing? Related to this is the following fact in [24].

Theorem 4.12. *At each umbilic point of order $\geqslant 2$, there are several dressing invariants on the induced metric.*

In summary, dressing is a powerful tool for generating new solutions from old ones; however, there are many dressing invariants in general.

5. Weierstrass type representation

In addition to the matrix splitting yielding the dressing action, there is another splitting. To be more precise, except at possibly some isolated points on the surface,

$$\begin{aligned}\Phi(z,\bar z,\lambda) &= \Phi_-(z,\bar z,\lambda)\Phi_+(z,\bar z,\lambda)\\ &= (\ldots+\Phi_{-1}(z,\bar z)\lambda^{-1}+I)(\Phi_0(z,\bar z)+\Phi_1(z,\bar z)\lambda+\ldots)\end{aligned}\tag{5.1}$$

From $\Phi_-=\Phi\Phi_+^{-1}$ one obtains that

$$\Phi_-^{-1}\partial_z\Phi_- = \Phi_+(\alpha_{-1}\lambda^{-1}+\alpha_0^-)\Phi_+^{-1}-\partial_z\Phi_+\cdot\Phi_+^{-1},\tag{5.2}$$

$$\Phi_-^{-1}\partial_{\bar z}\Phi_- = \Phi_+(\alpha_0^+ +\alpha_1\lambda)\Phi_+^{-1}-\partial_{\bar z}\Phi_+\cdot\Phi_+^{-1}.\tag{5.3}$$

These identities imply that

$$\partial_z\Phi_-(z,\bar z,\lambda)=\Phi_-(z,\bar z,\lambda)P(z,\bar z)\lambda^{-1},\tag{5.4}$$

$$\partial_{\bar z}\Phi_-(z,\bar z,\lambda)=0,\tag{5.5}$$

respectively. So,

$$\Phi_-=\Phi_-(z,\lambda)\tag{5.6}$$

and

$$\partial_z\Phi_-(z,\lambda)=\Phi_-(z,\lambda)P(z)\lambda^{-1}.\tag{5.7}$$

Moreover, we have the following result from [6].

Theorem 5.8. *The matrix P is meromorphic on $\widetilde{M}$. Any constant mean curvature surface can be constructed from such a "potential" on its universal covering by*

(1) *solving* (5.7) *for $\Phi_-(z,\lambda)$ with an appropriate initial condition,*

(2) *splitting $\Phi_-(z,\lambda) = \Phi(z,\bar{z},\lambda)G_+(z,\bar{z},\lambda)$ to get $\Phi(z,\bar{z},\lambda)$ and hence the surface.*

This is a Weierstrass type representation of the CMC surfaces. Warning: not every meromorphic P, which must be off-diagonal, gives a smooth CMC surface. For characterizations of admissible P's, see [3] and [26]; for the geometric meaning of P's, see [5] and [25]. Moreover, the following result also from [6] says that one only needs to use holomorphic functions if more terms in λ are allowed.

Theorem 5.9. *Each constant mean curvature surface can also be constructed from a "holomorphic potential"*

$$P_{-1}(z)\lambda^{-1} + P_0(z) + P_1(z)\lambda + \dots \tag{5.10}$$

as above.

As to graphing CMC surfaces using this representation, there is already a user-friendly software called DPWlab at the Center for Geometry, Analysis, Numerics and Graphics, University of Massachusetts at Amherst. The center's web address is given in the introduction.

In summary, with a Weierstrass type representation of the CMC surfaces at hand, we are ready to do more classifications and graphing of CMC surfaces.

Concluding remarks. We believe that the above Weierstrass type representation of the smooth solutions to the nonlinear PDE describing the CMC surfaces is an example of the following phenomena:

$$\begin{array}{c}\text{a nonlinear PDE}\\ + \\ \text{extra structure(s), i.e., symmetry(ies)}\end{array} \Longrightarrow \text{a representation of solutions.}$$

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