MORSE-SARD THEOREM FOR DELTA-CONVEX CURVES

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Abstract. Let $f\colon I \to X$ be a delta-convex mapping, where $I \subset \mathbb{R}$ is an open interval and X a Banach space. Let C_f be the set of critical points of f. We prove that $f(C_f)$ has zero 1/2-dimensional Hausdorff measure.

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Let Z and X be Banach spaces, $U \subset Z$ an open convex set and $f\colon U \to X$ a mapping. We say that f is a *delta-convex mapping* (d. c. mapping) if there exists a continuous convex function h on U such that $y^* \circ f + h$ is a continuous convex function for each $y^* \in Y^*, \|y^*\| = 1$. We say that $f\colon U \to X$ is locally d. c. if for each $x \in U$ there exists an open convex U' such that $x \in U' \subset U$ and $f|_{U'}$ is d. c.

This notion of d. c. mappings between Banach spaces (see [7]) generalizes Hartman's [3] notion of d. c. mappings between Euclidean spaces. Note that in this case it is easy to see that F is d. c. if and only if all its components are d. c. (i.e., they are differences of two convex functions).

For $f\colon U \to X$ we denote $C_f := \{x \in U\colon\ f'(x) = 0\}$.

A special case of [2, Theorem 3.4.3] says that for a mapping $f\colon \mathbb{R}^m \to X$ of class C^2, where X is a normed vector space, the set $f(C_f)$ has zero $(m/2)$-dimensional Hausdorff measure.

We will generalize this result in the case $m = 1$ showing that it is sufficient to suppose that f is d. c. on I (equivalently: f is continuous and f'_+ is locally of bounded variation on I).

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A similar generalization of the above mentioned result on C^2 mappings holds for $m = 2$ as is shown (by a completely different method) in [6] where it is proved that $f(C_f)$ has zero 1-dimensional Hausdorff measure for any d. c. mapping $f\colon \mathbb{R}^2 \to X$.

Whether $f(C_f)$ has zero $(m/2)$-dimensional Hausdorff measure for each d. c. mapping $f\colon \mathbb{R}^m \to X$ for $m \geqslant 3$ remains open even for X Euclidean space.

We denote α-dimensional Hausdorff measure (on a metric space X) by $\mathcal{H}^\alpha$ and for each $Y \subset X$ we put (see [5])

$$\mathcal{H}^\alpha_\infty(Y) = \frac{\omega_\alpha}{2^\alpha} \cdot \inf\left\{ \sum_{i=1}^\infty \operatorname{diam}^\alpha(M_i)\colon\ Y \subset \bigcup_{i=1}^\infty M_i \right\},$$

where $\omega_\alpha = (\Gamma(1/2))^\alpha \cdot (\Gamma(\alpha/2+1))^{-1}$.

For an open interval I, a Banach space X, $g\colon I \to X$ and $x \in I$, we denote

$$\operatorname{md}(g,x) := \lim_{r\to 0} \frac{\|g(x+r) - g(x)\|}{|r|}.$$

If g is Lipschitz, then $\operatorname{md}(g,x)$ exists a.e. on I. This fact is a special case of Kirchheim's theorem [4, Theorem 2] on a.e. metric differentiability of Lipschitz mappings (from $\mathbb{R}^n$ to X). In a standard way we obtain the following more general fact.

Lemma 1. *Let I be an open interval, X a Banach space, and let $g\colon I \to X$ have bounded variation on I. Then $\operatorname{md}(g,x)$ exists almost everywhere on I.*

Proof. We may suppose $I = \mathbb{R}$. Denote $s(x) = \bigvee\limits_0^x g$, $x \in \mathbb{R}$. By [2, 2.5.16.] there exists a Lipschitz mapping $H\colon \mathbb{R} \to X$ such that $g = H \circ s$. By [4, Theorem 2], $\operatorname{md}(H,x)$ exists a.e. on $\mathbb{R}$. Now, changing in the obvious way the last argument of [2, 2.9.22.], we obtain our assertion. $\square$

Theorem 2. *Let X be a Banach space, $I \subset \mathbb{R}$ an open interval and $f\colon I \to X$ a locally d. c. mapping. Let $C := \{x \in I\colon\ f'(x) = 0\}$. Then $\mathcal{H}^{1/2}(f(C)) = 0$.*

Proof. Note that f is continuous on I (see [7, Proposition 1.10.]). By [7, Theorem 2.3], f'_+ exists and has locally bounded variation on I. Consider an arbitrary interval $[a,b] \subset I$. It is clearly sufficient to prove $\mathcal{H}^{1/2}(f(C_1)) = 0$, where $C_1 := C \cap (a,b)$.

Let N_1 be the set of all isolated points of C_1 and $N_2 := \{x \in C_1\colon\ \operatorname{md}(f'_+,x)$ does not exist$\}$. Since N_1 is countable, $\mathcal{H}^{1/2}(f(N_1)) = 0$.

To prove $\mathcal{H}^{1/2}(f(N_2)) = 0$, consider an arbitrary $\varepsilon > 0$. By Lemma 1, we find a countable disjoint system of open intervals $\{(a_i,b_i)\colon\ i \in J\}$ such that

$$N_2 \subset \bigcup_{i\in J}(a_i,b_i) \subset (a,b), \quad \sum_{i\in J}(b_i - a_i) < \varepsilon \ \text{ and } \ (a_i,b_i)\cap N_2 \neq \emptyset,\ i \in J.$$

Clearly $\|f'_+(x)\| \leqslant \bigvee_{a_i}^{b_i} f'_+$ for each $i \in J$ and $x \in (a_i, b_i)$. Using the continuity of f and [1, Chap. I, par. 2, Proposition 3], we obtain

$$\operatorname{diam}(f((a_i, b_i))) \leqslant (b_i - a_i) \cdot \bigvee_{a_i}^{b_i} f'_+.$$

Therefore, using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
\mathcal{H}^{1/2}_\infty(f(N_2)) &\leqslant \frac{\omega_{1/2}}{2^{1/2}} \sum_{i \in J} \Bigl((b_i - a_i) \cdot \bigvee_{a_i}^{b_i} f'_+\Bigr)^{1/2} \\
&\leqslant \frac{\omega_{1/2}}{2^{1/2}} \Bigl(\sum_{i \in J} (b_i - a_i)\Bigr)^{1/2} \Bigl(\sum_{i \in J} \bigvee_{a_i}^{b_i} f'_+\Bigr)^{1/2} \leqslant \frac{\omega_{1/2}}{2^{1/2}} \varepsilon^{1/2} \Bigl(\bigvee_{a}^{b} f'_+\Bigr)^{1/2}.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\mathcal{H}^{1/2}_\infty(f(N_2)) = 0$; consequently (see [5, Lemma 4.6.]) we obtain $\mathcal{H}^{1/2}(f(N_2)) = 0$.

To complete the proof, it is sufficient to prove $\mathcal{H}^{1/2}(f(C_2)) = 0$, where $C_2 = C_1 \setminus (N_1 \cup N_2)$. Let $\varepsilon > 0$ be arbitrary. Clearly $\operatorname{md}(f'_+, x) = 0$ for each $x \in C_2$. Therefore, for each $x \in C_2$ we can choose $\delta_x > 0$ such that $[x - \delta_x, x + \delta_x] \subset (a, b)$ and $\|f'_+(y)\| \leqslant \varepsilon\, |y - x|$ for each $y \in [x - \delta_x, x + \delta_x]$. Using the continuity of f and [1, Chap. I, par. 2, Proposition 3], we obtain $\operatorname{diam}(f([x - \delta_x, x + \delta_x])) \leqslant 2\varepsilon(\delta_x)^2$.

Besicovitch's Covering Theorem (see [5]) easily implies that we can choose a countable set $A \subset C_2$ such that

$$C_2 \subset \bigcup_{x \in A} [x - \delta_x, x + \delta_x] \quad \text{and} \quad \sum_{x \in A} 2\delta_x \leqslant c(b - a),$$

where c is an absolute constant (not depending on ε). Since $\varepsilon > 0$ is arbitrary,

$$f(C_2) \subset \bigcup_{x \in A} f([x - \delta_x, x + \delta_x]),$$

and

$$\sum_{x \in A} (\operatorname{diam}(f([x - \delta_x, x + \delta_x])))^{1/2} \leqslant \sum_{x \in A} \sqrt{2\varepsilon}\, \delta_x \leqslant \sqrt{2\varepsilon}\, c(b - a),$$

we have $\mathcal{H}^{1/2}_\infty(f(C_2)) = 0$, hence $\mathcal{H}^{1/2}(f(C_2)) = 0$. $\square$

Remark 3. Since each C^2-function on I is a locally d.c. function (see [7]), [2, 3.4.4.] implies that the conclusion of Theorem 2 does not hold with $\mathcal{H}^\alpha$ $(\alpha < 1/2)$ in general.

References

[1] *N. Bourbaki*: Éléments de mathématique IX., Les structures fondamentales de l'analyse. Livre IV: Fonctions d'une variable réelle (théorie élémentaire). Act. Sci. et Ind. vol. 1074, Hermann, Paris, 1968.

[2] *H. Federer*: Geometric Measure Theory. Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer, New York, 1969.

[3] *P. Hartman*: On functions representable as a difference of convex functions. Pacific J. Math. *9* (1959), 707–713.

[4] *B. Kirchheim*: Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. Proc. Amer. Math. Soc. *121* (1994), 113–123.

[5] *P. Mattila*: Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability. Cambridge Studies in Advanced Mathematics, 44, Cambridge University Press, Cambridge, 1995.

[6] *D. Pavlica, L. Zajíček*: Morse-Sard theorem for d. c. functions and mappings on $\mathbb{R}^2$. Indiana Univ. Math. J. *55* (2006), 1195–1207.

[7] *L. Veselý, L. Zajíček*: Delta-convex mappings between Banach spaces and applications. Dissertationes Math. (Rozprawy Mat.) *289* (1989).

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