MAPLE TOOLS FOR THE KURZWEIL INTEGRAL

Peter Adams, Rudolf Výborný, University of Queensland

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Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. Riemann sums based on δ-fine partitions are illustrated with a Maple procedure.

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MSC 2000: 28-01, 28-02, 28-04, 28E99

1. Introduction

In 1957 Kurzweil [11] considered highly oscillatory differential equations, and introduced generalized solutions to such equations. If the differential equation is specified as

$$y' = f(x),$$

then the resulting integral is equivalent to an integral introduced by Perron in 1915. If the definition is formulated directly rather than through differential equations then it amounts to a small but ingenious modification of the Riemann definition. Henstock [6], [7], [8], [10], [9] rediscovered Kurzweil's approach and made further contributions. Today this integral is known under various names, such as Kurzweil's, Henstock's, Kurzweil-Henstock's and the generalized Riemann integral. We shall refer to it as Kurzweil's integral. It is becoming popular and widely used. McLeod's book [16] is intended for (university) teachers, Bartle's book [2] and Gordon's publication [5] are aimed at graduate level, however with different focuses. Bartle believed that Kurzweil's integral should become *the* integral. Lanzhou lectures by Lee Peng Yee [14] are also directed at an advanced audience. Muldowney [17] treats the subject in an abstract setting. In contrast, [3] and [4] contain an introduction to the Kurzweil theory as a part of a course on real analysis. The monograph by Lee and Výborný [13] tries to cater to all tastes and needs: Chapter 2 can be used as an elementary introduction to Kurzweil's theory, however advanced topics are included later. Moreover the last chapter describes applications to complex analysis, Fourier series and other topics. There are also resources in languages other than English, by Mawhin in [15], by Kurzweil himself in [12], and lecture notes in Czech by Schwabik.

In this paper we introduce *Maple* procedures for illustrating the definition of the Kurzweil integral. With some limitations (discussed below), given a function f, an interval $[a,b]$ and a strictly positive δ, the procedures produce a δ-fine partition and the corresponding Riemann sum numerically and graphically. The knowledge of Kurzweil's theory needed for understanding this paper is minimal; for example, the material expounded in the first half of the last chapter of [1] is sufficient.

2. Fine partitions

If

$$a = x_0 < x_1 < \ldots < x_n = b$$

and $\xi_i \in [x_{i-1}, x_i]$ then the set of couples

$$D \equiv \{(\xi_i, [x_{i-1}, x_i]);\ i = 1, \ldots, n\}$$

is called a *tagged division* of $[a,b]$, and the point ξ_i is the *tag* of the interval $[x_{i-1}, x_i]$. The term *partition* is used interchangeably with the term tagged division. If $\delta\colon [a,b] \mapsto (0,\infty)$ then by the usual definition the tagged division D is called δ*-fine* if

$$\xi_i - \delta(\xi_i) < x_{i-1} < x_i < \xi_i + \delta(\xi_i)$$

for $i = 1, \ldots, n$. We find it convenient to modify this definition by demanding instead

$$\xi_i - \delta(\xi_i) \leqslant x_{i-1} < x_i \leqslant \xi_i + \delta(\xi_i).$$

This new definition of δ-fine does not affect the definition of the integral (Definition 15.2 in [1] or Definition 2.4.1 in [13]). It is an important feature of Kurzweil's theory that for a suitable δ, a particular point of $[a,b]$ must become a tag of at least one subinterval of a δ-fine partition. This feature is preserved with our new definition of δ-fine, but perhaps a different δ must be used. For instance, if $\delta(x) = |x|$ (for non-zero x) then a traditional δ-fine partition of $[-1/2, 1/2]$ must have zero as a tag of at least one (and possibly two) subintervals.[1] With our definition of δ-fine,

[1] This is sometimes referred to as: anchoring the partition on the point, in this case on zero.

$\delta(x) = 0.9x$ (for non-zero x) or something similar will force zero to become a tag of a subinterval.

The theorem guaranteeing the existence of a δ-fine partition of a compact interval for a strictly positive δ is known as Cousin's lemma. Theorem 2.3.6 in [13] states the equivalence of Cousin's lemma and the completeness of the reals. This makes it clear that a computer cannot always produce a δ-fine partition for a given strictly positive δ. For the purposes of our programs, we satisfy ourselves with the following substitute for Cousin's lemma.

Theorem on approximate δ-fine partitions. *Let δ be a positive function on a compact interval $[a, b]$ and $\varepsilon > 0$. Then we can define a strictly increasing finite sequence*

$$\{x_0, x_1, \dots, x_N\}$$

such that

(i) $x_0 = a$,

(ii) (a) *either* $x_i \leqslant x_{i-1} + \delta(x_{i-1})$,

(b) *or* $x_{i-1} \geqslant x_i - \delta(x_i)$,

(c) *or* $0 < x_i - x_{i-1} \leqslant \varepsilon$ *for* $i = 1, 2, \dots, N$. *If neither* (a) *nor* (b) *occurs then the interval* (x_{i-1}, x_i) *contains a point* c *such that* $\liminf\limits_{x \to c} \delta(x) = 0$.

(iii) $x_N = b$. *If the set* S *of points* c *where* $\liminf\limits_{x \to c} \delta(x) = 0$ *is finite then all* x_i *with* $0 < i \leqslant N$ *satisfy either* (ii)(a) *or* (ii)(b).

P r o o f. Without loss of generality we assume that $\delta(x) < b - x$ for $x < b$. Let $n_0 = 0$, $x_{n_0} = a$ and if $x_{n_k} < b$ with $k \geqslant 0$ has been defined we define $x_{n_{k+1}}$ as follows. First, let $y_0 = x_{n_k}$ and $y_{m+1} = y_m + \delta(y_m)$ for a nonnegative integer m. The sequence $n \mapsto y_n$ is well defined, strictly increasing and bounded by b. Denote its limit by l. If $l < y_0 + \varepsilon$, $l < b$ and the set S is infinite we set $x_{n_{k+1}} = \min(b, y_0 + \varepsilon)$ otherwise let m be the first integer such that $y_m > l - \delta(l)$. We now set $x_{n_k + i} = y_i$ for $i = 1, \dots, m$ and $x_{n_{k+1}} = l$. If S is finite let d be the smallest distance between distinct points of S. Let us now assume, contrary to what we want to prove that the sequence $k \mapsto x_{n_k}$ is infinite. Since $x_{n_{k+1}} - x_{n_k} \geqslant \min(d, \varepsilon)$ if S is finite and $\geqslant \varepsilon$ otherwise for $k > 1$, the sequence must diverge to $+\infty$, this however contradicts the inequality $x_{n_k} \leqslant b$. Consequently there is a largest x_{n_k} and we denote it by x_N. If S is finite then the terms of the sequence $n \mapsto x_n$ by its very construction satisfy the requirements (ii)(a) or (ii)(b). If x_i satisfies neither (ii)(a) nor (ii)(b) then the interval (x_{i-1}, x_i) contains a limit l of some sequence of y's and clearly $l \in S$. $\square$

If the interval $[x_{i-1}, x_i]$ satisfies (ii)(a) then we tag this interval by x_{i-1}, and if it satisfies (ii)(b) then we tag it by x_i. If it doesn't satisfy either then we tag it by any point in the interval and justify this by saying that the interval is 'small'. Then the intervals $[x_{i-1}, x_i]$ together with their tags form an *approximate* δ*-fine* partition.

3. Our programs

The above theorem is the theoretical basis of our programs for producing an approximate δ-fine partition and the corresponding Riemann sum. However it is still not practical enough for implementation on a computer. If, for instance, $\delta(x) = x^2$ (for non-zero x) and $[a, b] = [-1/2, 1/2]$ then a program based only on the algorithm described in the theorem will quickly stall due to round-off errors. Similarly, if $\delta(x) = 10^{-11}$ then the sequence defined by $x_0 = -1$ and $x_i = x_{i-1} + \delta(x_{i-1})$ is diverging to ∞ but, on a computer calculating with 10 significant digits, the approximations to x_i will all equal -1. We therefore build into the program some safeguards preventing this and similar things happening, in particular $\delta(x_i)$ becoming too small. We also have to cater for the possible inability of the computer to calculate exactly the limit l mentioned in the proof of the theorem.

We have also incorporated improvements to the efficiency of the algorithm. For example, on the interval $[a, b]$, if the limit l equals b then the program finds an approximate δ-fine partition by repeatedly bisecting the interval $[a, b]$ and testing the left-half of the bisected subinterval. As soon as this subinterval is δ-fine, the program seeks to extend this subinterval to the right (still keeping it δ-fine) by evaluating δ at the midpoint and both endpoints of the subinterval. If $l < b$ then we apply the above procedure on $[a, l]$ then find the next limit $\bar{l}$ and continue the process on $[l, \bar{l}]$.

The set S mentioned in the proof of the theorem might be infinite or round-off and other computational limitations might preclude S being found. In these cases the program takes appropriate action, but might produce only an approximate δ-fine partition.

4. Examples

Here we present four examples. In all four examples our programs produce Riemann sums for proper δ-fine partitions, not approximate δ-fine partitions. More examples, ready for use, are provided on *Maple* worksheets on our websites.

Example 1. Let

$$f(x) = \begin{cases} (1 + 10x^2)^{-1} & \text{for } x < 0, \\ \frac{1}{2} & \text{for } x \geqslant 0 \end{cases}$$

and δ be defined by

$$\delta(x)=\begin{cases}\min(1.5, 0.15(1+f(x)+f'(x))^{-1}) & \text{for } x<0,\\ 0.01 & \text{for } x=0,\\ |x| & \text{for } x>0.\end{cases}$$

The integrand f in this example is a simple function that is continuous except at one point, so Kurzweil's integral is not really needed for integration of the function. We wish to illustrate that with a suitable δ the tagged division is adjusted to the behavior of the function, considered here on $[-3,1]$. Figure 1 shows the graph produced by our programs. The derivative f' in the definition of δ causes the subintervals to shrink where f is changing rapidly. Note that in the figure there is only one subinterval where the function is constant, but in contrast there are eight subintervals on $[-1.02,0]$.

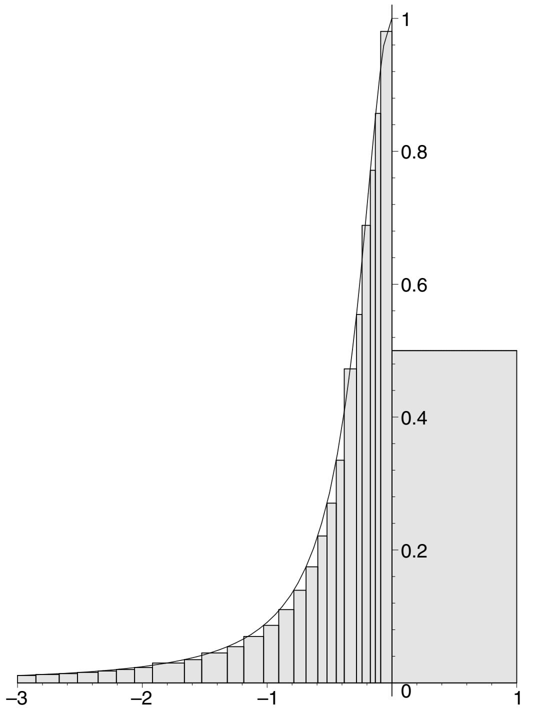

Figure 1. Adjusting the partition to the integrand

E x a m p l e 2. Let

$$f(x)=\begin{cases}\sqrt{x}\,|\cos(\pi x/50)| & \text{if } x \text{ is a complete square},\\ -1 & \text{otherwise}\end{cases}$$

and δ be defined by

$$\delta(x) = \begin{cases} 0.1 & \text{for } x = 0, \\ \min(0.9(\lceil \sqrt{x}\,\rceil^2 - x), 0.9(x - \lfloor \sqrt{x} \rfloor^2)) & \text{if } f(x) = -1, \\ \min\left(\frac{1}{2}, (1 + f(x))^{-1}\right) & \text{otherwise.} \end{cases}$$

(We chose $f(x) = -1$ rather than $f(x) = 0$ because the graph would have otherwise coincided with the x axis.) In this example δ was chosen so that all the points that are complete squares are chosen as tags. The graph on $[0, 100]$ is shown in Figure 2.

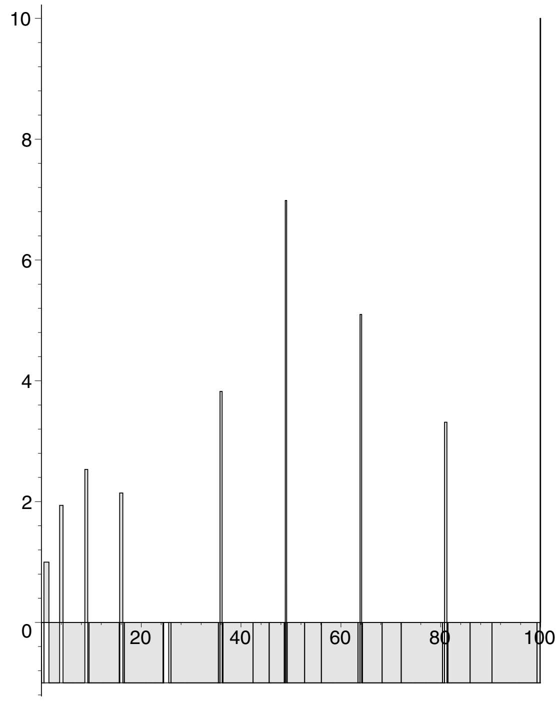

Figure 2. Anchoring the partition on complete squares

E x a m p l e 3. Let

$$f(x) = \begin{cases} |x|^{-\frac{1}{2}} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

and δ be defined by

$$\delta(x) = \begin{cases} 0.2|x| & \text{for } x \neq 0, \\ 0.005 & \text{for } x = 0. \end{cases}$$

The function f is Lebesgue integrable on $[-0.6, 0.6]$ so it is also Kurzweil integrable. The graph is shown in Figure 3. Now

$$\int_{-0.6}^{0.6} |x|^{-\frac{1}{2}}\,\mathrm{d}x = 3.098, \qquad \text{and the Riemann sum} = 2.773.$$

By similar reasoning to that in Example 2.4.3 of [13], it can be shown that for this choice of δ the error is at most 0.438, in agreement with the results above. The accuracy can easily be increased by changing δ appropriately but then the graph may not be easily legible. For example, if δ is changed so that $\delta(0) = 0.001$ then the Riemann sum equals 2.927, but the graph is difficult to read near $x = 0$.

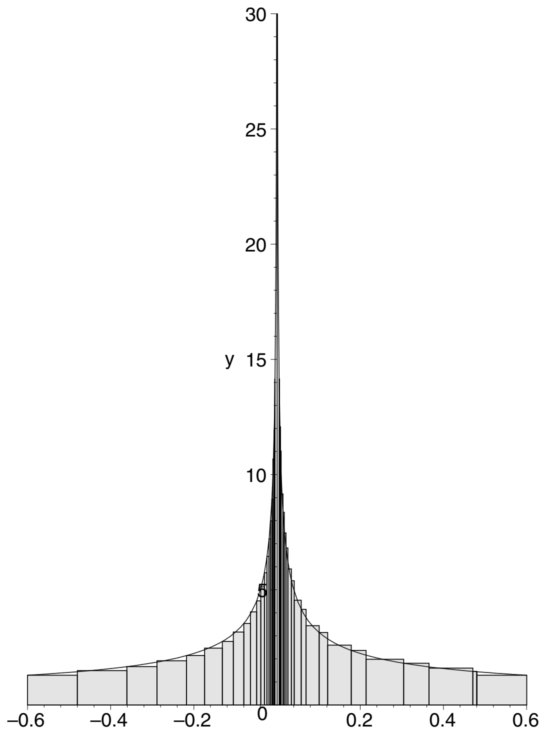

Figure 3. Riemann sums for a Lebesgue integrable function

Example 4. Let

$$f(x) = \begin{cases} (-1)^n n & \text{for } n \in \mathbb{N} \text{ and } \dfrac{1}{n+1} < x \leqslant \dfrac{1}{n}, \\ 0 & \text{for } x = 0 \end{cases}$$

and δ be defined by

$$\delta(x) = \begin{cases} \min\Big(\dfrac{1}{n} - x, x - \dfrac{1}{n+1}\Big) & \text{for } n \in \mathbb{N} \text{ and } \dfrac{1}{n+1} < x < \dfrac{1}{n}, \\ k^{-n} & \text{for } n \in \mathbb{N} \text{ and } x = n^{-1}, \\ \varepsilon & \text{for } x = 0. \end{cases}$$

If $D \equiv \{(\xi_i, [x_{i-1}, x_i]);\ i = 1, \dots, n\}$ is a δ-fine partition then

$$\Big| \sum_1^n f(\xi_i)(x_i - x_{i-1}) - \sum_1^\infty (-1)^i \frac{1}{i+1} \Big| < 2\varepsilon + 2\sum_{i=1}^\infty \frac{1}{k^i} \leqslant 2\varepsilon + 2\frac{1}{k-1}.$$

Since ε and k are arbitrary this proves the Kurzweil integrability of f and

$$\int_0^1 f = \sum_1^\infty (-1)^i \frac{1}{i+1}.$$

The Riemann sum for f and D are illustrated in Figure 4, where the interval of integration is $[0, 1/2]$ and $\varepsilon = 1/30$ and $k = 4$.

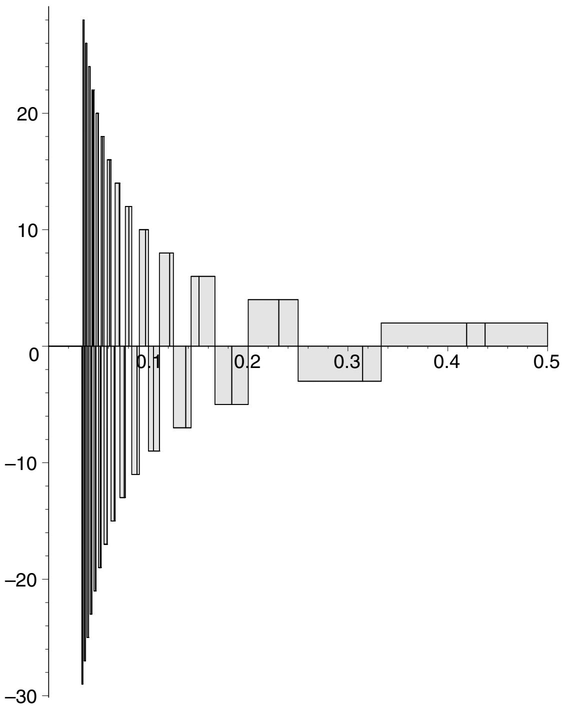

Figure 4. Riemann sums for a non-absolutely integrable f

5. Limitations

Maple already provides procedures for illustrating Riemann sums. For a given f, interval $[a, b]$ and natural number n these produce graphs or numerical values of the Riemann sums. The interval $[a, b]$ is divided into n subintervals and the function f is evaluated either at the left, or the right, or in the middle, of each subinterval. For instance, the commands `leftsum` and `leftbox` produce the Riemann sum and its graph for the function f and the interval $[a, b]$ subdivided into n subintervals with f evaluated at the left-hand end-points of the subintervals. Some caution is needed when illustrating convergence with any software because the computer is necessarily a finite-state device. For a given f and n there are infinitely many Riemann sums and *Maple* produces three of them. The definition of the Riemann integral requires that, for sufficiently large n, *all* and not merely three Riemann sums are close to the value of the integral.

In illustrating Riemann sums for a δ-fine tagged division there is an additional difficulty because the number of subintervals is controlled indirectly by the function δ. Again, for a given δ there are infinitely many δ-fine tagged divisions and consequently infinitely many associated Riemann sums. We feel that it is not good enough to produce a single Riemann sum, or even three of them. The produced sum should be indicative of the difficulties of getting the Riemann sum close to the integral, so it should be a 'typical' sum or even almost the worst possible sum. Producing a typical Riemann sum is possible with our program. All that is usually needed is to anchor (see [1] Remark 15.7 or [13] Remark 2.5.3) the partition on some 'troublesome' points. This was illustrated by Example 2 in the previous section. When teaching, we feel that involving the students in choosing an appropriate δ is valuable. Many elegant proofs in Kurzweil theory are facilitated by a judicious choice of δ.

6. Software availability

In accordance with the philosophy of open source software we offer our programs for download free of charge from our websites

`www.maths.uq.edu.au/~pa` or `www.maths.uq.edu.au/~rv`.

Readers are welcome to use these programs, but we request that our authorship be acknowledged and the programs not be modified without our permission. The websites also contain instructions on how to use our programs. We welcome suggested improvements and we shall acknowledge them if we incorporate them.

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Author's address: *Rudolf Výborný*, Department of Mathematics, The University of Queensland, Brisbane, Qld 4072, Australia, e-mail: `rv@maths.uq.edu.au`.