THE INDEPENDENT RESOLVING NUMBER OF A GRAPH

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To the Memory of W. T. Tutte

Abstract. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices in a connected graph G and a vertex v of G, the code of v with respect to W is the k-vector

$$c_W(v) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)).$$

The set W is an independent resolving set for G if (1) W is independent in G and (2) distinct vertices have distinct codes with respect to W. The cardinality of a minimum independent resolving set in G is the independent resolving number $\mathrm{ir}(G)$. We study the existence of independent resolving sets in graphs, characterize all nontrivial connected graphs G of order n with $\mathrm{ir}(G) = 1$, $n-1$, $n-2$, and present several realization results. It is shown that for every pair r, k of integers with $k \geqslant 2$ and $0 \leqslant r \leqslant k$, there exists a connected graph G with $\mathrm{ir}(G) = k$ such that exactly r vertices belong to every minimum independent resolving set of G.

Keywords: distance, resolving set, independent set

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1. Introduction

Independent sets of vertices in graphs is one of the most commonly studied concepts in graph theory. The independent sets of maximum cardinality are called *maximum independent sets* and these are the independent sets that have received the most attention. The number of vertices in a maximum independent set in a graph G is the *independence number* (or *vertex independence number*) of G and is denoted by $\beta(G)$. There are also certain independent sets of minimum cardinality that are of interest.

Ordinarily, a graph contains many independent sets. An independent set of vertices that is not properly contained in any other independent set of vertices is a *maximal independent set* of vertices. The minimum number of vertices in a maximal independent set is denoted by $i(G)$. This parameter is also called the *independent domination number* as it is a smallest cardinality of an independent set of vertices that dominate all vertices of G.

Some graphs contain (ordered) independent sets W such that the vertices of G are uniquely distinguished by their distances from the vertices of W. The goal of this paper is to study the existence of such independent sets in graphs and, when they exist, to investigate the minimum cardinality of such a set.

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u–v path in G. For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, we refer to the k-vector

$$c_W(v) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

as the *code* of v *with respect to* W. The set W is called a *resolving set* for G if distinct vertices have distinct codes. A *minimum resolving set* is also called a *basis* for G. The (*metric*) *dimension* $\dim(G)$ is the number of vertices in a basis for G. Resolving sets (and minimum resolving sets) have appeared previously. In [4], and later in [5], Slater introduced these ideas and used *locating set* for what we have called resolving set. He referred to the cardinality of a minimum resolving set in a graph G as its *location number* $\mathrm{loc}(G)$. Slater described the usefulness of these ideas when working with U.S. sonar and Coast Guard Loran (Long range aids to navigation) stations. Independently, Harary and Melter [3] discovered the concept of a location number as well but used the term metric dimension, the terminology that we have adopted. We refer to the book [1] for graph theoretical notation and terminology not described in this paper.

If G is a nontrivial connected graph of order n, then $1 \leqslant \dim(G) \leqslant n - 1$. Connected graphs of order $n \geqslant 2$ with dimension 1 or $n - 1$ are characterized in [3], [4], [5].

Theorem A. *Let G be a connected graph of order $n \geqslant 2$.*

(a) *Then $\dim(G) = 1$ if and only if $G = P_n$, the path of order n.*

(b) *Then $\dim(G) = n - 1$ if and only if $G = K_n$, the complete graph of order n.*

An *independent resolving set* W in a connected graph G is both resolving and independent. The cardinality of a minimum independent resolving set (or simply an ir-set) in a graph G is the *independent resolving number* $\mathrm{ir}(G)$. Let G be a connected graph of order n containing an independent resolving set. Since every independent resolving set of G is a resolving set, it follows that

$$1 \leqslant \dim(G) \leqslant \mathrm{ir}(G) \leqslant \beta(G) \leqslant n-1. \tag{1}$$

To illustrate this concept, consider the graph G of Figure 1(a). The set $W' = \{v_1, v_7, v_8\}$ shown in Figure 1(b) is a basis for G and so $\dim(G) = 3$. However, W' is not an independent resolving set for G. On the other hand, the set $W = \{v_1, v_4, v_5, v_6\}$ in Figure 1(c) is an independent resolving set. Indeed, the codes of the vertices of G with respect to W are

$$c_W(v_1) = (0,2,2,2),\ c_W(v_2) = (2,2,2,2),\ c_W(v_3) = (1,1,1,1),\ c_W(v_4) = (2,0,2,2),$$
$$c_W(v_5) = (2,2,0,2),\ c_W(v_6) = (2,2,2,0),\ c_W(v_7) = (3,1,1,2),\ c_W(v_8) = (3,2,1,1).$$

We can show, by a case-by-case analysis, that G contains no 3-element independent resolving set and so $\mathrm{ir}(G) = 4$. The set $\{v_1, v_2, v_4, v_5, v_6\}$ is a maximum independent set of G and so $\beta(G) = 5$. Thus the graph G of Figure 1(a) has $\dim(G) = 3$, $\mathrm{ir}(G) = 4$, and $\beta(G) = 5$.

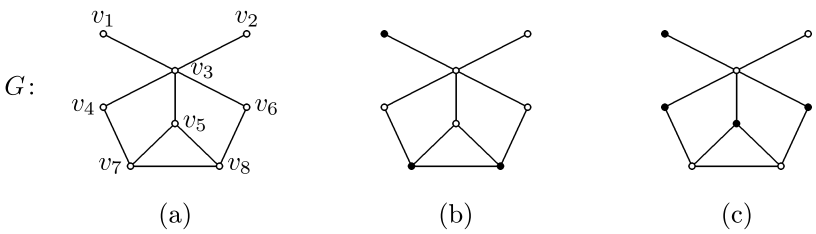

Figure 1. A graph G with $\dim(G) = 3$, $\mathrm{ir}(G) = 4$, and $\beta(G) = 5$

2. Preliminary results

Not all graphs have an independent resolving set, however, and so $\mathrm{ir}(G)$ is not defined for all graphs G. For example, the only independent sets of the complete graph K_n for $n \geqslant 3$ consist of a single vertex. Thus $\mathrm{ir}(K_n)$ is not defined for $n \geqslant 3$. Figure 2 shows the 3-regular graphs $K_{3,3}$, Q_3, and the Petersen graph P. A resolving set of $K_{3,3}$ contains at least two vertices from each partite sets of $K_{3,3}$. Since $\beta(K_{3,3}) = 3$, it follows that $\mathrm{ir}(K_{3,3})$ does not exist. On the other hand, $\mathrm{ir}(Q_3)$ and $\mathrm{ir}(P)$ are defined and, in fact, $\mathrm{ir}(Q_3) = \mathrm{ir}(P) = 3$. In Figure 2, the solid vertices represent a minimum independent resolving set for each of Q_3 and P.

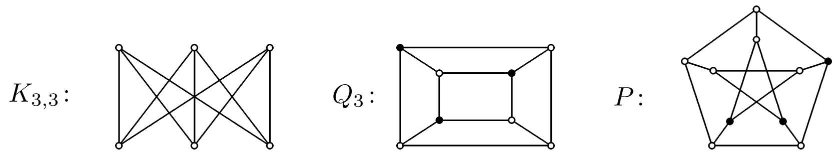

Figure 2. Three 3-regular graphs

Two vertices u and v in a connected graph G are *distance similar* if $d(u,x) = d(v,x)$ for all $x \in V(G) - \{u,v\}$. For a vertex v in a graph G, let $N(v)$ be the set of vertices adjacent to v and let $N[v] = N(v) \cup \{v\}$. Then two vertices u and v in a connected graph are distance similar if and only if (1) $uv \notin E(G)$ and $N(u) = N(v)$ or (2) $uv \in E(G)$ and $N[u] = N[v]$. Distance similarity in a graph G is an equivalence relation on $V(G)$. The following observation is useful.

Observation 2.1. *If U is a distance similar equivalence class in a connected graph G with $|U| = p \geqslant 2$, then every resolving set of G contains at least $p-1$ vertices from U. Thus if G has k distance similar equivalence classes and* $\mathrm{ir}(G)$ *is defined, then*

$$n - k \leqslant \dim(G) \leqslant \mathrm{ir}(G).$$

There exist graphs G such that every ir-set of G must contain all vertices of some distance similar equivalence class. For example, let G be the graph obtained from $K_{2,p}$, whose partite sets are $\{x,y\}$ and $U = \{u_1, u_2, \ldots, u_p\}$ with $p \geqslant 2$, by adding $p' \geqslant 2$ vertices v_i, $1 \leqslant i \leqslant p'$, and the pendant edges xv_i. Then G contains two distance similar equivalence classes of cardinality at least 2, namely U and $U' = \{v_1, v_2, \ldots, v_{p'}\}$. Since every ir-set of G has the form $(U \cup U') - \{w\}$ for some $w \in U \cup U'$, it follows that every ir-set of G contains either U or U'.

If U is a distance similar equivalence class of a connected graph G, then either U is an independent set in G or the subgraph $\langle U \rangle$ induced by U is complete in G. Thus we have the following observation.

Observation 2.2. *Let G be a connected graph and let U be a distance similar equivalence class in G with $|U| \geqslant 3$. If U is not independent in G, then* $\mathrm{ir}(G)$ *is not defined.*

The converse of Observation 2.2 is not true. For example, let $G = K_{3,3}$ with partite sets V_1 and V_2. We have seen that $\mathrm{ir}(G)$ is not defined. On the other hand, V_1 and V_2 are the only distance similar equivalence classes and they are both independent.

Proposition 2.3. *Let G be a connected graph of order $n \geqslant 6$ for which* $\mathrm{ir}(G)$ *is defined. If W is an independent resolving set of G, then* $\deg w \leqslant n-3$ *for every $w \in W$.*

P r o o f. Assume, to the contrary, that there exists $u \in W$ such that $\deg u \geqslant n-2$. Since W is independent, $|W| \leqslant 2$. On the other hand, since $\deg u \geqslant n-2 \geqslant 4$, it follows that $G \neq P_n$. Since P_n is the only connected graph of order n with dimension 1 by Theorem A, it follows that $|W| > 1$ and so $|W| = 2$. Let $W = \{u, v\}$. For each $x \in V(G) - W = N(u)$, the code $c_W(x) = (d(u,x), d(v,x)) = (1, d(v,x))$. Since $d(v,x)$ is one of $d(u,v)$, $d(u,v)+1$, and $d(u,v)-1$, there are at most three distinct codes for the vertices in $V(G) - W$. However, $|V(G) - W| = |N(u)| = n-2 \geqslant 4$, a contradiction. $\square$

The following corollary is a consequence of Proposition 2.3.

Corollary 2.4. *Let G be a connected graph of order $n \geqslant 6$.*
(a) *If G contains two nonadjacent vertices of degree $n-2$, then* $\mathrm{ir}(G)$ *is not defined.*
(b) *If G contains two vertices of degree $n-1$, then* $\mathrm{ir}(G)$ *is not defined.*

P r o o f. Assume, to the contrary, that $\mathrm{ir}(G)$ is defined, and let W be an independent resolving set of G. First, suppose that G contains two nonadjacent vertices x and y of degree $n-2$. Then x and y belong to the same distance similar equivalence class in G. By Observation 2.1, W contains at least one of x and y, which contradicts Proposition 2.3. Thus (a) holds.

Next, suppose that G contains two vertices x and y of degree $n-1$. Then x and y belong to the same distance similar equivalence class in G. Necessarily, W contains exactly one of x and y, which again contradicts Proposition 2.3. Thus (b) holds. $\square$

On the other hand, there exist graphs G of order $n \geqslant 6$ having two adjacent vertices of degree $n-2$ for which $\mathrm{ir}(G)$ is defined. For example, let G be the graph obtained from $\overline{K}_{n-2}$, where $V(\overline{K}_{n-2}) = \{v_1, v_2, \ldots, v_{n-2}\}$, and $P_2\colon x, y$ by adding the edges xv_i $(1 \leqslant i \leqslant n-3)$ and yv_j $(2 \leqslant j \leqslant n-2)$. The graph G is shown in Figure 3 for $n = 7$. Then $W = \{v_1, v_2, \ldots, v_{n-4}\}$ is a minimum independent resolving set of G and so $\mathrm{ir}(G) = n-4$.

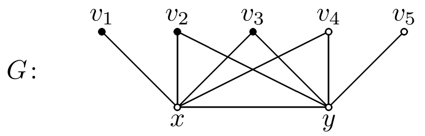

Figure 3. The graph G

Proposition 2.5. *Let G be a connected graph of order $n \geqslant 4$. Suppose that G contains two distinct distance similar equivalence classes U_1 and U_2 of cardinality at least 2. If some vertex of U_1 is adjacent to a vertex of U_2, then* $\mathrm{ir}(G)$ *is not defined.*

P r o o f. Suppose that $u_1u_2 \in E(G)$, where $u_1 \in U_1$ and $u_2 \in U_2$. Since U_1 and U_2 are distance similar equivalence classes, u_1 is adjacent to every vertex of U_2

and so every vertex of U_1 is adjacent to every vertex of U_2. By Observation 2.1, every resolving set of G must contain at least one vertex from each of U_1 and U_2. This implies, however, that no resolving set of G is independent and so $\mathrm{ir}(G)$ is not defined. □

The converse of Proposition 2.5 is not true. For example, let G be the graph obtained from two copies of K_4, whose vertex sets are $U_1 = \{u_1, u_2, u_3, u_4\}$ and $V_1 = \{v_1, v_2, v_3, v_4\}$ by adding the edge u_4v_4. Then $U_1 - \{u_4\}$ and $V_1 - \{v_4\}$ are two distinct distance similar equivalence classes of G. By Observation 2.2, $\mathrm{ir}(G)$ does not exist. However, no edge joins a vertex in $U_1 - \{u_4\}$ and a vertex in $V_1 - \{v_4\}$.

Let G be a connected graph with $\mathrm{ir}(G) = k$, let $W = \{w_1, w_2, \ldots, w_k\}$ be a minimum independent resolving set of G, and let $v \in V(G)$ with $\deg v = \Delta(G)$. Observe that if $u \in N(v)$, then $d(u, w_i)$ is one of $d(v, w_i)$, $d(v, w_i) + 1$, or $d(v, w_i) - 1$ for all i with $1 \leqslant i \leqslant k$. Thus there are at most $3^k - 1$ distinct codes of the vertices in $N(v)$ with respect to W. Therefore, $|N(v)| = \Delta(G) \leqslant 3^k - 1$. This observation gives the following bound for $\mathrm{ir}(G)$ of a connected graph G in terms of its maximum degree $\Delta(G)$.

Proposition 2.6. *If G is a nontrivial connected graph for which $\mathrm{ir}(G)$ is defined, then*

$$\mathrm{ir}(G) \geqslant \lceil \log_3(\Delta(G) + 1) \rceil.$$

The lower bound in Proposition 2.6 is sharp. In fact, for each pair (k, Δ) of integers such that $3^k = \Delta + 1$, there exists a connected graph G_k such that $\mathrm{ir}(G_k) = k$ and $\Delta(G_k) = \Delta = 3^k - 1$. For $k = 1$ ($\Delta = 2$) and $n \geqslant 3$, the graph $G = P_n$ has the desired properties. For $k = 2$ ($\Delta = 8$), we consider the graph G_2 of Figure 4. The maximum degree of G_2 is 8 with $\deg u_0 = 8$ and $N(u_0) = \{u_1, u_2, \ldots, u_8\}$. Then $W = \{v_1, v_2\}$ is an independent resolving set of G_2 and so $\mathrm{ir}(G_2) = 2$.

G_2:

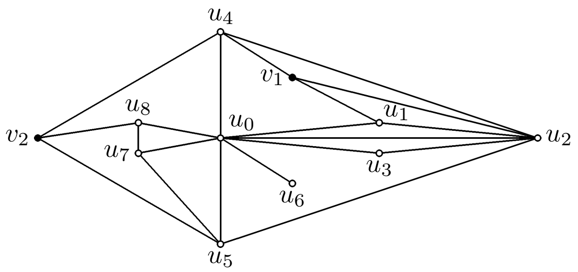

Figure 4. The graph G_2

For $k = 3$ ($\Delta = 26$), we construct the graph G_3 from the graph G_2 of Figure 4 by (I) replacing each vertex u_i ($0 \leqslant i \leqslant 8$) by the path u_{i_1}, u_i, u_{i_2} such that (a) u_0 is adjacent to all vertices u_{i_1} and u_{i_2} with $0 \leqslant i \leqslant 8$ and all u_j with $1 \leqslant j \leqslant 8$, (b) u_{0_1} and u_{0_2} are adjacent, respectively, to all vertices u_{i_1}, u_{i_2}, where $1 \leqslant i \leqslant 8$, and (c) v_j is adjacent to u_i, u_{i_1}, and u_{i_2} if and only if v_j is adjacent to u_i in G_2, where $0 \leqslant i \leqslant 8$ and $j = 1, 2$ and (II) adding a new vertex v_3 such that v_3 is adjacent to every vertex u_{i_1} for all $1 \leqslant i \leqslant 8$. This completes the construction of G_3 and certainly G_2 is a subgraph of G_3. Then $\Delta(G_3) = \deg u_0 = 26$. Since $W = \{v_1, v_2, v_3\}$ is a minimum independent resolving set of G_3, it follows that $\mathrm{ir}(G_3) = 3$. Repeating this procedure, we construct the graph G_k from G_{k-1} such that $\mathrm{ir}(G_k) = k$ and $\Delta(G_k) = 3^k - 1$.

3. Existence of independent resolving sets in some well-known graphs

In this section, we determine the existence of independent resolving sets in some well-known classes of graphs. Some additional definitions and notation are needed. A vertex of degree at least 3 in a graph G will be called a *major vertex*. An end-vertex u of G is said to be a *terminal vertex of a major vertex* v of G if $d(u,v) < d(u,w)$ for every other major vertex w of G. The *terminal degree* $\mathrm{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. Let $\sigma(G)$ denote the sum of the terminal degrees of the major vertices of G and let $\mathrm{ex}(G)$ denote the number of exterior major vertices of G. In fact, $\sigma(G)$ is the number of end-vertices of G. A connected graph with exactly one cycle is called a *unicyclic graph*. The graph $W_n = C_n + K_1$ is called the *wheel* of order $n+1$.

Theorem 3.1. *Let G be a connected graph of order $n \geqslant 3$.*

(a) *If G is a complete multipartite graph of order n, then $\mathrm{ir}(G)$ exists if and only if $G = K_{1,n-1}$. Furthermore, $\mathrm{ir}(K_{1,n-1}) = n-2$.*
(b) *If $G = C_n$ for $n \geqslant 5$, then $\mathrm{ir}(G) = 2$.*
(c) *If G is a tree, then $\mathrm{ir}(G) = 1$ if G is a path and $\mathrm{ir}(G) = \sigma(T) - \mathrm{ex}(T)$ otherwise.*
(d) *If G a unicyclic graph, then $\mathrm{ir}(G)$ exists.*
(e) *If $G = W_n$ for $n \geqslant 3$, then $\mathrm{ir}(W_n)$ does not exist for $3 \leqslant n \leqslant 5$, $\mathrm{ir}(W_6) = 3$, and $\mathrm{ir}(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for $n \geqslant 7$.*

Parts (a)–(d) in Theorem 3.1 are consequences of results from Section 2. Thus we will only verify (e) for $n \geqslant 7$. To do this, we need some additional definitions. In $W_n = C_n + K_1$, let $C_n\colon\ v_1,\ v_2, \ldots, v_n, v_1$ and let v be the central vertex of W_n. Let S be a set of two or more vertices of C_n, let v_i and v_j be two distinct vertices of S, and let P and P' denote the two distinct $v_i - v_j$ paths determined by C_n. If either P or P', say P, contains only two vertices of S (namely, v_i and v_j), then we refer to v_i and v_j as *neighboring vertices* of S and the set of vertices of P that belong to $C_n - \{v_i, v_j\}$ as the *gap* of S (determined by v_i and v_j). The two gaps of S determined by a vertex of S and its two neighboring vertices will be referred to as *neighboring gaps*. Consequently, if $|S| = r$, then S has r gaps, some of which may be empty. We first verify the following two claims.

Claim 1. Every ir-set W of W_n satisfies the following conditions (i)–(iii):

(i) Every gap of W contains at least one and at most three vertices of C_n.

(ii) At most one gap of W contains exactly three vertices.

(iii) If a gap of W contains at least two vertices, then any neighboring gap contains exactly one vertex.

Proof of Claim 1. Let W be an ir-set of W_n. Note that $|W| = \mathrm{ir}(W_n) \geqslant 3$ if $n \geqslant 7$. Since the central vertex v of W_n is adjacent to every other vertex of W_n, it follows that $v \notin W$. So W consists of vertices in C_n. If (i) is false, then either W is not independent, which is impossible, or there is a gap containing four consecutive vertices v_j, v_{j+1}, v_{j+2}, v_{j+3} of C_n, where $1 \leqslant j \leqslant n$ and addition is performed modulo n. In the latter case, $c_W(v_{j+1}) = c_W(v_{j+2}) = (2, 2, \ldots, 2)$, a contradiction. If (ii) is false, then there exist two distinct gaps $\{v_p, v_{p+1}, v_{p+2}\}$ and $\{v_q, v_{q+1}, v_{q+2}\}$. However, $c_W(v_{p+1}) = c_W(v_{q+1}) = (2, 2, \ldots, 2)$, a contradiction. If (iii) is false, then there exist five consecutive vertices v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}, of C_n such that v_{j+2} is the only vertex of W. However, $c_W(v_{j+1}) = c_W(v_{j+3})$, a contradiction. This completes the proof of Claim 1. $\square$

Claim 2. Any set of vertices of C_n that satisfies (i)–(iii) is a resolving set of W_n.

Proof of Claim 2. Let W be a set of vertices of C_n that satisfies (i)–(iii). We show that W is a resolving set of W_n. Let u be any vertex of $V(W_n) - W$. If $u = v$, $c_W(u) = (1, 1, \ldots, 1)$ and u is the only vertex of W_n with this code. Thus we may assume that $u \neq v$. There are three cases.

Case 1. Vertex u belongs to a gap of size 1 of W. Let v_i and v_j be the neighboring vertices of W that determine this gap. Then u is adjacent to v_i and v_j and has distance 2 to all other vertices of W. Since $n \geqslant 7$, no other vertices of W_n has this property and so $c_W(x) \neq c_W(u)$ for $x \neq u$.

Case 2. Vertex u belongs to a gap of size 2 of W. Then we may assume that v_j, $v_{j+1} = u$, v_{j+2}, v_{j+3} are vertices of C_n, where v_{j+1}, $v_{j+2} \notin W$ and v_j, $v_{j+3} \in W$. Then u is adjacent to v_j and has distance 2 from all other vertices of W. By property (iii), only u has this property and so $c_W(x) \neq c_W(u)$ for $x \neq u$.

Case 3. Vertex u belongs to a gap of size 3 of W. Then there exist vertices v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4} of C_n, only v_j and v_{j+4} of which belong to W. Assume first that $u = v_{j+1}$. Then u is adjacent to v_j and has distance 2 from all other vertices of W. By (iii), u is the only vertex of W_n with this property and so $c_W(x) \neq c_W(u)$ for $x \neq u$. Next, we assume that $u = v_{j+2}$. Then $c_W(u) = (2, 2, \ldots, 2)$. By properties (i) and (ii), no other vertex of of W_n has this representation. This completes the proof of Claim 2. $\square$

We are now prepared to prove Part (e) in Theorem 3.1 for $n \geqslant 7$.

Proof of Part (e) in Theorem 3.1 for $n \geqslant 7$. First we show that $\operatorname{ir}(W_n) \leqslant \lfloor \frac{2n+2}{5} \rfloor$ by constructing an independent resolving set W in W_n with $\lfloor \frac{2n+2}{5} \rfloor$ vertices.

(1) For $n \equiv 0 \pmod 5$, let $n = 5k$, where $k \geqslant 2$. Then $\lfloor \frac{2n+2}{5} \rfloor = 2k$. Then $W = \{v_{5i+1}, v_{5i+4} \colon 0 \leqslant i \leqslant k-1\}$ contains $2k$ vertices.

(2) For $n \equiv 1 \pmod 5$, let $n = 5k+1$, where $k \geqslant 2$. Therefore, $\lfloor \frac{2n+2}{5} \rfloor = 2k$. Then $W = \{v_{5i+1}, v_{5i+4} \colon 0 \leqslant i \leqslant k-2\} \cup \{v_{5k-4}, v_{5k}\}$ contains $2k$ vertices.

(3) For $n \equiv 2 \pmod 5$, let $n = 5k+2$, where $k \geqslant 1$. So $\lfloor \frac{2n+2}{5} \rfloor = 2k+1$. Then $W = \{v_{5i+1}, v_{5i+4} \colon 0 \leqslant i \leqslant k-1\} \cup \{v_{5k+1}\}$ contains $2k+1$ vertices.

(4) For $n \equiv 3 \pmod 5$, let $n = 5k+3$, where $k \geqslant 1$. In this case, $\lfloor \frac{2n+2}{5} \rfloor = 2k+1$. Then $W = \{v_{5i+1}, v_{5i+4} \colon 0 \leqslant i \leqslant k-2\} \cup \{v_{5k-4}, v_{5k}, v_{5k+2}\}$ contains $2k+1$ vertices.

(5) For $n \equiv 4 \pmod 5$, let $n = 5k+4$, where $k \geqslant 1$. Thus $\lfloor \frac{2n+2}{5} \rfloor = 2k+2$. Then $W = \{v_{5i+1}, v_{5i+4} \colon 0 \leqslant i \leqslant k-1\} \cup \{v_{5k+1}, v_{5k+3}\}$ contains $2k+2$ vertices.

In each case, W is independent and satisfies (i)–(iii). By Claim 2, W is an independent resolving set. Hence $\operatorname{ir}(W_n) \leqslant \lfloor \frac{2n+2}{5} \rfloor$.

It remains to show that $\operatorname{ir}(W_n) \geqslant \lfloor \frac{2n+2}{5} \rfloor$. Let W be an ir-set of W_n. We consider two cases.

Case 1. $|W| = 2\ell \geqslant 4$ for some integer ℓ. By (iii) in Claim 1 at most ℓ gaps of W contain one vertex and, by (i) and (ii) in Claim 1, all of them contain at most two vertices, except possibly one containing three vertices. So the number of vertices belonging to the gaps of W is at most $3\ell + 1$. Hence $n - 2\ell \leqslant 3\ell + 1$, which implies that $|W| = 2\ell \geqslant \lceil \frac{2}{5}n - \frac{2}{5} \rceil \geqslant \lfloor \frac{2n+2}{5} \rfloor$.

Case 2. $|W| = 2\ell + 1 \geqslant 3$ for some integer ℓ. By (iii) in Claim 1 at most ℓ gaps contain one vertex and, by (i) and (ii) in Claim 1, all contain at most two vertices except possibly one containing three vertices. So the number of vertices belonging to the gaps of W is at most $3\ell + 2$. Hence $n - 2\ell - 1 \leqslant 3\ell + 2$, which implies that $|W| = 2\ell + 1 \geqslant \lceil \frac{2}{5}n - \frac{6}{5} + 1 \rceil \geqslant \lfloor \frac{2n+2}{5} \rfloor$. $\square$

If G is a nontrivial connected graph of order n for which $\mathrm{ir}(G)$ exists, then by (1), $1 \leqslant \mathrm{ir}(G) \leqslant n-1$. The following result characterizes all nontrivial connected graphs G of order n for which $\mathrm{ir}(G) \in \{1, n-2, n-1\}$.

Theorem 4.1. *Let G be a nontrivial connected graph of order n for which $\mathrm{ir}(G)$ exists. Then*
(a) $\mathrm{ir}(G) = 1$ *if and only if* $G = P_n$,
(b) $\mathrm{ir}(G) = n-2$ *if and only if* $n \geqslant 3$ *and* $G = K_{1,n-1}$ *or* $n = 4$ *and* $G = (K_2 \cup K_1) + K_1$,
(c) $\mathrm{ir}(G) = n-1$ *if and only if* $n = 2$ *and* $G = K_2$.

Proof. Part (a) is an immediate consequence of the fact that P_n is the only connected graph of order n with dimension 1 by Theorem A.

For (b), it is straightforward to show that each graph G described in the theorem has order n and $\mathrm{ir}(G) = n-2$. To verify the converse, suppose that G is a nontrivial connected graph of order n such that $\mathrm{ir}(G) = n-2$ and that G is not a star. It is routine to show that $G = (K_2 \cup K_1) + K_1$ is the only connected graph of order $n \leqslant 4$ with $\mathrm{ir}(G) = n-2$. Thus, we may assume that $n \geqslant 5$. Let W be a minimum independent resolving set of G and let $V(G) - W = \{x, y\}$. Since W is independent and G is connected, every vertex in W is adjacent to at least one of x and y. Let W_1 be the set of vertices in W that are adjacent to x but not adjacent to y, let W_2 be the set of vertices in W that are adjacent to y but not adjacent to x, and let W_3 be the set of vertices in W that are adjacent to both x and y. Then $W = W_1 \cup W_2 \cup W_3$. Since $n \geqslant 5$, it follows that $|W| = n-2 \geqslant 3$. We consider four cases.

Case 1. $W = W_i$ for some $i \in \{1,2,3\}$. First, assume that $W = W_1$ or $W = W_2$, say $W = W_1$. Since G is connected, it follows that y is adjacent to x. This implies that G is a star with x as the central vertex, which is a contradiction. Therefore, $W = W_3$. Since $d(x,w) = d(y,w) = 1$ for all $w \in W = V(G) - \{x,y\}$, it follows that $\{x,y\}$ is a distance similar equivalence class. By Observation 2.1, W must contain at least one of x and y, a contradiction.

Case 2. $W = W_1 \cup W_2$ and $W_i \neq \emptyset$ for $i = 1,2$. Since W is independent and G is connected, it follows that x is adjacent to y. This implies that G is a double star with central vertices x and y. Since the order of G is at least 5, at least one of W_1 and W_2 contains two or more vertices, say $|W_1| \geqslant 2$. Let $u \in W_1$. Then $W' = W - \{u\}$ is an independent resolving set and so $\mathrm{ir}(G) \leqslant |W'| = n-3$, which is impossible.

C a s e 3. $W = W_i \cup W_3$, where $W_i \neq \emptyset$ for $i = 1, 2$ and $W_3 \neq \emptyset$. Assume, without loss of generality, that $W = W_1 \cup W_3$, where $W_1 \neq \emptyset$ and $W_3 \neq \emptyset$. Let $u \in W_1$, $v \in W_3$, and let $w \in W - \{u, v\}$. If $xy \notin E(G)$, then let $W' = W - \{v\}$. Since $d(x, u) = 1$, $d(y, u) = 3$, and $d(v, u) = 2$, it follows that W' is an independent resolving set of G of cardinality $n - 3$, which is impossible. If $xy \in E(G)$, then let $W'' = W - \{w\}$. Since (1) $d(x, u) = d(x, v) = 1$, (2) $d(y, u) = 2$ and $d(y, v) = 1$, and (3) $d(w, u) = d(w, v) = 2$, it follows that W'' is an independent resolving set of G of cardinality $n - 3$, a contradiction.

C a s e 4. $W = W_1 \cup W_2 \cup W_3$ and $W_i \neq \emptyset$ for $i = 1, 2, 3$. Let $u \in W_1$, $v \in W_2$ and let $W' = W - \{v\}$. If $xy \notin E(G)$, then $d(x, u) = 1$, $d(y, u) = 3$, and $d(v, u) = 4$; while if $xy \in E(G)$, then $d(x, u) = 1$, $d(y, u) = 2$, and $d(v, u) = 3$. In either case, W' is an independent resolving set of G of cardinality $n - 3$, a contradiction.

Therefore, for $n \geqslant 5$, the star $K_{1,n-1}$ is the only connected graph of order n with $\operatorname{ir}(G) = n - 2$ and so (b) holds.

For part (c), it is clear that $\operatorname{ir}(K_2) = 1$. For the converse, let G be a connected graph of order n with $\operatorname{ir}(G) = n - 1$. Then $\beta(G) = n - 1$ by (1) and so $G = K_{1,n-1}$. By (b), if $n \geqslant 3$, then $\operatorname{ir}(K_{1,n-1}) = n - 2$. Therefore, $n = 2$ and $G = K_2$. □

By Theorems 3.1 and 4.1, we are able to determine all pairs k, n of positive integers with $k \leqslant n$ that are realizable as the independent resolving number and the order of some connected graph. We omit the routine proof of the next result.

Theorem 4.2. *For each pair k, n of positive integers with $k \leqslant n$, there exists a connected graph G of order n with $\operatorname{ir}(G) = k$ if and only if $(k, n) = (1, 2)$ or $1 \leqslant k \leqslant n - 2$.*

Next, we show that certain pairs a, b are realizable as the dimension and the independent resolving number of some connected graph.

Theorem 4.3. *For every pair a, b of integers with $2 \leqslant a \leqslant b \leqslant \lfloor \frac{3}{2} a \rfloor$, there exists a connected graph G such that $\dim(G) = a$ and $\operatorname{ir}(G) = b$.*

P r o o f. For $a = b$, let $G = K_{1,a+1}$. Then $\dim(G) = \operatorname{ir}(G) = a$. Thus we may assume that $a < b$. Since $b \leqslant \lfloor \frac{3}{2} a \rfloor$, it follows that $3a - 2b + 1 \geqslant 1$. For each integer j with $1 \leqslant j \leqslant b - a$, let H_j be the graph of Figure 5. Then the graph G is obtained from the graphs H_j $(1 \leqslant j \leqslant b - a)$ by (1) identifying the $b - a$ vertices v_{j0} $(1 \leqslant j \leqslant b - a)$ and labeling the identified vertex v_0 and (2) adding the $3a - 2b + 1$ new vertices $x_1, x_2, \ldots, x_{3a-2b+1}$ and joining each vertex x_i $(1 \leqslant i \leqslant 3a - 2b + 1)$ to v_0. Let $X = \{x_1, x_2, \ldots, x_{3a-2b+1}\}$. We show that $\dim(G) = a$ and $\operatorname{ir}(G) = b$.

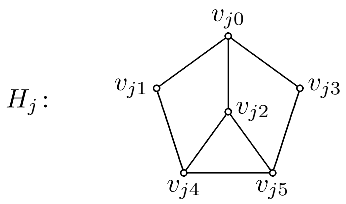

Figure 5. The graph H_j

First, we show that $\dim(G) = a$. Since $W = \{v_{j4}, v_{j5}\colon\ 1 \leqslant j \leqslant b-a\} \cup (X - \{x_1\})$ is a resolving set of G, it follows that $\dim(G) \leqslant |W| = 2(b-a) + (3a-2b) = a$. To show that $\dim(G) \geqslant a$, we verify the following claim. □

Claim 1. Every resolving set of G contains at least two vertices from each set

$$V_j = V(H_j) - \{v_0\} = \{v_{j1}, v_{j2}, v_{j3}, v_{j4}, v_{j5}\}$$

for $1 \leqslant j \leqslant b-a$.

P r o o f o f C l a i m 1. Assume, to the contrary, there exists a resolving set W of G such that W contains at most one vertex in V_j for some j with $1 \leqslant j \leqslant b-a$, say $j = 1$. Note that if u and u' are two distinct vertices of V_1 with $d(u, v_0) = d(u', v_0)$, then $d(u, v) = d(u', v)$ for all $v \in V(G) - V_1$. Since $d(v_{11}, v_0) = d(v_{12}, v_0) = d(v_{13}, v_0)$ and $d(v_{14}, v_0) = d(v_{15}, v_0)$, it follows that W must contain at least one vertex in V_1. So W contains exactly one vertex in V_1. We consider three cases.

C a s e 1. Vertex $v_{11} \in W$ or $v_{13} \in W$, say the former. Since $d(v_{12}, v_{11}) = 2 = d(v_{13}, v_{11})$ and $d(v_{12}, v) = d(v_{13}, v)$ for all $v \in V(G) - V_1$, it follows that $c_W(v_{12}) = c_W(v_{13})$.

C a s e 2. Vertex $v_{14} \in W$ or $v_{15} \in W$, say the former. Since $d(v_{11}, v_{14}) = 1 = d(v_{12}, v_{14})$ and $d(v_{11}, v) = d(v_{12}, v)$ for all $v \in V(G) - V_1$, it follows that $c_W(v_{11}) = c_W(v_{12})$.

C a s e 3. Vertex $v_{12} \in W$. Since $d(v_{14}, v_{12}) = 1 = d(v_{15}, v_{12})$ and $d(v_{14}, v) = d(v_{15}, v)$ for all $v \in V(G) - V_1$, it follows that $c_W(v_{14}) = c_W(v_{15})$.

In each case, W is not a resolving set of G, a contradiction. Therefore, every resolving set of G contains at least two vertices in $V(H_j) - \{v_0\}$ for $1 \leqslant j \leqslant b-a$. This completes the proof of Claim 1. □

By Claim 1, every basis of G must contain at least two vertices from each set V_j for $1 \leqslant j \leqslant b-a$. Moreover, by Observation 2.1, every basis of G contains at least $3a-2b$ vertices from X. It follows that $\dim(G) \geqslant 2(b-a) + (3a-2b) = a$. Therefore, $\dim(G) = a$.

Next, we show that $\operatorname{ir}(G) = b$. Since $W_0 = \{v_{j1},\, v_{j2},\, v_{j3}\colon\ 1 \leqslant j \leqslant b-a\} \cup (X - \{x_1\})$ is an independent resolving set, $\operatorname{ir}(G) \leqslant |W_0| = 3(b-2) + (3a-2b) = b$. In order to show that $\operatorname{ir}(G) \geqslant b$, we first verify the following claim.

Claim 2. No ir-set of G contains any vertex in $\{v_0, v_{j4},\, v_{j5}\colon\ 1 \leqslant j \leqslant b-a\}$.

Proof of Claim 2. We first show that no ir-set of G contains any vertex in $\{v_{j4}, v_{j5}\}$ for $1 \leqslant j \leqslant b-a$. Assume, to the contrary, that there exists an ir-set W of G such that W contains at least one vertex in $\{v_{j4}, v_{j5}\}$ for some j with $1 \leqslant j \leqslant b-a$, say $j = 1$. Since W is independent, W contains exactly one vertex in $\{v_{14}, v_{15}\}$, say $v_{14} \in W$. Since v_{11}, v_{12}, and v_{15} are adjacent to v_{14}, it follows that $v_{11}, v_{12}, v_{15} \notin W$. By Claim 1 then, $v_{13}, v_{14} \in W$. Since (1) $d(v_{11}, v_{13}) = 2 = d(v_{12}, v_{13})$, (2) $d(v_{11}, v_{14}) = 1 = d(v_{12}, v_{14})$, and (3) $d(v_{11}, v) = d(v_{12}, v)$ for all $v \in V(G) - V_1$, it follows that $c_W(v_{11}) = c_W(v_{12})$, a contradiction. Therefore, no ir-set of G contains any vertex in $\{v_{j4}, v_{j5}\}$ for $1 \leqslant j \leqslant b-a$. Furthermore, the vertex v_0 is adjacent to v_{11}, v_{12}, v_{13} in G and W must contain at least two of the three vertices v_{11}, v_{12}, v_{13}, which is impossible. Therefore, no ir-set of G contains v_0 and the proof of Claim 2 is complete. □

We now continue to show that $\operatorname{ir}(G) \geqslant b$. Assume, to the contrary, that $\operatorname{ir}(G) \leqslant b-1$. Let W' be an ir-set of G. Then $|W'| \leqslant b-1$. By Claim 1, the set W' contains at least two vertices in each set $\{v_{j1}, v_{j2}, v_{j3}, v_{j4}, v_{j5}\}$ for $1 \leqslant j \leqslant b-a$. By Claim 2, neither v_{j4} nor v_{j5} belongs to W' for $1 \leqslant j \leqslant b-a$. Also, by Observation 2.1, the set W' contains at least $3a-2b$ elements in X. Now let $T = X \cup \{v_{j1},\, v_{j2},\, v_{j3}\colon\ 1 \leqslant j \leqslant b-a\}$. Then $W' \subset T$. Since $|W'| \leqslant b-1$ and $|T| = b+1$, it follows that $|T - W'| \geqslant 2$. However, if $u_1, u_2 \in T - W'$, then $d(u_1, v) = d(u_2, v) = 2$ for all $v \in T$ and so $c_{W'}(u_1) = c_{W'}(u_2)$, which is a contradiction. Therefore, $\operatorname{ir}(G) = b$.

By Theorem 4.3 every pair a, b of positive integers with $2 \leqslant a \leqslant b \leqslant \lfloor \frac{3}{2}a \rfloor$ is realizable as the dimension and the independent resolving number of some connected graph. Furthermore, for each pair a, b of positive integers with $4 \leqslant a \leqslant b$, it can be shown that (1) there exists a connected graph F with $\dim(F) = \operatorname{ir}(F) = a$ and $\beta(F) = b$, (2) there exists a connected graph G with $\dim(G) = a$ and $\beta(G) = b$ such that $\dim(G) \neq \operatorname{ir}(G)$, and (3) there exists a connected graph H with $\operatorname{ir}(H) = a$ and $\beta(H) = b$ such that $\dim(H) \neq \operatorname{ir}(H)$. However, we do not have a complete solution for the following problem.

Problem 4.4. For which triples a, b, c of positive integers with $2 \leqslant a \leqslant b \leqslant c$, does there exist a connected graph G such that $\dim(G) = a$, $\operatorname{ir}(G) = b$, and $\beta(G) = c$?

To conclude this paper, we construct, for each pair k, r of integers with $k \geqslant 2$ and $0 \leqslant r \leqslant k$, a connected graph G with $\operatorname{ir}(G) = k$ such that exactly r vertices belong to every ir-set of G.

Theorem 4.5. *For every pair r, k of integers with $k \geqslant 2$ and $0 \leqslant r \leqslant k$, there exists a connected graph G with $\mathrm{ir}(G) = k$ such that exactly r vertices belong to every ir-set of G.*

P r o o f. For $r = 0$, let $G = K_{1,k+1}$. Since every ir-set of G consists of any k end-vertices of G, it follows that no vertex of G belongs to every ir-set of G. For $r = 1$, let G be obtained from $K_4 - e$, where $V(K_4 - e) = \{u_1, u_2, u_3, u_4\}$ and $e = u_1u_3$, by adding the k new vertices $v_1, v_2, \ldots, v_k$ and joining each vertex v_i $(1 \leqslant i \leqslant k)$ to u_2 and u_3. Then every ir-set of G consists of the vertex u_1 and any $k - 1$ vertices from the set $\{v_1, v_2, \ldots, v_k\}$. Thus u_1 is the only vertex that belongs to every ir-set of G.

Now let $2 \leqslant r \leqslant k$. First, we construct a graph F of order $r + 2^r$ with $V(F) = U \cup W$, where $U = \{u_0, u_1, \ldots, u_{2^r-1}\}$ and the ordered set $W = \{w_{r-1}, w_{r-2}, \ldots, w_0\}$ are disjoint. The induced subgraph $\langle U \rangle$ of F is complete, while W is independent. To define the adjacencies between W and U, let each integer j $(0 \leqslant j \leqslant 2^r - 1)$ be expressed in its base 2 (binary) representation. Thus, each such j can be expressed as a sequence of r coordinates, that is, an r-vector, where the rightmost coordinate represents the value (either 0 or 1) in the 2^0 position, the coordinate to its immediate left is the value in the 2^1 position, etc. For integers i and j, with $0 \leqslant i \leqslant r - 1$ and $0 \leqslant j \leqslant 2^r - 1$, we join w_i and u_j if and only if the value in the 2^i position in the binary representation of j is 1. This completes the construction of F. Then the graph G is obtained from F by adding $k - r$ copies $u_{01}, u_{02}, \ldots, u_{0,k-r}$ of u_0 and joining each of the $k - r$ vertices $u_{01}, u_{02}, \ldots, u_{0,k-r}$ to every neighbor of u_0 in F. Let $U_0 = \{u_{01}, u_{02}, \ldots, u_{0,k-r}\}$. Then the set $U_0 \cup \{u_0\}$ is an independent set in G and each of vertices in U_0 has the same neighborhood as that of u_0 in G. For $r = 3$ and $k = 5$, the edges joining W and $U \cup U_0$ in the graph G just constructed are shown in Figure 6.

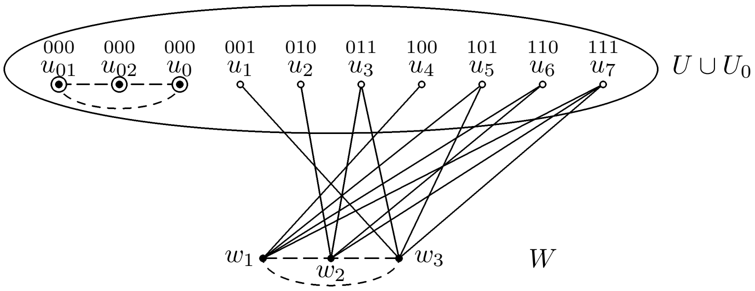

Figure 6. The graph G for $r = 3$ and $k = 5$

Notice that (1) every two vertices of U are adjacent, (2) every vertex in U_0 is adjacent to every vertex in $U - \{u_0\}$, (3) there is no edge between any two vertices in $U_0 \cup \{u_0\}$, and (4) there is no edge between any two vertices in W. By an extensive case-by-case analysis, it can be shown that every ir-set consists of W and any $k-r$ vertices of $U_0 \cup \{u_0\}$. Therefore, exactly r vertices in G, namely the r vertices in W, belong to every ir-set of G. $\square$

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