ON MCSHANE-TYPE INTEGRALS WITH RESPECT TO SOME DERIVATION BASES

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Dedicated to Prof. Jaroslav Kurzweil on the occasion of his 80th birthday

Abstract. Some observations concerning McShane type integrals are collected. In particular, a simple construction of continuous major/minor functions for a McShane integrand in $\mathbb{R}^n$ is given.

Keywords: McShane integral, Kurzweil-Henstock integral, Perron integral, basis

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1. Introduction

Kurzweil-Henstock integral with respect to different derivation bases was considered in numerous publications (see for example [2], [3], [5], [11]). At the same time comparatively less attention was given to McShane type integrals with respect to bases different from the usual full interval basis. In this connection the paper [7] introducing approximate McShane integral is of interest. Being motivated by this paper (and also by its review [6]), we investigate here McShane type integrals with respect to more general bases. We obtain a condition put on McShane basis under which the corresponding McShane integral is absolute and therefore coincides (in the class of measurable functions) with the Lebesgue integral (Section 3). Considering Perron and McShane bases associated with the so-called local systems, we discuss in Section 4 the relation between Kurzweil-Henstock and McShane integrals defined with respect to the related bases. Section 5 is devoted to Perron type integrals with respect to McShane bases (*strong* Perron integrals, in terminology used in [7]) which are equivalent to the corresponding McShane type integrals. We give a (surprisingly)

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simple construction of continuous major/minor functions for a McShane integrand in $\mathbb{R}^n$. Then, this method is discussed in application to (one-dimensional) McShane integrals associated with local systems.

2. Preliminaries

We use the following notation and definitions. By an *interval* in $\mathbb{R}^n$ we mean the Cartesian product of any n compact nondegenerate subintervals of the real line $\mathbb{R}$.

By a *tagged interval* (a *free tagged interval*) we mean a pair (I, x) where I is an interval in $\mathbb{R}^n$ and $x \in I$ ($x \in \mathbb{R}^n$, respectively). By a *basis* (a *McShane basis*) in $\mathbb{R}^n$ we understand any nonempty collection $\mathcal{B} = \{\beta\}$ of families β of tagged (free tagged, respectively) intervals which has the *filter base property*: $\emptyset \notin \mathcal{B}$, and for every $\beta_1, \beta_2 \in \mathcal{B}$ there exists $\beta \in \mathcal{B}$ such that $\beta \subset \beta_1 \cap \beta_2$. Obviously, each basis is a McShane basis (note that what we call here a basis, is sometimes referred to as a Perron basis, see [8]).

By a *free tagged division* we mean a finite collection of free tagged intervals (I, x) in which intervals I are pairwise nonoverlapping. If $x \in E$ for all (I, x), then we say that a free tagged division is *tagged in* a set $E \subset \mathbb{R}^n$. A free tagged division is called a *free tagged partition of* an interval J if the union of intervals I from this division is J, and all the tags belong to J. Free tagged divisions will be denoted by $\mathcal{P}$, while free tagged partitions usually by π. For a function $f\colon \mathbb{R}^n \to \mathbb{R}$ and a free tagged division $\mathcal{P}$ we denote

$$\sigma(\mathcal{P}, f) = \sum_{(I,x)\in\mathcal{P}} f(x)|I|.$$

We say that a free tagged division $\mathcal{P}$ is *β-fine* if $\mathcal{P} \subset \beta \in \mathcal{B}$.

Given a McShane basis $\mathcal{B}$, by a $\mathcal{B}$-interval we mean any interval I such that $(I, x) \in \beta \in \mathcal{B}$ for some x and β. The collection of all $\mathcal{B}$-intervals we denote with $\mathcal{I}_{\mathcal{B}}$. We say that $\mathcal{B}$ has the *partitioning property* if

(i) for each $I \in \mathcal{I}_{\mathcal{B}}$ and every $\beta \in \mathcal{B}$ there exists a free tagged partition of I that is β-fine;

(ii) for any two $I, J \in \mathcal{I}_{\mathcal{B}}$ the closure of difference $I \setminus J$ can be expressed as a union of finitely many nonoverlapping $\mathcal{B}$-intervals.

McShane bases only with the partitioning property are considered in the sequel.

Definition 2.1. Let $\mathcal{B}$ be a McShane basis and let $I \in \mathcal{I}_{\mathcal{B}}$. We call a function $f\colon I \to \mathbb{R}$, *$\mathcal{B}M$-integrable* if there exists a real number $\mathbf{I}$ (its *$\mathcal{B}M$-integral*) such that for any $\varepsilon > 0$ there is a $\beta \in \mathcal{B}$ such that for every β-fine free tagged partition π of I,

$$|\sigma(\pi, f) - \mathbf{I}| < \varepsilon. \tag{1}$$

If $\mathcal{B}$ is a basis then f is called *$\mathcal{B}H$-integrable*, $\mathbf{I}$ being its *$\mathcal{B}H$-integral.*

Due to the filter base and the partitioning properties of $\mathcal{B}$, the value of integral is unique.

The family of all $\mathcal{B}$-intervals $J \subset I$ we denote as $\mathcal{I}_{\mathcal{B},I}$. With the aid of (ii) one proves that if $f\colon I \to \mathbb{R}$ is $\mathcal{B}M$-integrable, then it is $\mathcal{B}M$-integrable on each $J \in \mathcal{I}_{\mathcal{B},I}$. So, the *indefinite integral* F of f is defined as a function $F\colon \mathcal{I}_{\mathcal{B},I} \to \mathbb{R}$ by

$$F(J) = \int_J f.$$

2.1. McShane-$\mathcal{B}$-Perron integrals.

Fix a McShane basis $\mathcal{B}$ and an interval $I \in \mathcal{I}_{\mathcal{B}}$. By the *upper McShane-$\mathcal{B}$-derivative* of a function $G\colon \mathcal{I}_{\mathcal{B},I} \to \mathbb{R}$ at $x \in I$ we mean the value

$$\overline{D}_{\mathcal{B}}G(x) = \inf_{\beta\in\mathcal{B}} \sup_{(J,x)\in\beta} \frac{G(J)}{|J|}.$$

In a similar way the *lower McShane-$\mathcal{B}$-derivative* $\underline{D}_{\mathcal{B}}G(x)$ is defined. When $\mathcal{B}$ is a basis, then $\overline{D}_{\mathcal{B}}G(x)$ and $\underline{D}_{\mathcal{B}}G(x)$ are called respectively the *upper* and the *lower $\mathcal{B}$-derivative* of G at $x \in I$.

We will say that a function $G\colon \mathcal{I}_{\mathcal{B},I} \to \mathbb{R}$ is *additive* if $G(J) = \sum\limits_{i=1}^{l} G(J_i)$ whenever the interval $J \in \mathcal{I}_{\mathcal{B},I}$ is the union of nonoverlapping intervals $J_1, \ldots, J_l \in \mathcal{I}_{\mathcal{B},I}$. Similarly for *subadditivity* and *superadditivity.*

Definition 2.2. We say that an additive function $M\colon \mathcal{I}_{\mathcal{B},I} \to \mathbb{R}$ is a *McShane-$\mathcal{B}$-major function* for $f\colon I \to \mathbb{R}$ if at each point $x \in I$ we have

$$\underline{D}_{\mathcal{B}}M(x) \geqslant f(x). \tag{2}$$

We say that an additive function $m\colon \mathcal{I}_{\mathcal{B},I} \to \mathbb{R}$ is a *McShane-$\mathcal{B}$-minor function* for f if at each point $x \in I$ we have

$$\overline{D}_{\mathcal{B}}m(x) \leqslant f(x).$$

Definition 2.3. We say that a function $f\colon I \to \mathbb{R}$ is *McShane-$\mathcal{B}$-Perron integrable* if

$$\inf_M M(I) = \sup_m m(I),$$

where M ranges over the set of all McShane-$\mathcal{B}$-major and m ranges over all McShane-$\mathcal{B}$-minor functions for f. The common value is taken as the integral of f. When $\mathcal{B}$ is a basis, then we say that f is just *$\mathcal{B}$-Perron integrable.*

Theorem 2.4. *$\mathcal{B}M$-integral and McShane-$\mathcal{B}$-Perron integral are equivalent.*

Proof. The proof is standard and follows like proofs of [7, Theorems 3.7&3.8]. □

Corollary 2.5. *$\mathcal{B}H$-integral and $\mathcal{B}$-Perron integral are equivalent.*

2.2. Examples of bases.

2.2.1. Full bases in $\mathbb{R}^n$. Any positive function δ defined on $\mathbb{R}^n$ is called a *gauge*. Having fixed a gauge δ we say that a free tagged (or tagged) interval (I, x) is δ-fine, if I is contained in the $\delta(x)$-neighbourhood of x (we use the sup metric in $\mathbb{R}^n$ throughout the paper). Denote respectively by α_δ and β_δ families of all free tagged and tagged intervals in $\mathbb{R}^n$ that are δ-fine.

$$\mathcal{A}_{\text{full}} = \{\alpha_\delta\colon \ \delta \text{ a gauge}\}, \qquad \mathcal{B}_{\text{full}} = \{\beta_\delta\colon \ \delta \text{ a gauge}\}$$

form respectively a McShane basis and a basis in $\mathbb{R}^n$ both with partitioning property.

2.2.2. One-dimensional bases related to local systems. By a *local system* (see [10]) we mean a family $\Delta = \{\Delta(x)\}_{x\in\mathbb{R}}$ such that each $\Delta(x)$ is a nonvoid collection of subsets of $\mathbb{R}$ with the properties:

(i) $\{x\} \notin \Delta(x)$,
(ii) if $S \in \Delta(x)$ then $x \in S$,
(iii) if $S \in \Delta(x)$ and $R \supset S$ then $R \in \Delta(x)$,
(iv) if $S \in \Delta(x)$ and $\delta > 0$ then $(x - \delta, x + \delta) \cap S \in \Delta(x)$.

We say that Δ is *filtering down* if for each x and any $R, S \in \Delta(x)$, $R \cap S \in \Delta(x)$. Only such Δ's will be considered here. Any S belonging to $\Delta(x)$ is called a *path* leading to x. A function $\mathcal{C}$ on $\mathbb{R}$ such that $\mathcal{C}(x) \in \Delta(x)$ for each x is called a *choice*. Given a choice $\mathcal{C}$, we write $(I, x) \in \beta_{\mathcal{C}}$ ($(I, x) \in \tilde{\beta}_{\mathcal{C}}$) and say that a tagged interval (a free tagged interval) (I, x) is *$\beta_{\mathcal{C}}$-fine* (*$\tilde{\beta}_{\mathcal{C}}$-fine*, respectively) or *$\mathcal{C}$-fine* for short, if both endpoints of I are in $\mathcal{C}(x)$. The basis and the McShane basis induced by a local system Δ are defined respectively as

$$\mathcal{B}_\Delta = \{\beta_{\mathcal{C}}\colon \ \mathcal{C} \text{ a choice}\}, \qquad \widetilde{\mathcal{B}}_\Delta = \{\tilde{\beta}_{\mathcal{C}}\colon \ \mathcal{C} \text{ a choice}\}.$$

We say that a local system Δ satisfies the *intersection condition* (abbr. `IC`) if for every choice $\mathcal{C}$, there exists a gauge δ on $\mathbb{R}$ such that if $0 < y - x < \min\{\delta(x), \delta(y)\}$, then $\mathcal{C}(x) \cap \mathcal{C}(y) \cap [x, y] \neq \emptyset$. Thomson has proved in [10] that if Δ is bilateral, i.e., if each member of each $\Delta(x)$ has x as a bilateral accumulation point, and if it satisfies `IC`, then *each subinterval of the real line has a $\mathcal{C}$-fine partition for any choice $\mathcal{C}$.*

In what follows, for any basis $\mathcal{B}_\Delta$ associated with a local system Δ, the partitioning property will be always meant in this stronger version.

Examples of local systems are the *full* local system (consisting of families of neighbourhoods), the *density* local system [12, Example 2], the $\mathcal{I}$-*density* local system [4].

A slightly different notion is a *path system.* In this case a set $E_x \ni x$ is attached to each $x \in \mathbb{R}$ so that x is an accumulation point of E_x. Clearly, the collection $E(x) = \{(x-\delta, x+\delta) \cap E_x \colon \delta > 0\}$, $x \in \mathbb{R}$, does not form a local system since the condition (iii) is not satisfied. However, we remove this obstacle by defining an auxiliary local system Δ by $\Delta(x) = \{S \subset \mathbb{R} \colon S \supset R \in E(x)\}$, $x \in \mathbb{R}$, which we call the *local system induced by the path system* $E = \{E_x\}_{x\in\mathbb{R}}$. Anyway, the collection $\mathcal{B}_E = \{\beta_{E,\delta} \colon \delta \text{ a gauge}\}$ where $\beta_{E,\delta} = \{([a,b],x) \colon x-\delta(x) < a \leqslant x \leqslant b < x+\delta(x),\ a,b \in E_x\}$, forms a basis and it is apparent that the $\mathcal{B}_E H$- and $\mathcal{B}_\Delta H$-integrals are equivalent. The same with $\widetilde{\mathcal{B}}_E M$- and $\widetilde{\mathcal{B}}_\Delta M$-integrals. Thus, $\mathcal{B}_E H$- and $\widetilde{\mathcal{B}}_E M$-integrals can be considered as a case of $\mathcal{B}_\Delta H$- and $\widetilde{\mathcal{B}}_\Delta M$-integrals respectively.

3. When $\mathcal{B}M$-integral is absolute?

Let $\mathcal{B}$ be a McShane basis. If for each gauge δ there is a $\beta \in \mathcal{B}$ such that all members of β are δ-fine, then clearly the $\mathcal{B}M$-integral includes the ordinary McShane integral, i.e., includes the Lebesgue integral. We consider now if this generalization is strict. One checks easily that the $\mathcal{B}M$-integral is equivalent (in the class of measurable functions) to the McShane integral iff it is absolute, i.e., iff the integrability of a function f yields the integrability of $|f|$.

Theorem 3.1. *Assume that a McShane basis $\mathcal{B}$ satisfies the following condition: for each $\beta \in \mathcal{B}$ and any two $(I,x),(J,y) \in \beta$, either I and J are nonoverlapping or the intersection $I \cap J$ is expressible as the union of some nonoverlapping intervals $K_1,\dots,K_k$ with $(K_i,x),(K_i,y) \in \beta$ for $i = 1,\dots,k$.*

Then, the $\mathcal{B}M$-integral is absolute.

Proof. Let a function f on an n-dimensional interval I be $\mathcal{B}M$-integrable to a value $\mathbf{I}$. For $\varepsilon > 0$ take a suitable $\beta \in \mathcal{B}$ such that for any β-fine free tagged partition π of I the inequality (1) holds. Consider any two β-fine free tagged partitions of I: $\{(I_i,x_i)\}_i$ and $\{(J_j,y_j)\}_j$. Denote $K_{ij} = I_i \cap J_j$ (only nondegenerate intervals K_{ij} are taken into account). According to the condition assumed, $K_{ij} = \bigcup_{k=1}^{s_{ij}} K_{ijk}$, where K_{ijk}'s are pairwise nonoverlapping and $\{(K_{ijk},x_i)\}_{i,j,k}$ and $\{(K_{ijk},y_j)\}_{i,j,k}$ are also β-fine free tagged partitions of I. With Saks-Henstock lemma for the $\mathcal{B}M$-integral, for any collection $\mathcal{R} = \{(i,j)\}$ of pairs (i,j) for which K_{ij} have been defined, we get

$$\Bigl|\sum_{(i,j)\in\mathcal{R}} \sum_{k=1}^{s_{ij}} \Bigl(f(x_i)|K_{ijk}| - \int_{K_{ijk}} f\Bigr)\Bigr| < 2\varepsilon, \quad \Bigl|\sum_{(i,j)\in\mathcal{R}} \sum_{k=1}^{s_{ij}} \Bigl(f(y_j)|K_{ijk}| - \int_{K_{ijk}} f\Bigr)\Bigr| < 2\varepsilon,$$

whence

$$\Big|\sum_{(i,j)\in\mathcal{R}}\sum_{k=1}^{s_{ij}}(f(x_i)-f(y_j))|K_{ijk}|\Big| < 4\varepsilon. \tag{3}$$

Then

$$\begin{aligned}\Big|\sum_i |f(x_i)||I_i| - \sum_j |f(y_j)||J_j|\Big| &= \Big|\sum_{i,j,k} |f(x_i)||K_{ijk}| - \sum_{i,j,k} |f(y_j)||K_{ijk}|\Big|\\ &\leqslant \sum_{i,j,k} |f(x_i)-f(y_j)||K_{ijk}| = \sum_{l=1}^{2}\Big|\sum_{(i,j)\in\mathcal{R}_l}\sum_{k=1}^{s_{ij}}(f(x_i)-f(y_j))|K_{ijk}|\Big|,\end{aligned}$$

where

$$\begin{aligned}\mathcal{R}_1 &= \{(i,j)\colon\ K_{ij}\neq\emptyset \text{ and } f(x_i)\geqslant f(y_j)\},\\ \mathcal{R}_2 &= \{(i,j)\colon\ K_{ij}\neq\emptyset \text{ and } f(x_i)< f(y_j)\}.\end{aligned}$$

Apply (3) separately to $\mathcal{R}_1$ and $\mathcal{R}_2$, and get

$$\Big|\sum_i |f(x_i)||I_i| - \sum_j |f(y_j)||J_j|\Big| < 8\varepsilon.$$

So, for $|f|$ the Cauchy criterion for $\mathcal{B}M$-integrability is fulfilled. □

Let $\beta_\delta^{\mathrm{d}}$ be the family of all δ-fine free tagged *dyadic* intervals, that is intervals of the kind $([j/2^n,(j+1)/2^n],x)$, $j\in\mathbb{Z}$, $n\in\mathbb{N}$. The McShane basis

$$\{\beta_\delta^{\mathrm{d}}\colon\ \delta \text{ a gauge}\}$$

has the partitioning property and satisfies the assumption of the foregoing theorem.

4. On McShane integral with respect to local systems

The $\widetilde{\mathcal{B}}_\Delta M$-integral (related to a local system Δ) is in general not absolute.

Theorem 4.1. *Let Δ be a local system with the partitioning property. Assume that for some $x\in\mathbb{R}$ there is a path $S\in\Delta(x)$ which is dense (metrically) in no neighbourhood of x. Then there exists a function f nonintegrable in the ordinary McShane sense, but $\widetilde{\mathcal{B}}_\Delta M$-integrable.*

P r o o f. We may assume that S is dense in no left neighbourhood of x. Let $(a_n)_{n=1}^{\infty}$ be an increasing sequence of points that converges to x, such that $S \cap (a_{2n-1}, a_{2n}) = \emptyset$, $n = 1, 2, \ldots$. Define f on $[a_1, x]$ in the following way:

$$f = \begin{cases} \dfrac{1}{(a_{2n} - a_{2n-1})n} & \text{on } (a_{2n-1}, \frac{1}{2}(a_{2n-1} + a_{2n})),\ n = 1, 2, \ldots, \\ -\dfrac{1}{(a_{2n} - a_{2n-1})n} & \text{on } (\frac{1}{2}(a_{2n-1} + a_{2n}), a_{2n}),\ n = 1, 2, \ldots, \\ 0 & \text{elsewhere on } [a_1, x]. \end{cases}$$

Since $\int_{a_{2n-1}}^{a_{2n}} |f| = 1/n$, f is not McShane integrable on $[a_1, x]$. We are to justify that f is $\widetilde{\mathcal{B}}_\Delta M$-integrable. For each $y \in (a_n, a_{n+1})$, $n = 1, 2, \ldots$, take a number $\delta(y) > 0$ with $(y - \delta(y), y + \delta(y)) \subset (a_n, a_{n+1})$. Put $\delta(a_n) = \min\{|a_{n+1} - a_n|, |a_n - a_{n-1}|, 2^{-n}\}$, assuming $|a_1 - a_0| = 1$. Let $\varepsilon > 0$. Since f is Riemann integrable to zero on each interval $[a_{2n-1}, a_{2n}]$, there exist numbers η_n such that $|\sigma(\pi, f)| < \varepsilon 2^{-n}$ for each η_n-fine free tagged partition π of $[a_{2n-1}, a_{2n}]$. We can assume that $\delta \leqslant \eta_n$ on $[a_{2n-1}, a_{2n}]$. Define a choice $\mathcal{C}$ on $[a_1, x]$ by putting $\mathcal{C}(x) = S$ and $\mathcal{C}(y) = \big(y - \delta(y), y + \delta(y)\big)$ for $y \in [a_1, x)$, and consider any $\mathcal{C}$-fine free tagged partition $\tilde{\pi}$ of $[a_1, x]$. For each member (I, y) of $\tilde{\pi}$ there are four possibilities:

- $(*)$ $y = x$; then (I, y) contributes nothing to $\sigma(\tilde{\pi}, f)$.
- $(**)$ $y \neq x$ and $I \subset [a_{2n}, a_{2n+1}]$; then $y \in [a_{2n}, a_{2n+1}]$ thanks to the definition of $\delta(y)$, whence (I, y) contributes nothing to $\sigma(\tilde{\pi}, f)$ too.
- $(***)$ $y \neq x$ and $I \subset [a_{2n-1}, a_{2n}]$; then, since $\mathcal{C}(x)$ misses (a_{2n-1}, a_{2n}), (I, y) is a member of a free tagged partition $\pi_n \subset \tilde{\pi}$ of the interval $[a_{2n-1}, a_{2n}]$. Since π_n is η_n-fine, $|\sigma(\pi_n, f)| < \varepsilon 2^{-n}$.
- $(****)$ $y \neq x$ with I meeting two intervals: (a_{n-1}, a_n) and (a_n, a_{n+1}); then $y = a_n$ (by the definition of δ) and (I, a_n) can be split at a_n into two intervals (I', a_n) and (I'', a_n) with the same contribution to $\sigma(\tilde{\pi}, f)$ as (I, a_n), one of them being of the type $(**)$, the other of the type $(***)$.

For these reasons $|\sigma(\pi, f)| < \sum\limits_{n=1}^{\infty} \varepsilon 2^{-n} = \varepsilon$. Thus, f is $\widetilde{\mathcal{B}}_\Delta M$-integrable to zero. $\square$

Now we turn to examples of Kurzweil-Henstock integrable but not $\mathcal{B}_\Delta M$-integrable functions.

Lemma 4.2. *Let Δ be a local system with the partitioning property. Assume that a function $f\colon [a, b] \to \mathbb{R}$ is $\widetilde{\mathcal{B}}_\Delta M$-integrable with the indefinite integral $F\colon [a, b] \to \mathbb{R}$, $F(x) = \int_a^x f$. Then for each x there exists a path $S \in \Delta(x)$ such that $F \restriction S$ is a* VB*-function.*

P r o o f. Suppose it is not true. Then there is an $x \in [a, b]$ such that $F \restriction S$ has unbounded variation for all $S \in \Delta(x)$. Obviously, the function $\hat{f}$ defined by $\hat{f}(x) = 0$, $\hat{f}(t) = f(t)$ if $t \in [a, b] \setminus \{x\}$, is $\widetilde{\mathcal{B}}_\Delta M$-integrable with the indefinite integral F. There exists a choice $\mathcal{C}$ such that for any $\mathcal{C}$-fine free tagged partition π of $[a,b]$ one has $|\sigma(\pi,\hat{f})-F(b)|<1$. There are points $a_1,a_2,\ldots,a_{2N}\in\mathcal{C}(x)$, $a_1<a_2<\ldots<a_{2N}$, with

$$\sum_{i=1}^{N}|F(a_{2i})-F(a_{2i-1})|>2.$$

The free tagged division $\{([a_{2i-1},a_{2i}],x)\}_{i=1}^{N}$ is $\mathcal{C}$-fine. Saks-Henstock lemma for the $\widetilde{\mathcal{B}}_{\Delta}M$-integral implies that

$$\sum_{i=1}^{N}|F(a_{2i})-F(a_{2i-1})|=\sum_{i=1}^{N}|\hat{f}(x)(a_{2i}-a_{2i-1})-(F(a_{2i})-F(a_{2i-1}))|\leqslant 2,$$

giving the desired contradiction. □

With the aid of the above lemma it is easy to give examples of local systems Δ for which there are functions integrable in the Kurzweil-Henstock sense, while not being $\widetilde{\mathcal{B}}_{\Delta}M$-integrable.

P a t h s y s t e m s. Let $E=\{E_x\}_{x\in\mathbb{R}}$ be a path system. Take a decreasing sequence $a_1=1>a_2>a_3>\ldots$ converging to 0, $a_n\in E_0$ for $n\geqslant 2$, and define a function F on $[0,1]$ by putting

$$F(x)=\begin{cases}0 & \text{for } x=0 \text{ and } x=a_{2i+1},\ i=0,1,2,\ldots,\\ 1/i & \text{for } x=a_{2i},\ i=1,2,\ldots,\\ \text{linear} & \text{on intervals } [a_{i+1},a_i],\ i=1,2,\ldots.\end{cases}$$

The so defined F is the indefinite Kurzweil-Henstock integral of F'. For any neighbourhood I of 0, the set $I\cap E_0$ contains almost all points from the sequence $(a_n)_{n=1}^{\infty}$ and so the restriction $F\restriction(I\cap E_0)$ has unbounded variation. According to Lemma 4.2, F is not an indefinite $\widetilde{\mathcal{B}}_{E}M$-integral. Since F' is Riemann integrable on every interval $[c,1]$, $1>c>0$, there is no other indefinite $\widetilde{\mathcal{B}}_{E}M$-integral for F'; hence F' is not $\widetilde{\mathcal{B}}_{E}M$-integrable.

T h e d e n s i t y l o c a l s y s t e m. Define a function F on $[0,1]$ by

$$F(x)=\begin{cases}0 & \text{for } x=0 \text{ and } x=2^{-2i},\ i=0,1,\ldots,\\ 1/i & \text{for } x=2^{-2i-1},\ i=0,1,\ldots,\\ \text{linear} & \text{on intervals } [2^{-i-1},2^{-i}],\ i=0,1,\ldots.\end{cases}$$

It is the indefinite Kurzweil-Henstock integral of F'. We are to check that F is not an indefinite $\widetilde{\mathcal{B}}_{\Delta}M$-integral, with Δ being the density local system. Take any (measurable) $S\in\Delta(0)$. According to the definition of Δ, the set S has density 1 at 0, hence there exists $h>0$ such that $|(0,t)\cap S|/t>\frac{7}{8}$ for each $t\in(0,h)$. Take i_0

with $2^{-2i_0} < h$. For each $i \geqslant i_0$ we can choose points

$$s_1^i \in S \cap \Big(\frac{1}{2^{2i+1}}, \frac{1}{2^{2i+1}} + \frac{1}{4}\Big(\frac{1}{2^{2i}} - \frac{1}{2^{2i+1}}\Big)\Big),$$
$$s_2^i \in S \cap \Big(\frac{1}{2^{2i}} - \frac{1}{4}\Big(\frac{1}{2^{2i}} - \frac{1}{2^{2i+1}}\Big), \frac{1}{2^{2i}}\Big).$$

This is possible since both the foregoing intervals have length $\frac{1}{8}$ of the length of $(0, 2^{-2i})$. Intervals $\{[s_1^i, s_2^i]\}_{i=i_0}^{\infty}$ are pairwise nonoverlapping and have endpoints in S. Moreover, thanks to the way the points s_1^i, s_2^i were chosen, $F(s_1^i) - F(s_2^i) > \frac{1}{2}(F(2^{-2i-1}) - F(2^{-2i})) = \frac{1}{2}i^{-1}$. So $\sum\limits_{i=i_0}^{\infty} |F(s_1^i) - F(s_2^i)| = \infty$ and $F \restriction S$ is not a VB-function. According to Lemma 4.2, F is not an indefinite $\widetilde{\mathcal{B}}_\Delta M$-integral and so (like in the previous example) F' is not $\widetilde{\mathcal{B}}_\Delta M$-integrable.

Remark 4.3. A similar 'density' argument can be used to give an analogous example for $\mathcal{I}$-density local system (we do not want to involve the reader into extensive technical details needed for this). It is not clear if for *any* local system with the partitioning property one can go along arguments alike those used in the above examples. But there is a more interesting problem: is the converse of Lemma 4.2 true? Precisely,

Question 4.4. Let Δ be a local system with the partitioning property and assume that a function $f\colon [a,b] \to \mathbb{R}$ is $\mathcal{B}_\Delta H$-integrable with the indefinite integral $F\colon [a,b] \to \mathbb{R}$. Suppose that F has the following property: there is a choice $\mathcal{C}$ such that for each $x \in [a,b]$, $F \restriction \mathcal{C}(x)$ is a VB-function. Must f be $\widetilde{\mathcal{B}}_\Delta M$-integrable?

5. A simple construction of major/minor functions for the McShane-Perron integral

In this section we shall deal with some modifications of Definition 2.3. The first to be considered is the one with a continuity assumption put on major/minor functions. Given a McShane basis $\mathcal{B}$, a $\mathcal{B}$-interval I, and a function $G\colon \mathcal{I}_{\mathcal{B},I} \to \mathbb{R}$, we say that G is *$\mathcal{B}$-continuous at $x \in I$* if for each $\varepsilon > 0$ there exists $\beta \in \mathcal{B}$ such that $|G(J)| < \varepsilon$ for every $(J,x) \in \beta$. The function G is said to be *$\mathcal{B}$-continuous* if it is $\mathcal{B}$-continuous at each $x \in I$.

Definition 5.1. We say that a function $f\colon I \to \mathbb{R}$ is *McShane-$\mathcal{B}^{\mathrm{c}}$-Perron integrable* if

$$\inf_M M(I) = \sup_m m(I),$$

where M ranges over the set of all $\mathcal{B}$-continuous McShane-$\mathcal{B}$-major and m ranges over all $\mathcal{B}$-continuous McShane-$\mathcal{B}$-minor functions for f. This common value is taken as the integral of f. If $\mathcal{B}$ is a basis, then f is called simply *$\mathcal{B}^{\mathrm{c}}$-Perron integrable.*

Clearly, if f is McShane-$\mathcal{B}^{\mathrm{c}}$-Perron integrable then it is McShane-$\mathcal{B}$-Perron integrable with the same integral. A question with an old background is whether the converse is true. For some results on $\mathcal{B}^{\mathrm{c}}$-Perron integrals see [1, 9]. Our concern here is the McShane-$\mathcal{B}^{c}$-Perron integral.

With a standard argument one shows that the McShane-$\mathcal{A}_{\mathrm{full}}$-Perron integral (*McShane-Perron integral* for short) is equivalent to the ordinary McShane integral ($\mathcal{A}_{\mathrm{full}}M$-integral) (Theorem 2.4). Modifying slightly this argument we will show that for e a c h McShane integrable function f and each $\varepsilon > 0$ there exists a c o n t i n u o u s McShane-$\mathcal{A}_{\mathrm{full}}$-major function (*McShane-major function* for short) M such that $|M(I) - \int_I f| < \varepsilon$.

Let I be an n-dimensional interval, $\mathcal{I}$ the family of all its subintervals. Suppose we have a McShane integrable function $f\colon I \to \mathbb{R}$. Fix ε and let δ be a gauge such that the inequality $|\sigma(\pi, f) - \int_I f| < \varepsilon$ holds for any δ-fine free tagged partition of I. Take $J \in \mathcal{I}$ and define

$$\Phi^f_\delta(J) = \sup_{\mathcal{P}} \sum_{(K,t)\in\mathcal{P}} f(t)|K|, \tag{4}$$

where sup is taken over all δ-fine free tagged divisions $\mathcal{P}$ in I such that the intervals from $\mathcal{P}$ form a partition of J; i.e., $K \subset J$, but not necesserily $t \in J$. By Saks-Henstock lemma we have $|\Phi^f_\delta(J) - \int_J f| \leqslant 2\varepsilon$. This implies that Φ^f_δ is bounded as an interval function. So there exists B such that $|\Phi^f_\delta(J)| \leqslant B$ for all $J \in \mathcal{I}$.

We are to check three properties of $\Phi^f_\delta\colon \mathcal{I} \to \mathbb{R}$: being taken as M, it satisfies (2) ($\mathcal{B} = \mathcal{A}_{\mathrm{full}}$, $\mathcal{I}_{\mathcal{B},I}$ is $\mathcal{I}$ here), it is additive (and so it is a McShane-major function for f), and it is continuous (which is the same as being $\mathcal{A}_{\mathrm{full}}$-continuous). For any interval J from the $\delta(x)$-neighbourhood of $x \in I$, the one-element division $\mathcal{P} = \{(J, x)\}$ is in the domain of sup in (4) and so $\Phi^f_\delta(J) \geqslant f(x)|J|$. Hence $\underline{D}_{\mathcal{A}_{\mathrm{full}}}(\Phi^f_\delta)(x) \geqslant f(x)$ and (2) is satisfied. Obviously Φ^f_δ is *superadditive*, i.e., $\Phi^f_\delta(J) \geqslant \sum\limits_{i=1}^{l} \Phi^f_\delta(J_i)$ whenever the interval $J \in \mathcal{I}$ is the union of some nonoverlapping intervals $J_1, \ldots, J_l \in \mathcal{I}$. To prove the converse inequality, take any division $\mathcal{P}$ in I which is in the domain of sup in (4) for $\Phi^f_\delta(J)$. Then, the divisions $\mathcal{P}_i = \{(K \cap J_i, t)\colon (K,t) \in \mathcal{P}\}$, $i = 1, \ldots, l$, are in domains of sup for $\Phi^f_\delta(J_i)$ respectively. Moreover, $\sigma(\mathcal{P}, f) = \sum\limits_{i=1}^{l} \sigma(\mathcal{P}_i, f) \leqslant \sum\limits_{i=1}^{l} \Phi^f_\delta(J_i)$ and since $\mathcal{P}$ is arbitrary we get $\Phi^f_\delta(J) \leqslant \sum\limits_{i=1}^{l} \Phi^f_\delta(J_i)$.

Finally, assume that Φ^f_δ is discontinuous at some $x \in I$. That means, there exists $\varepsilon > 0$ such that for an arbitrarily small $\eta > 0$ there is an interval $J_1 \in \mathcal{I}$ with $x \in \operatorname{int} J_1$, $\operatorname{diam} J_1 < \eta$, and $\Phi^f_\delta(J_1) > \varepsilon$ or $\Phi^f_\delta(J_1) < -\varepsilon$. With no restriction of generality assume the former case holds for all η. Pick any such J_1 with $\operatorname{diam} J_1 < \delta(x)$. There exists a free tagged division $\mathcal{P}$ in I which is in the domain of sup in (4) for $\Phi^f_\delta(J_1)$, such that $\sigma(\mathcal{P}, f) > \varepsilon$. Denote $\mathcal{R} = \{(K,t) \in \mathcal{P}\colon K \ni x\}$. Take an open interval $L \ni x$ so small that it meets only intervals from $\mathcal{R}$ and

$2^n|f(t)||K\cap L| < \sigma(\mathcal{P},f)-\varepsilon$ for each $(K,t)\in\mathcal{R}$. Divide each difference $K\setminus L$, $(K,t)\in\mathcal{R}$, into finitely many nonoverlapping intervals $K_1^K,\dots,K_{m_K}^K$ and define a new free tagged division:

$$\mathcal{P}_1=(\mathcal{P}\setminus\mathcal{R})\cup\bigcup_{(K,t)\in\mathcal{R}}\bigcup_{i=1}^{m_K}\{(K_i^K,t)\}.$$

Estimate (there are at most 2^n members of $\mathcal{R}$)

$$\begin{aligned}|\sigma(\mathcal{P},f)-\sigma(\mathcal{P}_1,f)| &= \Bigl|\sigma(\mathcal{R},f)-\sum_{(K,t)\in\mathcal{R}}\sum_{i=1}^{m_K}f(t)|K_i^K|\Bigr|\\ &= \Bigl|\sigma(\mathcal{R},f)-\sum_{(K,t)\in\mathcal{R}}f(t)|K\setminus L|\Bigr|\\ &\leqslant \sum_{(K,t)\in\mathcal{R}}|f(t)||K\cap L| < \sum_{(K,t)\in\mathcal{R}}\frac{\sigma(\mathcal{P},f)-\varepsilon}{2^n}\leqslant\sigma(\mathcal{P},f)-\varepsilon.\end{aligned}$$

So,

$$\varepsilon<|\sigma(\mathcal{P},f)|-|\sigma(\mathcal{P}_1,f)-\sigma(\mathcal{P},f)|\leqslant\sigma(\mathcal{P}_1,f).$$

Next, take an interval $J_2\subset L$ with $x\in\operatorname{int}J_2$ and $\Phi_\delta^f(J_2)>\varepsilon$. Like for J_1, find a free tagged division $\mathcal{P}_2$ with intervals contained in J_2 but missing x such that $\varepsilon<\sigma(\mathcal{P}_2,f)$. Then find a J_3 with $\mathcal{P}_3$ and so on. There is an integer M with $M\varepsilon>B$. Consider the free tagged division $\mathcal{S}=\bigcup_{i=1}^{M}\mathcal{P}_i$. We can complete it to a δ-fine free tagged division π from the domain of sup in (4) for $\Phi_\delta^f(J_1)$, attaching to every complementary interval the tag x. Since all the complementary intervals are subsets of J_1, $|J_1|<\eta$, and η could have been chosen arbitrarily small at the start of the construction of $\mathcal{P}_i$'s, we may assume that $|\sigma(\mathcal{S},f)-\sigma(\pi,f)|<M\varepsilon-B$. We get

$$\sigma(\pi,f)>\sigma(\mathcal{S},f)-|\sigma(\mathcal{S},f)-\sigma(\pi,f)|>M\varepsilon-M\varepsilon+B=B,$$

which contradicts the definition of B. By this, continuity of Φ_δ^f is established.

In a similar way one proves that the function $\varphi_\delta^f\colon\mathcal{I}\to\mathbb{R}$ defined by

$$\varphi_\delta^f(J)=\inf_{\mathcal{P}}\sum_{(K,t)\in\mathcal{P}}f(t)|K|,$$

where inf is taken over all δ-fine free tagged divisions $\mathcal{P}$ in I such that the intervals from $\mathcal{P}$ form a partition of J, is a continuous McShane-minor function for f.

For any two δ-fine free tagged partitions π_1,π_2 of I we have $|\sigma(\pi_1,f)-\sigma(\pi_2,f)|<2\varepsilon$, whence $\Phi_\delta^f(I)-\varphi_\delta^f(I)\leqslant2\varepsilon$. This obviously implies the McShane-$\mathcal{A}^{\mathrm{c}}_{\mathrm{full}}$-Perron integrability of f. So we have proved

Theorem 5.2. *The McShane integral and the McShane-$\mathcal{A}^{\mathrm{c}}_{\mathrm{full}}$-Perron integral in $\mathbb{R}^n$ are equivalent.*

From Theorems 2.4 and 5.2 we also get

Corollary 5.3. *The McShane-Perron and the McShane-$\mathcal{A}^{\mathrm{c}}_{\mathrm{full}}$-Perron integral in $\mathbb{R}^n$ are equivalent.*

Remark 5.4. Notice that for a (one-dimensional) Kurzweil-Henstock integration, i.e., with $t \in K$ for any tagged interval (K,t), the definition of a *major function* (McShane-$\mathcal{B}_{\mathrm{full}}$-major function in notation of the present paper) analogous to (4), namely

$$\widetilde{\Phi}^f_\delta(J) = \sup_\pi \sum_{(K,t)\in\pi} f(t)|K|,$$

where sup is taken over all δ-fine tagged partitions π of I, does not suit the purpose. Since we are not allowed to pick t's outside of J (even not outside of K), the so defined $\widetilde{\Phi}^f_\delta$ can fail to be additive. Actually, put f on $[0,1]$ by $f=0$ on $[0,\frac12)$, $f=1$ on $[\frac12,1]$. For any gauge δ, for a $z \in (\frac12-\delta(\frac12),\frac12)$, one has $\widetilde{\Phi}^f_\delta([0,z])=0$, $\widetilde{\Phi}^f_\delta([z,1])=1-z$, while $\widetilde{\Phi}^f_\delta([0,1])=\frac12+\delta(\frac12)>1-z$. On the other hand, it is a standard matter to check that the function $\Psi\colon [a,b]\to\mathbb{R}$ defined by $\Psi([c,d])=\widetilde{\Phi}^f_\delta([a,d])-\widetilde{\Phi}^f_\delta([a,c])$ is a major function for f (it is additive); however (in the foregoing situation), it is not continuous at $\frac12$. The known constructions of continuous major/minor functions for a Kurzweil-Henstock integrand use differentiability and variational arguments; see for example [9].

5.1. Local systems' case. Consider a local system Δ with the partitioning property. As a particular case of Theorem 2.4 we have

Theorem 5.5. *The $\widetilde{\mathcal{B}}_\Delta M$-integral is equivalent to the McShane-$\widetilde{\mathcal{B}}_\Delta$-Perron integral.*

This statement has been proved in [7] in case of the density local system. A question is if the definition with the use of $\mathcal{B}_\Delta$-continuous McShane-$\widetilde{\mathcal{B}}_\Delta$-major/minor functions gives us a notion equivalent to the McShane-$\widetilde{\mathcal{B}}_\Delta$-Perron integral. Having left this question open we just point out that the technique of defining major/minor functions employed before, does not work here anymore.

Consider a $\widetilde{\mathcal{B}}_\Delta M$-integrable function $f\colon I\to\mathbb{R}$. Let $\mathcal{C}$ be a choice such that the inequality $|\sigma(\pi,f)-\int_I f|<\varepsilon$ is fulfilled for any $\mathcal{C}$-fine free tagged partition of I. Take $J\subset I$ and define

$$\Phi^f_{\mathcal{C}}(J) = \sup_{\mathcal{P}} \sum_{(K,t)\in\mathcal{P}} f(t)|K|, \tag{5}$$

where sup is taken over all $\mathcal{C}$-fine free tagged divisions $\mathcal{P}$ in I, such that the intervals from $\mathcal{P}$ form a partition of J. The value $\Phi^f_{\mathcal{C}}(J)$ is finite up to the choice of $\mathcal{C}$. As above, one checks that $M = \Phi^f_{\mathcal{C}}\colon \mathcal{I} \to \mathbb{R}$ satisfes the condition (2) (with $\mathcal{B} = \widetilde{\mathcal{B}}_\Delta$). The question is if it is additive. Unlike in the full local system case, usually the answer is *not*. Suppose that a local system Δ with the partitioning property has an $S \in \Delta(x)$ for some x, such that $(c,d) \cap S = \emptyset$ for some $c, d \in S$. Suppose that $x < c < d$ and define a function f on $[x,d]$ by $f(x) = 1$ and $f = 0$ elsewhere. Take the following choice $\mathcal{C}\colon$ $\mathcal{C}(x) = S$, $\mathcal{C}(t) = [x,d]$ at $t \in (x,d)$, $\mathcal{C}(d) = (x,\infty)$. Observe that $\Phi^f_{\mathcal{C}}([x, \frac12(c+d)]) = c - x$, $\Phi^f_{\mathcal{C}}([\frac12(c+d), d]) = 0$, $\Phi^f_{\mathcal{C}}([x,d]) = d - x > c - x$.

In both Definitions 2.3 and 5.1, one can change the meaning of a *McShane-$\mathcal{B}$-major/minor function* by replacing additivity with superadditivity (for McShane-$\mathcal{B}$-major) and subadditivity (for McShane-$\mathcal{B}$-minor function). For many bases it is known that this extension of the integral is not strict, but in general and even in some particular cases the problem of strictness is open.

The concluding example is related to the so changed definitions in the case of the McShane-$\widetilde{\mathcal{B}}_\Delta M$-Perron integral. Even if we allow McShane-$\widetilde{\mathcal{B}}_\Delta M$-major/minor functions not to be additive, only super-/sup-additive, the interval function $\Phi^f_{\mathcal{C}}$ need not be $\mathcal{B}_\Delta$-continuous. Let Δ be the local system induced by the dyadic path system $\{E_x\colon x \in \mathbb{R}\}$ [2]. Consider the function $f\colon [0,1] \to \mathbb{R}$ and the choice $\mathcal{C}$ defined as follows. For an $n \in \mathbb{N}$ put $a_n = \frac12 - \frac{1}{2^{n+1}}$ and pick a point $b_n < a_n$ such that $2^{n+1}(a_n - b_n) < \frac{1}{2^n}$. We may assume that $0 \leqslant b_1 < b_2 < b_3 < \ldots$. Put $f(b_n) = 2^{n+1}$, $f = 0$ elsewhere, and $\mathcal{C}(b_n) = E_{b_n} \cap [b_n - (a_n - b_n), \frac12]$, $\mathcal{C}$ equals anything elsewhere. Take any neighbourhood I of $\frac12$. Let a_l be the first element of the sequence that belongs to I. We have that $a_l, \frac12 \in \mathcal{C}(b_l)$, whence

$$\Delta\Phi^f_{\mathcal{C}}(I) \geqslant f(b_l)\Big(\frac12 - a_l\Big) = 1.$$

On the other hand, since for each n, $\mathcal{C}(b_n) \cap [a_n, \frac12] = \{a_n, \frac12\}$ and $a_n - \inf \mathcal{C}(b_n) \leqslant 2(a_n - b_n)$, for any $\mathcal{C}$-fine free tagged partition π of $[0,1]$, the value $\sigma(\pi, f)$ does not exceed

$$\sup_{N \geqslant 1} \left\{ \sum_{i=1}^{N} 2f(b_i)(a_i - b_i) + f(b_N)\Big(\frac12 - a_N\Big) \right\} = \sum_{i=1}^{\infty} 2f(b_i)(a_i - b_i) + 1 < 3.$$

So, $\Phi^f_{\mathcal{C}}\colon \mathcal{I} \to \mathbb{R}$ is properly defined but $\mathcal{B}_\Delta$-discontinuous.

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