Asymptotic preserving error estimates for numerical solutions of compressible Navier-Stokes equations in the low Mach number regime

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Introduction - I

Let $\rho, \mathbf{u}, p$ be the fluid density, velocity, pressure.

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{1a}$$

$$(\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla p = \mathrm{div} \mathbb{S}(\nabla \mathbf{u}). \tag{1b}$$

$$\rho(\mathbf{x}, 0) = \rho_0 > 0, \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0. \tag{1c}$$

$$\mathbb{S} = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3} \mathrm{div} \mathbf{u} \mathbf{I}) + \eta \mathrm{div} \mathbf{u} \mathbf{I}, \quad \mu > 0, \quad \eta \geq 0.$$

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$$\begin{aligned} &p \in C^2(0,\infty) \cap C^1[0,\infty), p(0) = 0, p'(\rho) > 0, \text{ for all } \rho > 0, \\ &\lim_{\rho \to \infty} \frac{p'(\rho)}{\rho^{\gamma-1}} = p_\infty > 0, \lim_{\rho \to 0^+} \frac{p'(\rho)}{\rho^\alpha} = p_0 > 0, \gamma \geq 1, \alpha \leq 1. \end{aligned} \tag{1d}$$

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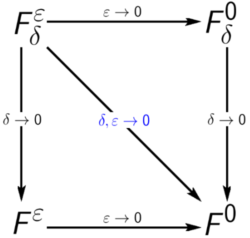
$$p = \rho^\gamma$$

Asymptotic preserving[3]



$$\bar{\rho}(\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla_x \mathbf{V}) + \nabla_x \pi = \mu \Delta \mathbf{V}, \mathrm{div}_x \mathbf{V} = 0, \bar{\rho} > 0, \tag{2a}$$

$$\mathbf{V}(0) = \mathbf{V}_0. \tag{2b}$$

Introduction - II

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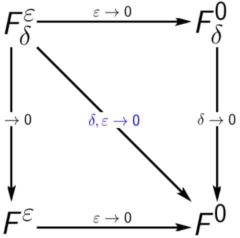

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$$\partial_t^\ell \mathbf{V} \in C^\ell([0, T]; W^{k-\ell,2}(\Omega; R^3)), \ell = 0, 1, 2,$$
$$\partial_t^j \Pi \in C^j([0, T]; W^{k-1-j}(\Omega)), j = 0, 1, k \geq 4,$$
$$V_0 \in W^{k,2(\Omega; R^3)}, \mathrm{div}_x \mathbf{V}_0 = 0.$$

Numerical Scheme

Piecewise linear Crouzeix-Raviart element for velocity.

$$V_{0,h} \equiv \{\mathbf{v}_h \in L^2(\Omega_h); \quad \mathbf{v}_h|_K \in \mathcal{P}^1(K), \forall K \in \Omega_h; \\ \int_\Gamma [\![\mathbf{v}_h]\!] = 0, \forall \Gamma \in \mathcal{E}_{int}; \quad \int_\Gamma \mathbf{v}_h = 0, \forall \Gamma \in \partial\mathcal{T}\}.$$

Piecewise constant element for density, pressure and temperature

$$Q_h \equiv \{\phi_h \in L^2(\Omega_h); \phi_h|_K \in \mathcal{P}^0(K), K \in \Omega_h\}.$$

- Element K, L, interface $\Gamma = K \cap L$.
- $\mathbf{n}_{\Gamma,K}$ be the outer normal, pointing from K to L

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Upwind flux

$$\mathcal{F}^{up}(f,\mathbf{u})|_\Gamma = \begin{cases} f_K & \text{if } s_{\Gamma,K} \geq 0, \\ f_L & \text{else,} \end{cases} \qquad s^n_{\Gamma,K} = \mathbf{u}^n|_\Gamma \cdot \mathbf{n}_{\Gamma,K} = s^{n,+}_{\Gamma,K} + s^{n,-}_{\Gamma,K}$$

Jump, average on element & edge

$$[\![f]\!]_\Gamma = f_L - f_K. \quad \hat{f}_K = \frac{1}{|K|}\int_K f dx. \quad \{f\}_\Gamma = \frac{1}{2}(f_K + f_L).$$

A convergent scheme (Karper [4])

Find $\{\rho_h^n, \mathbf{u}_h^n\}_{n=1}^{n_T} \subset (Q_h \times V_{0,h})$ such that for any $(\phi_h, \mathbf{v}_h) \in (Q_h \times V_{0,h})$

$$\sum_{K\in\mathcal{T}} |K| \frac{\rho_K^n - \rho_K^{n-1}}{\Delta t}\phi_h - \sum_{K\in\mathcal{T}}\sum_{\Gamma\in\partial K}|\sigma|\rho_\sigma^{n,up}(\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K})\phi_h = 0, \tag{3a}$$

$$\begin{aligned}\sum_{K\in\mathcal{T}} \int_K \frac{\rho_h^n \hat{\mathbf{u}}_h^n - \rho_h^{n-1}\hat{\mathbf{u}}_h^{n-1}}{\Delta t}\mathbf{v}_h - \sum_{K\in\mathcal{T}}\sum_{\Gamma\in\partial K}|\sigma|(\rho\hat{\mathbf{u}})_\sigma^{n,up}(\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}) \cdot \mathbf{v}_h \\ - \frac{1}{\varepsilon^2}\sum_{K\in\mathcal{T}} p_h^n \sum_{\Gamma\in\partial K} \mathrm{div}_h \mathbf{v}_h + \mu \sum_{K\in\mathcal{T}} \int_K \nabla_h \mathbf{u}_h^n : \nabla_h \mathbf{v}_h \\ + (\frac{\mu}{3} + \eta) \sum_{K\in\mathcal{T}} \int_K \mathrm{div}_h \mathbf{u}_h^n \mathrm{div}_h \mathbf{v}_h = 0.\end{aligned} \tag{3b}$$

Relative Energy

Relative energy inequality (Feireisl et al.[1])

$$\mathcal{E}(\rho, \mathbf{u}|z, \mathbf{V})|_0^\tau + \int_0^\tau \int_\Omega \big(\mathbb{S}(\nabla \mathbf{u}) - \mathbb{S}(\nabla \mathbf{V})\big) : (\nabla \mathbf{u} - \nabla \mathbf{V}) dxdt \leq \int_0^\tau \mathcal{R}(\rho, \mathbf{u}, z, \mathbf{V}) dt$$

$$\mathcal{E}(\rho, \mathbf{u}|z, \mathbf{V}) = \int_\Omega (\rho|\mathbf{u} - \mathbf{V}|^2 + E(\rho|z)) dx$$

$$E(\rho|z) = H(\rho) - H'(z)(\rho - z) - H(z), H(\rho) = \rho \int_1^\rho \frac{p(s)}{s^2} ds$$

$H(\rho)$ convex for $\rho \in (0, \infty)$, $E(\rho, z) \geq 0$ and $E(\rho, z) = 0 \Leftrightarrow \rho = z$.

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$$\mathcal{E}_\varepsilon(\rho,\mathbf{u}|z,\mathbf{V}) = \int_\Omega (\rho|\mathbf{u}-\mathbf{V}|^2 + \frac{1}{\varepsilon^2}E(\rho|z))\,dx$$

Theorem

Let p satisfy (1d) *with* $\gamma \geq 3/2$. *Let* $\{\rho^n, \mathbf{u}^n\}_{0\leq n\leq N}$ *be a family of numerical solutions constructed by the scheme* (3) *and the mesh be regular, and initial data* $(\rho^0, \mathbf{u}^0)$ *obey*

$$\mathcal{E}_\varepsilon(\rho_\varepsilon^0, \mathbf{u}_\varepsilon^0|\bar{\rho}, \mathbf{V}(0)) \leq E_0 < \infty, \; M_0/2 \leq \int_{\mathcal{T}} \rho_\varepsilon^0 dx \leq 2M_0, \; M_0 = \bar{\rho}|\mathcal{T}| \quad (4)$$

Moreover, suppose that $[\Pi, \mathbf{V}]$ *is a classical solution to* (2) *emanating from the initial data* $\mathbf{V}_0 \in W^{k,2}(\Omega; R^3), divV_0 = 0, k \geq 4$. *Then there exists a positive number independent of* $h, \Delta t, \varepsilon$ *such that*

$$\sup_{1\leq n\leq N} \mathcal{E}_\varepsilon(\rho_\varepsilon^n, \mathbf{u}_\varepsilon^n|\bar{\rho}, \mathbf{V}(t_n,\cdot)) + \Delta t \sum_{1\leq n\leq N} \int_{\mathcal{T}} |\nabla_h \mathbf{u}^n - \nabla_x \mathbf{V}(t_n,\cdot)|^2$$
$$\leq c\left(\sqrt{\Delta t} + h^a + \varepsilon + \mathcal{E}_\varepsilon(\rho_\varepsilon^0, \mathbf{u}_\varepsilon^0|\bar{\rho}, \mathbf{V}_0)\right) \quad (5)$$

where $a = \min\{\frac{2\gamma-3}{\gamma}, 1\}$.

- Step 1: uniform estimates (Karper[4], Gallouët et al.[2])

$$\boxed{(3b)|_{\mathbf{v}_h=\mathbf{u}_K^n} + (3a)|_{\phi_h=\frac{-|\hat{\mathbf{u}}_K^n|^2}{2}} + (3a)|_{\phi_h=H'(\rho_K^n)}}$$

sum up for each time step $\implies$

$$\sum_{K\in\mathcal{T}}|K|\left(\frac{1}{2}\rho_K^m|\hat{\mathbf{u}}_K^m| + \frac{1}{\varepsilon^2}E(\rho_K^m|\bar{\rho})\right) - \sum_{K\in\mathcal{T}}|K|\left(\frac{1}{2}\rho_K^0|\hat{\mathbf{u}}_K^0| + \frac{1}{\varepsilon^2}E(\rho_K^0|\bar{\rho})\right)$$
$$+\Delta t\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}\int_K\left(\mu\int_K|\nabla_x\mathbf{u}^n|^2 + (\frac{\mu}{3}+\eta)\int_K|\mathrm{div}_x\mathbf{u}^n|^2\right) + D_i^m = 0$$

Idea of Proof - I

- Step 1: uniform estimates (Karper[4], Gallouët et al.[2])

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Remark: not valid for linear scheme yet

- Step 2: Relative energy inequality (Gallouët et al.[2])

$$\boxed{(3b)\big|_{\mathbf{v}_h=\mathbf{u}_K^n} + (3a)\big|_{\phi_h=\frac{-|\hat{\mathbf{u}}_K^n|^2}{2}} + (3a)\big|_{\phi_h=H'(\rho_K^n)}}$$

$$-\boxed{(3b)\big|_{\mathbf{v}_h=\mathbf{V}_K^n} + (3a)\big|_{\phi_h=\frac{-|\hat{\mathbf{V}}_K^n|^2}{2}} + (3a)\big|_{\phi_h=H'(z_K^{n-1})}}$$

sum up for each time step $\Longrightarrow$

$$\int_\Omega \left(\rho^m|\hat{\mathbf{u}}^m - \hat{\mathbf{V}}_h^m|^2 + \frac{1}{\varepsilon^2}E(\rho^m|\bar{\rho})\right)dx - \int_\Omega \left(\rho^0|\hat{\mathbf{u}}^0 - \hat{\mathbf{V}}_h^0|^2 + \frac{1}{\varepsilon^2}E(\rho^0|\bar{\rho})\right)dx$$

$$+\Delta t\sum_{n=1}^m\sum_{K\in\Omega}\left(\int_K|\nabla_x(\mathbf{u}^n - \mathbf{V}_h^n)|^2 + (\frac{\mu}{3}+\eta)\int_K|\mathrm{div}(\mathbf{u}^n - \mathbf{V}_h^n)|^2\right) \le \sum_{i=1}^3 S_i(\rho_h) + R_1^m + G^m$$

$$|R_1^m| \le c(\sqrt{\Delta t} + h^a), \quad |G^m| \le c\Delta t\sum_{n=1}^m \mathcal{E}_\varepsilon(\rho^n, \mathbf{u}^n|\bar{\rho}, \mathbf{V}^n).$$

Idea of Proof - III

- Step 3: Control of S_i

$$S_1 = \Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K \mu \nabla_x \mathbf{V}_h^n : \nabla_x (\mathbf{V}_h^n - \mathbf{u}^n) dx,$$

$$S_2 = \Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} |K| \bar{\rho} \frac{\mathbf{V}_{h,K}^n - \mathbf{V}_{h,K}^{n-1}}{\Delta t} \cdot (\mathbf{V}_{h,K}^n - \mathbf{u}_K^n) dx,$$

$$S_3 = \Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \sum_{\Gamma \in \partial K} |\sigma| \bar{\rho} (\hat{\mathbf{V}}_{h,\sigma}^{n,up} - \hat{\hat{\mathbf{u}}}_{\sigma}^{n,up}) \cdot (\mathbf{V}_{h,\sigma}^n - \mathbf{V}_{h,K}^n) \hat{\mathbf{V}}_{h,\sigma}^{n,up} \cdot \mathbf{n}_{\sigma,K} dx.$$

Navier-Stokes (2) test with $(\mathbf{V}^n - \mathbf{u}^n) \Longrightarrow$

$$\sum_{i=1}^{3} S_i + R_2^m = 0, \quad |R_2^m| \leq c(h^b + \Delta t + \varepsilon), \quad b = \min\{\frac{5\gamma - 6}{2\gamma}, 1\}.$$

Error Estimates

$$\sup_{1\leq n\leq N} \mathcal{E}_\varepsilon\Big(\rho_\varepsilon^n, \mathbf{u}_\varepsilon^n | \bar{\rho}, \mathbf{V}(t_n,\cdot)\Big) + \Delta t \sum_{1\leq n\leq N} \int_{\mathcal{T}} |\nabla_h \mathbf{u}^n - \nabla_x \mathbf{V}(t_n,\cdot)|^2$$

$$\leq c\Big(\sqrt{\Delta t} + h^a + \varepsilon + \mathcal{E}_\varepsilon(\rho_\varepsilon^0, \mathbf{u}_\varepsilon^0 | \bar{\rho}, \mathbf{V}_0)\Big), \quad a = \min\{\frac{2\gamma-3}{\gamma}, 1\}$$

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Remark

Theorem holds also in the 2D case for any $0 \leq a < \frac{2\gamma-2}{\gamma}$ *if* $\gamma \in (1,2]$ *and* $a = 1$ *if* $\gamma > 2$. *Note that in this case the limit system* (2) *admits global-in-time smooth solutions as long as the initial data are regular.*

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Theorem holds also in the 2D case for any $0 \leq a < \frac{2\gamma-2}{\gamma}$ if $\gamma \in (1,2]$ and $a = 1$ if $\gamma > 2$. Note that in this case the limit system (2) admits global-in-time smooth solutions as long as the initial data are regular.

$$\begin{aligned}
&p = \rho^\gamma, z = (1+\varepsilon^2\Pi)^{1/\gamma}\\
&e_{\mathcal{E}} = \sup_{1\leq n\leq N} \mathcal{E}_\varepsilon\Big(\rho_\varepsilon^n, \mathbf{u}_\varepsilon^n|\bar{\rho}, \mathbf{V}(t_n,\cdot)\Big)\\
&e_{\nabla \mathbf{u}} = \|\nabla_h \mathbf{u}^n - \nabla_x \mathbf{V}(t_n,\cdot)\|_{L^2(L^2)}\\
&e_{\mathbf{u}} = \|\mathbf{u}^n - \mathbf{V}(t_n,\cdot)\|_{L^2(L^2)}\\
&e_\rho = \|\rho - z\|_{L^2(L^2)},\\
&e_p = \|p - 1 - \varepsilon^2\Pi\|_{L^2(L^2)}.
\end{aligned}$$

Experiment - I

$$u_1(x,y,0) = \sin^2(\pi x)\sin(2\pi y)$$
$$u_2(x,y,0) = -\sin(2\pi x)\sin^2(\pi y)$$
$$\rho(x,y,0) = 1 - \frac{\varepsilon^2}{2}\tanh(y-0.5)$$

Table : Error with respect to the numerical solution of NS[5]

h	$e_{\mathcal{E}}$	EOC	$e_{\nabla \mathbf{u}}$	EOC	$e_{\mathbf{u}}$	EOC	e_ρ	EOC	e_p	EOC
1/8	1.12e-03	–	4.91e-01	–	1.82e-03	–	3.65e-04	–	5.11e-04	–
1/16	3.74e-04	1.58	2.55e-01	0.95	1.18e-03	0.63	7.90e-05	2.21	1.11e-04	2.20
1/32	1.09e-04	1.78	2.29e-01	0.16	7.87e-04	0.58	1.50e-05	2.40	2.09e-05	2.41
1/64	1.91e-05	2.51	1.26e-01	0.86	3.24e-04	1.28	3.31e-06	2.18	4.64e-06	2.17
1/128	4.50e-06	2.09	4.41e-02	1.51	1.41e-04	1.20	8.73e-07	1.92	1.22e-06	1.93
1/256	1.14e-06	1.98	1.54e-02	1.52	6.32e-05	1.16	2.06e-07	2.08	2.88e-07	2.08

(a) $\varepsilon = h, \mu = 0.01$

h	$e_{\mathcal{E}}$	EOC	$e_{\nabla \mathbf{u}}$	EOC	$e_{\mathbf{u}}$	EOC	e_ρ	EOC	e_p	EOC
1/8	5.42e-03	–	4.66e-01	–	2.99e-03	–	8.81e-04	–	1.23e-03	–
1/16	1.34e-03	2.02	1.73e-01	1.43	2.17e-03	0.46	1.79e-04	2.30	2.51e-04	2.29
1/32	3.66e-04	1.87	6.58e-02	1.39	1.17e-03	0.89	3.78e-05	2.24	5.30e-05	2.24
1/64	1.06e-04	1.79	2.37e-02	1.47	6.13e-04	0.93	8.21e-06	2.20	1.15e-05	2.20
1/128	2.96e-05	1.84	8.26e-03	1.52	2.78e-04	1.14	1.81e-06	2.18	2.54e-06	2.18
1/256	7.96e-06	1.89	2.97e-03	1.48	1.31e-04	1.09	4.09e-07	2.15	5.73e-07	2.15

(b) $\varepsilon = h, \mu = 1$

Exact solution of unsteady Taylor vortex

$$V_1(x,y,t) = \sin(2\pi x)\cos(2\pi y)\, e^{-8\pi^2 \mu t}$$
$$V_2(x,y,t) = -\cos(2\pi x)\sin(2\pi y)\, e^{-8\pi^2 \mu t}$$
$$\Pi(x,y,t) = \tfrac{1}{4}\left(\cos(4\pi x) + \cos(4\pi y)\right) e^{-16\pi^2 \mu t}$$

$$\rho(x,y,0) = 1 + \varepsilon^2 \Pi(x,y,0)$$
$$u_1(x,y,0) = V_1(x,y,0)$$
$$u_2(x,y,0) = V_2(x,y,0)$$

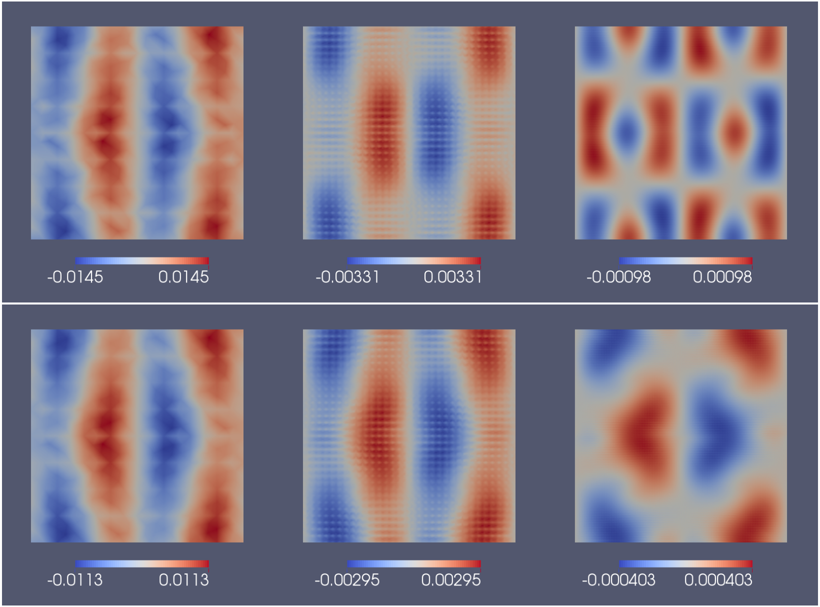
h	$e_{\mathcal{E}}$	EOC	$e_{\nabla \mathbf{u}}$	EOC	$e_{\mathbf{u}}$	EOC	e_ρ	EOC	e_p	EOC
1/8	3.60e-02	–	3.57e-01	–	7.56e-03	–	3.66e-04	–	4.22e-04	–
1/16	3.04e-03	3.57	1.94e-01	0.88	2.35e-03	1.69	8.67e-05	2.08	9.08e-05	2.22
1/32	2.98e-04	3.35	1.30e-01	0.58	9.44e-04	1.32	1.92e-05	2.17	1.76e-05	2.37
1/64	5.26e-05	2.50	8.35e-02	0.64	3.25e-04	1.54	4.36e-06	2.14	3.45e-06	2.35
1/128	1.46e-05	1.85	4.46e-02	0.90	1.11e-04	1.55	1.08e-06	2.01	1.06e-06	1.70
1/256	3.88e-06	1.91	2.22e-02	1.01	4.05e-05	1.45	2.65e-07	2.03	2.34e-07	2.18

(a) $\varepsilon = h, \mu = 0.01, \gamma = 3$

h	$e_{\mathcal{E}}$	EOC	$e_{\nabla \mathbf{u}}$	EOC	$e_{\mathbf{u}}$	EOC	e_ρ	EOC	e_p	EOC
1/8	1.54e-03	–	4.38e-02	–	2.20e-03	–	1.26e-04	–	7.06e-04	–
1/16	3.63e-04	2.08	2.14e-02	1.03	1.04e-03	1.08	3.02e-05	2.06	1.20e-04	2.56
1/32	1.18e-04	1.62	9.99e-03	1.10	4.29e-04	1.28	1.03e-05	1.55	2.41e-05	2.32
1/64	3.95e-05	1.58	4.84e-03	1.05	1.92e-04	1.16	2.70e-06	1.93	5.22e-06	2.21
1/128	1.22e-05	1.69	2.40e-03	1.01	9.26e-05	1.05	6.89e-07	1.97	1.25e-06	2.06
1/256	3.45e-06	1.82	1.20e-03	1.00	4.59e-05	1.01	1.74e-07	1.99	3.10e-07	2.01

(b) $\varepsilon = h, \mu = 1, \gamma = 3$

Error of velocity component $u_1 - V_1$ for different mesh sizes and Mach numbers; $h = \frac{1}{16}, \frac{1}{64}, \frac{1}{256}$ (left to right), $\varepsilon = 0.1,\ 0.001$ (top to bottom).



For more details, see Feireisl et al. 2016.
www.math.cas.cz/fichier/preprints/IM_20160916140850_71.pdf

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Thank you for your attention!