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DIAMETER-INVARIANT GRAPHS

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Abstract. The diameter of a graph G is the maximal distance between two vertices of G. A graph G is said to be diameter-edge-invariant, if $d(G-e) = d(G)$ for all its edges, diameter-vertex-invariant, if $d(G-v) = d(G)$ for all its vertices and diameter-adding-invariant if $d(G+e) = d(e)$ for all edges of the complement of the edge set of G. This paper describes some properties of such graphs and gives several existence results and bounds for parameters of diameter-invariant graphs.

Keywords: extremal graphs, diameter of graph

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1. Introduction

Let G be an undirected, finite graph without loops or multiple edges. Then we denote by: $V(G)$ the vertex set of G; $E(G)$ the edge set of G; $\overline{G}$ the complement of G with the edge set $E(\overline{G})$; $d_G(u,v)$ (or simply $d(u,v)$) the distance between two vertices u,v in G; $e(u)$ the eccentricity of u. The radius $r(G)$ is the minimum of the vertex eccentricities, the diameter $d(G)$ is the maximum of the vertex eccentricities; $\deg_G(v)$ is the degree of vertex v in G and $\Delta(G)$ is the maximum degree of G. The notions and notations not defined here are used accordingly to the book [2].

Harary [9] introduced the concept of changing and unchanging of a graphical invariant i, asking for characterization of graphs G for which $i(G-v), i(G-e)$ or $i(G+e)$ either differ from $i(G)$ or are equal to $i(G)$ for all $v \in V(G), e \in E(G)$ or $e \in E(\overline{G})$ respectively. Some of the most important invariants (for example in communications) are the radius and the diameter of a graph.

Even earlier, in late sixties A. Kotzig initiated the study of graphs for which $d(G-e) > d(G)$ for all $e \in E(G)$. These graphs are called *diameter-minimal*, for

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example see the papers of Glivjak, Kyš and Plesník [6], [7], [12]. Later on S. M. Lee [10], [11] initiated the study of graphs for which $d(G-e)=d(G)$ for all $e\in E(G)$ and he called them diameter-edge-invariant.

From the practical point of view we need to study the stability of the radius and the diameter of a graph G, especially when an arbitrary edge or vertex is removed from G. This operation can represent a single failure of communication line or any communication center (processor, etc.). The papers [1], [3], [5], [13] examine several properties of graphs in which radii do not change under these two conditions, and moreover, when an arbitrary edge is added to the graph G. These graphs are defined as follows:

Definition 1.1. A graph G is:

(1) *radius-edge-invariant* (r.e.i.) if $r(G-e)=r(G)$ for every $e\in E(G)$;
(2) *radius-vertex-invariant* (r.v.i.) if $r(G-v)=r(G)$ for every $v\in V(G)$;
(3) *radius-adding-invariant* (r.a.i.) if $r(G+e)=r(G)$ for every $e\in E(\overline{G})$.

According to this definition and to the previous study of diameter-edge-invariant graphs [10], [11], [13] we can define the following classes of graphs:

Definition 1.2. A graph G is:

(1) *diameter-edge-invariant* (d.e.i.) if $d(G-e)=d(G)$ for every $e\in E(G)$;
(2) *diameter-vertex-invariant* (d.v.i) if $d(G-v)=d(G)$ for every $v\in V(G)$;
(3) *diameter-adding-invariant* (d.a.i.) if $d(G+e)=d(G)$ for every $e\in E(\overline{G})$.

Following this definition, in the beginning of Section 2 we will prepare some auxiliary results concerning operations on diameter-invariant graphs. Then, using them we will construct several d.e.i., d.v.i. and d.a.i. graphs. We will also characterize the d.v.i. and d.a.i. graphs of diameter 2. In Section 3 we will try to find some bounds for diameter-invariant-graphs.

2. Existence results

We first give some preliminary results about operations on graphs.

Recall that the join of graphs G and H is denoted $G+H$ and consists of $G\cup H$ and all edges of the form u_iv_j where $u_i\in G$, $v_j\in H$. It is obvious that $d(G+H)=1$ if G and H are complete graphs and $d(G+H)=2$ otherwise. Also $\deg_{G+H}(v)=\deg_G(v)+|V(H)|$ for all $v\in V(G)$ and $\deg_{G+H}(u)=\deg_H(u)+|V(G)|$ for all $u\in V(H)$. Lee [10] gave several results for d.e.i. graphs.

Theorem 2.1. *The join of graphs G, H is diameter-vertex-invariant*

(1) *of diameter* 1 *if and only if* $G = K_n, H = K_m$, $m \cdot n \neq 1$,

(2) *of diameter* 2 *if and only if there are at least two edges in* $E(\overline{G}) \cup E(\overline{H})$ *not joined to the same vertex and*

  a) $G = K_1$ *(or* $H = K_1$*) and* $d(H) = 2$ $(d(G) = 2)$, *or*

  b) $|V(G)| > 1$ *and* $|V(H)| > 1$.

P r o o f. (1) The first case is obvious, as every complete graph is d.v.i., except K_1 and K_2. $G + H$ is a complete graph if and only if G is a complete graph and H is a complete graph.

(2) If $d(G+H) = 2$ and all edges in $E(\overline{G}) \cup E(\overline{H})$ are connected to a single vertex v then $d(G + H - v) = 1$, a contradiction.

a) Now let $G = K_1 = \{v\}$. Then $d(G+H-v) = d(G+H)$ if and only if $d(H) = 2$. For all vertices $u \in V(H)$ we have $d(G + H - u) = 2$, as there exists at least one edge $ab \in E(\overline{H - u})$ and $d(G + H - u) \leqslant 2r(G + H - u) \leqslant 2e(v) = 2$.

b) Let G and H have both at least 2 vertices. Consider $v \in V(G+H)$ and a graph $G + H - v$. For all $u, w \in V(G + H - v)$ we have $d(u, w) = 1$ if $u \in G$, $w \in H$ and $d(u, v) \leqslant 2$ if $u, w \in H$ (or $u, w \in G$). The fact that $E(\overline{G + H - v}) \geqslant 1$ implies that $d(G + H - v) = 2$. □

The next observation is obvious.

Theorem 2.2. *The join of graphs G, H is diameter-adding-invariant of radius* 2 *if and only if* $|E(\overline{G})| + |E(\overline{H})| \geqslant 2$.

Consider a finite connected graph I. Let $\{G_i\colon\ i \in V(I)\}$ be a class of graphs indexed by a finite set $V(I)$.

The Sabidussi sum $S^+(\{G_i\colon\ i \in V(I)\})$ (or shortly S^+) of $\{G_i\colon\ i \in V(I)\}$ is a graph defined as follows:

$$\begin{aligned} V(S^+(\{G_i\colon\ i \in V(I)\})) &= \bigcup\{V(G_i)\colon\ i \in V(I)\},\ E(S^+(\{G_i\colon\ i \in V(I)\})) \\ &= \bigcup\{E(G_i)\colon\ i \in V(I)\} \cup \{xy\colon\ x \in V(G_i), y \in V(G_j), ij \in E(I)\}. \end{aligned}$$

Sabidussi sum is sometimes called *X-join*. One can show that $d(S^+(\bigcup\{G_i\colon\ i \in V(I)\})) = d(I)$.

Lee [11] gives the following theorem.

Theorem 2.3. *Let I be a graph of diameter $d \geqslant 2$. For any class of connected graphs $\{G_i\colon i \in V(I)\}$ with $|V(G_i)| \geqslant 2$ for all i, the Sabidussi sum $S^+(\{G_i\colon i \in V(I)\})$ is diameter-edge-invariant with diameter d. Moreover, if I is diameter-edge-invariant then $S^+(\{G_i\colon i \in V(I)\})$ is diameter-edge-invariant without the restriction of $|V(G_i)| \geqslant 2$.*

However, the assumption that G_i be connected is unnecessary for $d \geqslant 3$.

Theorem 2.4. *Let I be a graph of diameter $d \geqslant 3$. For any class of graphs $\{G_i\colon i \in V(I)\}$ with $|V(G_i)| \geqslant 2$ for all i, the Sabidussi sum $S^+(\{G_i\colon i \in V(I)\})$ is diameter-edge-invariant with diameter d.*

Proof. It is sufficient to show that in any $S^+ - e$ there are no vertices u, v at distance greater than $d \geqslant 3$. If u, v are from the same graph G_i or if $u \in V(G_i), v \in V(G_j), d(i,j) > 1$, then there are at least 2 edge-disjoint paths of length at most d joining u and v. Therefore $d_{S^+-e}(u,v) \leqslant d$ for all $e \in E(S^+)$.

Let $u \in V(G_i), v \in V(G_j)$ be two vertices such that $ij \in E(I)$ and suppose that there is no other path of length at most d joining u, v. Since $d(I) > 2$, we have at least one vertex $w \in I$ adjacent to i (or j), some other vertex $a \in V(G_i)$ (or $a \in V(G_j)$) and some vertex $b \in V(G_w)$. But then we have at least two edge-disjoint paths of length at most three joining u and v—the edge uv and the path u-a-b-v. Therefore $d_{S^+-e}(u,v) \leqslant 3 \leqslant d$ for all $e \in E(S^+)$. □

We can prove similar result for d.v.i. graphs:

Theorem 2.5. *Let I be a graph of diameter $d \geqslant 2$. For any class of graphs $\{G_i\colon i \in V(I)\}$ with $|V(G_i)| \geqslant 2$ for all i, the Sabidussi sum $S^+(\{G_i\colon i \in V(I)\})$ is diameter-vertex-invariant with diameter d. Moreover, if I is diameter-vertex-invariant then $S^+(\{G_i\colon i \in V(I)\})$ is diameter-vertex-invariant without the restriction of $|V(G_i)| \geqslant 2$.*

Proof. If $|V(G_i)| \geqslant 2$ then for any two vertices u, v at distance $d(u,v) \geqslant 2$, there are at least two vertex-disjoint paths of length $d(u,v)$. Therefore $d_{S^+-w}(u,v) \leqslant d$ for all $w \neq u, v$. Let i, j be two vertices of graph I such that $d(i,j) = d(I)$. As $V(G_i) \geqslant 2$ and $V(G_j) \geqslant 2$, for all $w \in V(S^+)$ there are at least two vertices at distance d in $S^+ - w$. Finally, $d(S^+ - w) = d(S^+)$ and S^+ is d.v.i. The second part of the result is obvious. □

Theorem 2.6. *Let I be a diameter-adding-invariant graph of diameter $d \geqslant 2$. For any class of graphs $\{G_i \colon i \in V(I)\}$, the Sabidussi sum $S^+(\{G_i \colon i \in V(I)\})$ is diameter-adding-invariant with diameter d.*

Proof. We will prove this theorem by contradiction. Let S^+ be not a d.a.i. graph. It is clear that for all vertices $a, b \in G_k$ there is $d(S^+ + ab) = d(S^+) = d(I)$. Thus we have two vertices $v \in G_i, u \in G_j$ such that $d(S^+ + uv) < d(S^+) = d(I)$. But then $d(I + ij) \leqslant d(S^+ + uv) < d(S^+) = d(I)$, a contradiction. □

The corona $G \circ H$ of graphs G and H was defined by Frucht and Harary ([4], see also [2]) as the graph obtained by taking one copy of G of order p_G and p_G copies of H, and then joining the i'th vertex of G to every vertex in the i'th copy of H. If the i'th vertex is named v, then the copy belonging to v will be named H_v.

It is clear that if $p_G > 1$, $r(G) = r_G$, $d(G) = d_G$, then $r(G \circ H) = r_G + 1$, $d(G \circ H) = d_G + 2$ and v is a central vertex of $G \circ H$ if and only if v is a central vertex of G. Moreover, $h \in H_v$ is a peripheral vertex of $G \circ H$ if and only if v is a peripheral vertex in G. Since $d(G \circ H - v) = \infty$ for $v \in G$ and $e_{G \circ H - hv}(h) > d(G \circ H)$ for the peripheral vertex v of the graph G and $h \in H_v$, the corona of two graphs will never be d.e.i. or d.v.i.

The paper [1] gives the following theorem:

Theorem 2.7. *For any graphs G, H, such that $|V(G)| \geqslant 3$, the corona $G \circ H$ is radius-adding-invariant if and only if G is radius-adding-invariant.*

For the diameter of $G \circ H$ the following theorem holds:

Theorem 2.8. *For any graphs G, H, such that $|V(G)| \geqslant 3, H \neq K_1$ the corona $G \circ H$ is diameter-adding-invariant if and only if G is diameter-adding-invariant.*

Proof. ($\Longrightarrow$) Suppose that $G \circ H$ is d.a.i., but G is not d.a.i. Let $e \in E(\overline{G})$ be an edge such that $d(G + e) < d(G)$. Therefore

$$d(G \circ H + e) = d((G + e) \circ H) = d(G + e) + 2 < d(G) + 2 = d(G \circ H),$$

a contradiction.

($\Longleftarrow$) We consider various possibilities for an edge $e \in E(\overline{G \circ H})$.

(1) If $e \in E(\overline{G})$, then

$$d(G \circ H + e) = d(G + e) + 2 = d(G) + 2 = d(G \circ H).$$

(2) If $e \in E(\overline{H_v})$ for any $v \in V(G)$, then for all $w \in V(G \circ H)$ we have $e_{G \circ H}(w) = e_{G \circ H + e}(w)$ and thus $d(G \circ H) = d(G \circ H + e)$.

(3) Suppose $e = uh_v$ where $u \in V(G)$, $h_v \in H_v$, $v \neq u$. Let $d(G \circ H + e) < d(G \circ H)$. If x and y are two peripheral vertices of $G \circ H$ such that $d(x, y) = d(G \circ H)$, then the x-y geodesic in $G \circ H + e$ must contain e. Moreover, if $x \notin H_v$ and $y \notin H_v$ then u-h_v-v is a part of the x-y geodesic in $G \circ H + e$. But then for all such pairs $d_{G \circ H + uv}(x, y) < d(G \circ H)$.

On the other hand let, for example, $x \in H_v$. It is clear that for all $z \in H_v$, $z \neq x$ we have $d_{G \circ H + e}(y, z) \geqslant d_{G \circ H + e}(y, x) + 1$. But then again $d_{G \circ H + uv}(x, y) < d(G \circ H)$ and $d_{G \circ H + uv}(x, h_v) < d(G \circ H)$. This leads to the case (1) which was discussed above.

(4) Finally, suppose $e = h_u h_v$ where $u, v \in V(G)$, $h_u \in H_u$, $h_v \in H_v$, $v \neq u$. Let $d(G \circ H + e) < d(G \circ H)$. It is obvious that for all $h'_u \in H_u$, $h'_v \in H_v$, $h'_u \neq h_u$, $h'_v \neq h_v$ we have $e_{G \circ H + e}(h'_u) \geqslant e_{G \circ H + e}(h_u)$ and $e_{G \circ H + e}(h'_v) \geqslant e_{G \circ H + e}(h_v)$. Thus if x and y are two peripheral vertices of $G \circ H$ different from h_u, h_v such that $d(x, y) = d(G \circ H)$, then the x-y geodesic in $G \circ H + e$ must contain e. Moreover, the x-y geodesic must contain a subpath of length three of the form u-h_u-h_v-v, h''_u-h_u-h_v-v or h''_u-h_u-h_v-h''_v.

Consider the graph $G \circ H + uv$. To obtain an x-y path of length less than $d(G \circ H)$ it is sufficient to take u-v instead of u-h_u-h_v-v, h''_u-u-v instead of h''_u-h_u-h_v-v or h''_u-u-v-h''_v instead of h''_u-h_u-h_v-h''_v in the x-y geodesic formed in $G \circ H + h_u h_v$. Thus $d_{G \circ H + h_u h_v}(x, y) \geqslant d_{G \circ H + uv}(x, y)$ and since $d_{G \circ H + uv}(h'_u, h'_v) = d_{G \circ H + uv}(h_u, h_v) = d_{G \circ H + uv}(h_u, h'_v) = d_{G \circ H + uv}(h'_u, h_v)$ we have $d_{G \circ H + uv}(a, b) < d(G \circ H)$ for all $a, b \in V(G \circ H)$. Therefore $d(G \circ H + uv) < d(G \circ H)$. But this is the case (1) which was discussed above. □

If $H = K_1$ and G is d.a.i. having $|V(G)| \geqslant 3$ then $G \circ H$ is not necessarily d.a.i.: Consider the group $\mathbb{Z}_{2r+1}$ and define a graph $G_{\mathbb{Z}_{2r+1}}$ in the following way:

$$V(G) = \{(i, j);\ i, j \in \mathbb{Z}_{2r+1}\},$$
$$(i_1, j_1)(i_2, j_2) \in E(G) \iff |i_1 - i_2| \leqslant 1 \wedge |j_1 - j_2| \leqslant 1.$$

If (i_1, j_1) and (i_2, j_2) are two vertices of $G_{\mathbb{Z}_{2r+1}}$, then $d((i_1, j_1), (i_2, j_2)) = \max\{\min\{|i_1 - i_2|, 2r + 1 - |i_1 - i_2|\}, \min\{|j_1 - j_2|, 2r + 1 - |j_1 - j_2|\}\} \leqslant r$. Since for each vertex $u = (i, j)$, there are $8r$ vertices $u_k = (i_k, j_k), i_k = i + r \mod(2r+1) \vee i_k = i + r + 1 \mod(2r + 1) \vee j_k = j + r \mod(2r + 1) \vee j_k = j + r + 1 \mod(2r + 1)$ such that $d(u, u_k) = r$, the graph $G_{\mathbb{Z}_{2r+1}}$ is self-centered of radius r.

Now, consider a graph G' obtained in the following way: Suppose $V(G') = V(G_{\mathbb{Z}_{2r+1}}) + v$, $E(G') = E(G_{\mathbb{Z}_{2r+1}}) + uv$ where $u = (i, j) \in V(G_{\mathbb{Z}_{2r+1}})$. We have $e_{G'}(v) = d(G_{\mathbb{Z}_{2r+1}}) + 1 = d(G')$. Let $f \in E(\overline{G'})$ be an arbitrary edge. If $f \in E(\overline{G_{\mathbb{Z}_{2r+1}}})$, then $e_{G_{\mathbb{Z}_{2r+1}}}(w) = e_{G_{\mathbb{Z}_{2r+1}} + f}(w)$ for all $w \in V(G_{\mathbb{Z}_{2r+1}})$ and thus $d(G') = e_{G'}(v) = e_{G' + f}(v) = d(G' + f)$. If $f \notin E(G_{\mathbb{Z}_{2r+1}})$, then f is of type $v(i', j')$ where $i \neq i'$, or $j \neq j'$. It is sufficient to take the vertex $a = (i + r \mod(2r + 1), j' +$

$r \bmod (2r+1)$) to obtain a vertex such that $d(a,v) = d(G_{\mathbb{Z}_{2r+1}}) + 1 = d(G')$. Thus G' is d.a.i.

Now consider a graph $G' \circ K_1$. Let $H_v = \{b\}$ be a copy of K_1 belonging to $v \in G'$. One can show that $d(G' \circ K_1 + bu) = d(G') + 1 < d(G' \circ K_1)$. Thus $G' \circ K_1$ is not d.a.i.

Consider the two following graphs I_1, I_2:

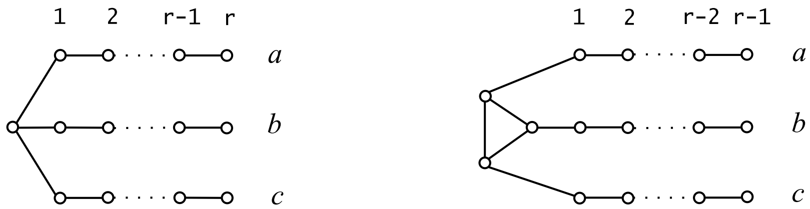

Figure 1

In the first case $d = 2r$, in the second $d = 2r - 1$. Since in both graphs there are three pairs of vertices $\{a,b\}$, $\{b,c\}$, $\{c,a\}$ at distance d, and adding a single edge may change at most two of these distances, both graphs are d.a.i. of diameter d for all $r \geqslant 1$.

Lee [10] showed, as a consequence of Theorem 2.3, that any connected graph is an induced subgraph of a d.e.i. graph of diameter $d \geqslant 2$. Walikar et. al. [3] showed that for every graph G, the graph H formed as $K_2 + G + K_2$ is d.e.i. As a consequence they got that every graph could be embedded in a d.e.i. graph. Later in this section we will show that for each graph G, there is an d.e.i., d.v.i. and d.a.i. graph H of diameter d having G as an induced subgraph.

Lemma 2.9. *Let G be a graph with at least two vertices. Then the graph $H = K_2 + G + K_2$ is diameter-vertex-invariant and diameter-adding-invariant of diameter 2.*

P r o o f. One can show that $d(H) = 2$. As $|E(\overline{H})| > 1$, it is clear that H is d.a.i. We can write $H = (K_2 + G) + K_2$. Thus by Theorem 2.1 H is d.v.i. $\square$

Theorem 2.10. *Every graph G can be embedded as an induced subgraph in a diameter-edge-invariant, diameter-vertex-invariant and diameter-adding-invariant graph H of diameter $d \geqslant 2$.*

P r o o f. Suppose G has at least two vertices. It is sufficient to take the graph $K_2 + G + K_2$ for $d = 2$ and the Sabidussi sum $S^+(\{G_i \equiv G\colon\ i \in V(I)\})$ where I is a graph I_1 if $d = 2k$ or I_2 if $d = 2k+1$. It follows from the results of the previous section that S^+ is d.e.i., d.v.i. and d.a.i.

If $G = K_1$ then it is a subgraph of any graph, and as for each d there exists d.e.i., d.v.i. and d.a.i. graph H, the theorem holds. □

Because of the previous theorem, we cannot obtain a forbidden subgraph characterization for all d.e.i., d.v.i., and d.a.i. graphs.

Bálint and Vacek in [1] constructed several r.e.i., r.v.i. and r.a.i. graphs. We will now show that there are graphs which radius and diameter are both invariant.

Theorem 2.11. *Let r, d be natural numbers such that $2 \leqslant r < d \leqslant 2r$. Let G be a graph with at least two vertices. Then there exists a radius-edge-invariant, diameter-edge-invariant, radius-vertex-invariant and diameter-vertex-invariant graph H such that $r(H) = r$, $d(H) = d$, $C(H) = V(G)$ and G is an induced subgraph of H.*

[1] gives a somewhat weaker result with similar graph construction for radius-invariant graphs only.

P r o o f. For $d \neq 2r - 1$ consider the following graph Q:

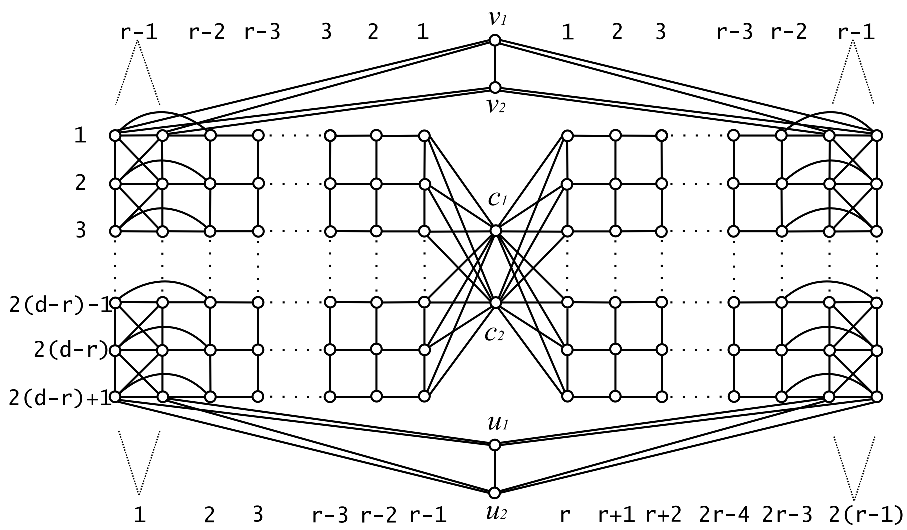

Figure 2

Q is formed by 2 central vertices c_1, c_2; by $2(d-1)+1$ rows of vertices in $2(r-1)$ columns and by 4 additional vertices v_1, v_2, u_1, u_2. Every column except 1 and $2(r-1)$ (counted from the left side) has $2(d-r)+1$ vertices. Columns 1 and $2(r-1)$ have $2(2(d-r)+1)$ vertices. Vertices c_1, c_2 are adjacent to all vertices in columns $r-1$ and r. Vertices v_1, v_2 (u_1, u_2) are adjacent and joined to all vertices in row 1 $(2(d-r)+1)$ and columns 1 and $2(r-1)$. Vertex in row k and column l is adjacent to all vertices in row k and columns $l-1$, $l+1$ and to all vertices in column l and rows $k-1$, $k+1$ except the case when $l=r-1$ or $l=r$.

It is clear that $e(c_1)=e(c_2)=r$, $e(v)>r$ otherwise, and $d(u_i,v_j)=\min\{d(v_i,c_1)+d(c_1,u_j),2(d-r)+2\}$. Since $d\neq 2r-1$ we have $2(d-r)+2\leqslant d$ or $2r\leqslant d$ and thus $d(u_i,v_j)\leqslant d$. For any other vertex x, $x\neq c_i$, $x\neq u_i$ (or v_i) we have $d(x,v_i)\leqslant\min\{2(d-r)+1,2r-2\}\leqslant d$. Now, let y, z be arbitrary vertices except u_i, v_i, c_i. When y, z belong to the same row and the same half (right or left) of Q we obviously have $d(y,z)<r<d$. Consider a shortest cycle F such that $y,z\in F$. The length of the cycle F can be at most $2+2(d-r)+2(r-1)=2d$ if it is made as a sequence of $y-c_1$, c_1-z, $z-u_i$ (or $z-v_i$), u_i-y (or v_i-y) geodesics or less otherwise. This implies $d(x,y)\leqslant d$. Thus for all $w\in V(Q)$ we have $e(w)\leqslant d$.

To obtain vertices o, p such that $d(o,p)=d$ it is sufficient to take the vertex o in row 1 and column 1 and the vertex p in row $2(d-r)+1$ and column $d+1$. This implies that $r(Q)=r$ and $d(Q)=d$. Note: There are more than one pair of such vertices.

Since for every vertex a, $a\neq c_i$ there are at least two edge and vertex-disjoint c_1-a (or c_2-a) paths, and, in addition there are four vertices in the graph Q at distance r from c_1, c_2, we have $r(Q-e)=r(Q-b)=r$ for all $e\in E(Q)$, $b\in V(Q)$, Q is r.e.i. and r.v.i.

Next, we will show that Q is also d.e.i. and d.v.i. We have already proved that $e_{Q-e}(c_i)=e_{Q-b}(c_i)=r$. Consider the eccentricity of the vertices v_i (u_j). Let s be any vertex except $v_i(u_j)$ and suppose s does not belong to row 1 (or $2(d-r)+1$). Thus there are at least two edge and vertex-disjoint u_i-s geodesics. It is clear that $d_{Q-u_1u_2}(u_1,u_2)=2$ and for all vertices t in row 1 we have $d(u_i,t)\leqslant(r-1)+2\leqslant d$. Thus for all $e\in E(Q)$, $b\in V(Q)$ we have $e_{Q-e}(u_i)\leqslant d$ and $e_{Q-b}(u_i)\leqslant d$.

Now let y, z be arbitrary vertices except u_i, v_i, c_i. One can show that if vertices y, z do not lie in the same row and the same half of the graph Q, then the length of at most one of the y-c_1, c_1-z, z-u_i (z-v_i), u_i-y (v_i-y) geodesics is different in Q and in $Q-e$ $(Q-b)$. It follows directly from the construction of Q that the difference in lengths of these paths can be at most 1. Consider a shortest cycle F' such that $y,z\in F'$. The length of the cycle F' can be at most $2+2(d-r)+2(r-1)+1=2d+1$ if it is made as a sequence of y-c_1, c_1-z, z-u_i (or z-v_i), u_i-y (or v_i-y) geodesics in $Q-e$ $(Q-b)$. Thus $d_{Q-e}(y,z)\leqslant d$ and $d_{Q-b}(y,z)\leqslant d$.

We can obtain vertices $o,p\in V(Q-b)$ such that $d(o,p)=d$ in the same way as in Q. Finally, for $d\neq 2r-1$ the graph Q is r.e.i., r.v.i., d.e.i. and d.v.i. of radius r and diameter d.

For $d=2r-1$ it is sufficient to take only $d-1$ rows of vertices. It is clear that $d(u_i,v_j)=d$. All other facts could be proved similarly as above and we leave the details to the reader.

The desired graph H is obtained from the graph Q by substituting the graph G instead of the vertices c_1, c_2. □

Theorem 2.12. *Let r, d be natural numbers such that $r \leqslant d \leqslant 2r$. Then there exists a radius-adding-invariant and diameter-adding-invariant graph G such that $r(G) = r$ and $d(G) = d$.*

P r o o f. It is sufficient to take the tree I_1 if $d = 2r$ and the following tree for $d = 2r - 1$.

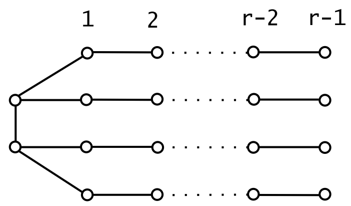

Figure 3

Otherwise the desired graph can be constructed as follows: Denote $G_0 = G_{\mathbb{Z}_{2k+1}}$ where $k = 2r - d \geqslant 2$. From [1] we have that G_0 is r.a.i. Since G_0 is self-centered and $r(G_0 + e) \leqslant d(G_0 + e) \leqslant d(G_0) = r(G_0)$ it is also d.a.i.

We will construct a graph G_{i+1} from the graph G_i as $G_{i+1} = G_i \circ H$, $H \neq K_1$. From Theorem 2.7 and from Theorem 2.8 it follows directly that every graph G_i is r.a.i. and d.a.i. For $i = d - r$ we have an r.a.i. and d.a.i. graph G_{d-r} such that $r(G_{d-r}) = i \cdot 1 + r(G_0) = (d - r) + (2r - d) = r$ and $d(G_i) = i \cdot 2 + d(G_0) = 2(d - r) + (2r - d) = d$. □

Walikar, Buckley and Itagi [13] showed that any graph G of diameter 2 is d.e.i. if and only if every edge of G is contained in a triangle and if there are at least two geodesics for all vertices v, w at distance 2. As we have already stated, a graph G of diameter $d = 2$ is d.a.i. if and only if $E(\overline{G}) \geqslant 2$. For d.v.i. graphs we have the following result.

Theorem 2.13. *Suppose that a graph G has diameter 2. Then G is diameter-vertex-invariant if and only if*

(1) *for all $u, v \in V(G)$ such that $d(u, v) = 2$ there are at least two u-v geodesics,*
(2) *there are at least two edges $a_1a_2, b_1b_2 \in E(\overline{G})$ not incident with the same vertex.*

P r o o f. $(\Longrightarrow)$
(1) Suppose there is only one such geodesic u-x-v. Then $d_{G-x}(u, v) \geqslant 3$, a contradiction.

(2) Let all edges in $E(\overline{G})$ have one joint incident vertex v. Then $G-v$ is a complete graph. Therefore $d(G-v)=1$ which is again a contradiction.

($\Longleftarrow$) Consider an arbitrary vertex $w \in V(G)$ and the graph $G-w$. From (2) it follows that we have $E(\overline{G-w}) \geqslant 1$, and thus $d(G-w) > 1$. For any two vertices $u, v \in V(G-w)$ there is $d_G(u,v) \leqslant 2$. If $d_G(u,v)=2$, then from (1) it follows that there must be some path u-a-v in $G-w$. Therefore $d(u,v)=2$. $\square$

3. Some bounds

A k-depth spanning tree (k-DST) of a graph G is a spanning tree of G of height k. It must be true that $k \leqslant d$, and if $k=d$, such trees must be rooted at a peripheral vertex. A breadth first search algorithm beginning with any vertex v such that $e(v)=k$ will always produce a k-DST. Moreover, if $d(u,v)=i$ then the vertex u belongs to level i. We will consider only breadth first search distance spanning trees later in this paper.

Theorem 3.1. *Let G be a diameter-edge-invariant graph with n vertices and diameter d. Then for all $v \in V(G)$*

(1) $2 \leqslant \deg(v) \leqslant n - \frac{1}{2}(3d-6)$ *(except $d=2$ where it is $2 \leqslant \deg(v) \leqslant n-1$) if d is even and*

(2) $2 \leqslant \deg(v) \leqslant n - \frac{1}{2}(3d-5)$ *if d is odd.*

Moreover, all these bounds are sharp.

Proof. The lower bound is obvious as G has no bridges. Consider a d-DST rooted at a peripheral vertex x.

There must be at least one vertex y on level d. As G is d.e.i. there are at least two edge-disjoint x-y paths of length d in G. Thus there are no levels i, $i+1$ both with only one vertex. Because of this we have at most $\frac{1}{2}d+1$ levels with only one vertex if d is even and at most $\frac{1}{2}(d+1)$ levels with only one vertex if d is odd.

Any vertex v on level i can be adjacent only to vertices on levels $i-1$, i, $i+1$. Thus there are at least $d-2$ remaining levels with vertices which are not adjacent to v. At most $\frac{1}{2}d$ ($\frac{1}{2}(d-1)$ if d is odd) of these levels have only one vertex.

Therefore
$$\deg(v) \leqslant n - 1 - 2\left(\frac{d}{2} - 2\right) + \frac{d}{2} = n - \frac{3d-6}{2}$$
if d is even and
$$\deg(v) \leqslant n - 1 - 2(d-2) + \frac{d-1}{2} = n - \frac{3d-5}{2}$$
if d is odd.

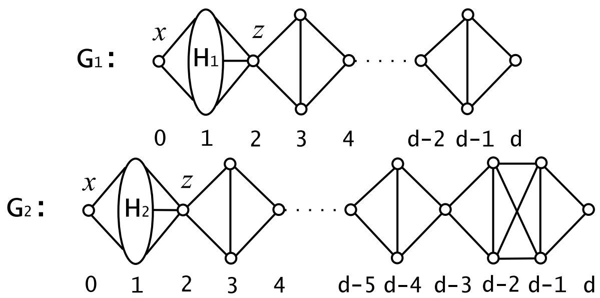

Figure 4

There is one exception. For $d=2$ it is $\frac{1}{2}(3d-6)=0$. But for any graph G it must hold $\deg(v)\leqslant n-1$.

To obtain a graph which reaches the bound it is sufficient to take $H_1=K_{n-\frac{3}{2}d+1}$ in the graph G_1 if d is even and $H_2=K_{n-(3d-1)/2}$ in the graph G_2 if d is odd. In both graphs x has the minimal and z the maximal possible degree. □

Lee [11] gave the bound for the minimal number of vertices in d.e.i. graphs of diameter d which is $\frac{3}{2}d+1$ vertices if d is even and $\frac{3}{2}(d+1)$ vertices if d is odd.

Theorem 3.2. *Let G be a diameter-vertex-invariant graph with n vertices and diameter d. Then for all $v\in V(G)$*

(1) $\deg(v)=n-1$, *if* $d=1$,
(2) $2\leqslant\deg(v)\leqslant n-1$ *if* $d=2$,
(3) $2\leqslant\deg(v)\leqslant n-3$ *if* $d=3$,
(4) $2\leqslant\deg(v)\leqslant n-4$ *if* $d=4$ *unless* $n=2d+2=10$, *for which it is* $2\leqslant\deg(v)\leqslant 5$,
(5) $2\leqslant\deg(v)\leqslant n-2d+3$ *if* $d\geqslant 5$.

These bounds are sharp.

Proof. The first two statements are obvious. If $d=3$ then there is no vertex v such that $e(v)=n-2$. Otherwise there is a unique vertex u such that $d(u,v)=2$. Thus $d(G-u)\leqslant 2r(G-u)=2e_{G-u}(v)=2$, a contradiction.

Suppose that $d(G)\geqslant 4$. Consider two vertices u, v such that $d(u,v)=d$ and two d-DST T_1, T_2 rooted at peripheral vertices v and u. Since G has no cut-vertices, each of these trees has at least 2 vertices on each of the levels $1,\dots,d-1$. We will prove the bound by a contradiction.

Let there be a vertex w such that $\deg(w)>n-2d+3$. If it belongs to level i, then it could be adjacent only to vertices on levels $i-1$, i, $i+1$ (if such exist). Since $\deg(w)>n-2d+3$, for $d-2$ levels there remain at most $2d-5$ vertices. Thus

(1) w is adjacent to every vertex on level $i-1$, i, $i+1$, or

(2) for all trees T_1, T_2 there is exactly 1 vertex on each of the levels 0 and d and 2 vertices on every other level except $i-1$, i, $i+1$.

Moreover, it is clear that there is a diametral path P such that $w \in P$.

(1) At least one tree T_i contains the vertex w on level $i \geqslant \lceil \frac{1}{2}d \rceil$. Let it be the tree T_1 and let it contain only one vertex (for example u) on level d. Then we can prove that $d(G-u) = d-1$: Let a_1, a_2 be two vertices on levels higher than i and b_1, b_2 be two vertices on levels lower than i. Therefore $d(a_i, b_k) < d(u, b_k) \leqslant d$. As $d(a_i, w) < \frac{1}{2}d$ we have $d(a_1, a_2) < d$. Moreover, G is d.v.i., and thus the vertices b_1, b_2 lie on a cycle. The vertex w is adjacent to all vertices on level $i-1$ and therefore the length of this cycle must be less than $2d$. Thus $d(b_1, b_2) < d$. Finally, $d(G-u) = d-1$, a contradiction. As a result of this part we already get that $\Delta(G) \leqslant n - 2d + 4$.

Let the tree T_1 contain two vertices on level d and let $\Delta(G) = n - 2d + 4$. Thus there are exactly 2 vertices on each level $1, \ldots, i-2$. Let us mark the vertices on level 2 as c_1, c_2. It must be $\deg(c_1) > 2$ and $\deg(c_2) > 2$. Otherwise, if $xc_j \in E(G), x \neq v$ then

$$d(G-x) \geqslant e_{G-x}(c_j) \geqslant d(c_i, u) = d(c_i, v) + d(v, u) = d+1 > d.$$

If $c_1c_2 \in E(G)$ or if $i-1 > 2$ (and thus there are only 2 vertices on level 2), then in $G-v$ all vertices on levels lower than i lie on a cycle of length less than $2d$. Similarly as in previous part $d(G-v) = d-1$.

Now, consider the case in which $c_1c_2 \in E(G)$ and $i-1 = 2$. Then $d_{G-v}(c_1, c_2) \leqslant 4$ and thus for any vertex $y \in V(G-v)$ we have $e_{G-v}(y) \leqslant \max\{4, d-1\}$. Finally, it holds $\Delta(G) \leqslant n - 2d + 3$ with the exception of $d = 4$. In that case we cannot use the same arguments as those given in the previous paragraph. Therefore, we obtain only the inequality $\Delta(G) \leqslant n - 2d + 4 = n - 4$.

If $n = 2d + 2 = 10$, then there are at most 3 vertices on level 2. In that case $d_{G-v}(c_1, c_2) \leqslant 2$ and thus $e_{G-v}(y) \leqslant \max\{2, d-1\} < d$ for all $y \in V(G-v)$. Therefore $\Delta(G) \leqslant n - 2d + 3 = 5$.

(2) Suppose $\Delta(G) \geqslant n - 2d + 4$. We can use the same arguments and notations as above. If, for example $d(u, w) < \frac{1}{2}d$ then $d(G-u) = d-1$. If $d(u, w) = d(w, v) = \frac{1}{2}d$ then for a tree T_1 rooted at central vertex v with the vertex w on level i either w is adjacent to every vertex on level $i-1$ or w is adjacent to every vertex on level $i+1$. Thus $d(G-v) = d-1$ in the first case or $d(G-u) = d-1$ in the second case.

Suppose $4 \neq d \geqslant 3$ or $2d + 2 = 10 = n$. The graph G (where $H = K_{n-2d}$, see Figure 5) certifies that our bounds are sharp. The following graph (see Figure 6) is for $d = 4$, $n \neq 10$ ($H = K_{n-10}$).

For $d = 2$ it is sufficient to take C_4 and substitute any vertex of C_4 with K_{n-3}. □

Similarly as the previous theorem we can prove the following result:

Theorem 3.3. *Diameter-vertex-invariant graph of diameter $d \geqslant 3$ has at least $2d+2$ vertices.*

To obtain a d.v.i. graph with $2d+2$ vertices is sufficient to take K_2 instead of H in Figure 5.

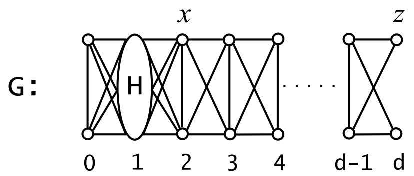

Figure 5

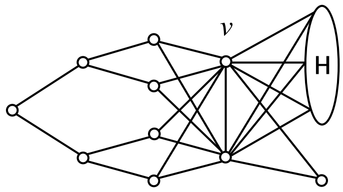

Figure 6

Theorem 3.4. *Let G be a diameter-adding-invariant graph with n vertices and diameter $d \geqslant 3$. Then for all $v \in V(G)$*

(1) $\deg(v) \leqslant n - \frac{3}{2}d + 2$ *if d is even,*

(2) $\deg(v) \leqslant n - \frac{3}{2}(d+1) + 3$ *if d is odd.*

These bounds are sharp.

Proof. Consider a diametral u-v path and the cycle F of length $d+1$ in the graph $G+uv$ formed by the u-v path and the edge uv. The eccentricity of every vertex w in the subgraph F is $\lceil \frac{1}{2}d \rceil$. Also $d_F(s,t) = d_{G+uv}(s,t)$ for all $s,t \in F$. Moreover, since G is d.a.i., there are at least two vertices $x, y \in V(G+uv)$ such that $d_{G+uv}(x,y) = d$.

Case 1: $x \in F$

Let z be the last joint vertex of the x-y geodesic and of the cycle F. One can prove that $d_{G+uv}(z,y) \geqslant \lfloor \frac{1}{2}d \rfloor$. For every $a \in V(G+uv)$ we have:

(1) a is adjacent to at most 3 successive vertices of F. Otherwise $d_G(u,v) < d(G)$.

(2) a is adjacent to at most 3 successive vertices of any z-y geodesic. Otherwise $d_{G+uv}(x,y) < d(G)$.

(3) a is adjacent to at most 4 vertices of the cycle F and of some z-y geodesic together. (Only if a is adjacent to z and its neighbours.) Otherwise $d_{G+uv}(x,y) < d(G)$.

(4) if $a = z$ then it is adjacent to at most 3 vertices of the cycle F and of some z-y geodesic together.

Case 2: $x \notin F$, $y \notin F$

It is clear that the x-y geodesic contains at most $\lceil \frac{1}{2}d \rceil$ vertices of cycle F. If two vertices b, c belong to F and to the x-y geodesic, then some b-c geodesic belongs to F. For every $a \in V(G + uv)$ we have:

(1) a is adjacent to at most 3 successive vertices of F. Otherwise $d(u, v)_G < d(G)$.

(2) a is adjacent to at most 3 successive vertices of any x-y geodesic. Otherwise $d_{G+uv}(x, y) < d(G)$.

(3) If the cycle F and the x-y geodesic have $\lceil \frac{1}{2}d \rceil$ vertices in common, then a is adjacent to at most 4 vertices of the cycle F and the x-y geodesic together. If the cycle F and the x-y geodesic have $\lceil \frac{1}{2}d \rceil - i$ vertices in common, then a is adjacent to at most $4 + i$ vertices of the cycle F and the x-y geodesic together. Otherwise $d_{G+uv}(x, y) < d(G)$.

(4) If a belongs both to x-y geodesic and to the cycle F then it is adjacent to at most 3 vertices of the cycle F and the x-y geodesic together.

Thus a is adjacent to at most $n - 1 - (d + 1 + \lceil \frac{1}{2}d \rceil - 4)$ vertices which is the same as the bounds.

To obtain a graph which certifies that the bounds are the best possible it is sufficient to take the graphs I_1 (I_2) and substitute some central vertex with the graph $K_{n-3d/2}$ (or $K_{n-(3d+1)/2}$). □

The next bound follows immediately from the proof of the previous theorem.

Theorem 3.5. *Diameter-adding-invariant graph of diameter d has at least*

(1) $\frac{3}{2}d + 1$ *vertices if d is even,*

(2) $\frac{1}{2}(3d + 1)$ *vertices if d is odd.*

References

[1] *V. Bálint, O. Vacek*: Radius-invariant graphs. Math. Bohem. *129* (2004), 361–377.

[2] *F. Buckley, F. Harary*: Distance in Graphs. Addison-Wesley, Redwood City, 1990.

[3] *R. D. Dutton, S. R. Medidi, R. C. Brigham*: Changing and unchanging of the radius of graph. Linear Algebra Appl. *217* (1995), 67–82.

[4] *R. Frucht, F. Harary*: On the corona of two graphs. Aequationes Math. *4* (1970), 322–325.

[5] *F. Gliviak*: On radially extremal graphs and digraphs, a survey. Math. Bohem. *125* (2000), 215–225.

[6] *F. Glivjak*: On certain classes of graphs of diameter two without superfluous edges. Acta Fac. Rer. Nat. Univ. Comenianae, Math. *21* (1968), 39–48.

[7] *F. Glivjak, P. Kyš, J. Plesník*: On the extension of graphs with a given diameter without superfluous edges. Mat. Cas. Slovensk. Akad. Vied *19* (1969), 92–101.

[8] *N. Graham, F. Harary*: Changing and unchanging the diameter of a hypercube. Discrete Appl. Math. *37/38* (1992), 265–274.
[9] *F. Harary*: Changing and unchanging invariants for graphs. Bull Malaysian Math. Soc. *5* (1982), 73–78.
[10] *S. M. Lee*: Design of diameter e-invariant networks. Congr. Numerantium *65* (1988), 89–102.
[11] *S. M. Lee, R. Tanoto*: Three classes of diameter e-invariant graphs. Comment. Math. Univ. Carolin. *28* (1987), 227–232.
[12] *J. Plesník*: Critical graphs of given diameter. Acta Fac. Rer. Nat. Univ. Comenianae, Math. *30* (1975), 71–93.
[13] *H. B. Walikar, F. Buckley, K. M. Itagi*: Radius-edge-invariant and diameter-edge-invariant graphs. Discrete Math. *272* (2003), 119–126.

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