ABOUT DIFFERENTIAL INEQUALITIES FOR NONLOCAL BOUNDARY VALUE PROBLEMS WITH IMPULSIVE DELAY EQUATIONS

Alexander Domoshnitsky, Irina Volinsky, Ariel

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Abstract. We propose results about sign-constancy of Green's functions to impulsive nonlocal boundary value problems in a form of theorems about differential inequalities. One of the ideas of our approach is to construct Green's functions of boundary value problems for simple auxiliary differential equations with impulses. Careful analysis of these Green's functions allows us to get conclusions about the sign-constancy of Green's functions to given functional differential boundary value problems, using the technique of theorems about differential and integral inequalities and estimates of spectral radii of the corresponding compact operators in the space of essential bounded functions.

Keywords: impulsive equation; nonlocal boundary value problem; Green's function; positivity of Green's function; negativity of Green's function; estimates of solutions

MSC 2010: 34K45, 34K06, 34K10, 34K12, 34K38

1. Introduction

Various mathematical models with impulsive differential equations attract to this topic attention of many authors [3], [16], [19], where various results on boundary value problems and stability of these equations were presented. One of possible approaches to study impulsive equations is the theory of generalized differential equations allowing researchers to consider systems with continuous as well as systems with discontiniuos solutions and discrete systems in the frame of the same theory [2], [14]. In this paper we use the concept of the approach proposed in the monograph [3], and developed then, for example, in the papers [6]–[10].

Various comparison theorems for solutions of the Cauchy and periodic problems for ordinary differential equations with impulses have been obtained in [7], [17], [18].

On the basis of the comparison theorems, tests of stability are proved in [1]. Theory of impulsive differential equations and inclusions was presented in the book [4].

Nonlocal problems have naturally appeared in mathematical models of many processes in applications. For non-impulsive functional differential equations nonlocal problems were considered in Chapter 15 of the book [1]. Existence results for nonlocal boundary value problems with impulsive equations were studied in the papers [5], [13]. There are almost no results on sign-constancy of Green's function for impulsive boundary value problems. Concerning nonlocal impulsive boundary value problems as far as we know, there are no results about positivity/negativity of Green's function.

In this paper we propose results about differential inequalities for sign-constancy of Green's functions of nonlocal impulsive boundary value problems.

2. Main results

We consider the impulsive equation

$$(Lx)(t) = x'(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i(t)) = f(t), \quad t \in [a,b] \tag{2.1}$$

$$x(t_j) = \beta_j x(t_j - 0), \quad j = 1, \ldots, k, \tag{2.2}$$

$$x(\zeta) = 0, \quad \zeta \notin [a,b], \tag{2.3}$$

where $f \in L_\infty$ is a measurable essentially bounded function and $\tau_i \geqslant 0$, $i = 1, \ldots, m$ are measurable functions such that $\operatorname{mes}\{t\colon\ t - \tau_i(t) = \text{const}\} = 0$ for $i = 1, \ldots, m$, $\beta_j > 0$, $j = 1, \ldots, k$, $a = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} = b$.

Consider the following variants of boundary conditions:

$$lx = \int_a^b \varphi(s)x'(s)\,\mathrm{d}s + \theta x(a) = c, \tag{2.4}$$

where $\varphi \in L_\infty[a,b]$; $\theta, c \in \mathbb{R}$;

$$x(a) = c, \tag{2.5}$$

$$x(b) = c, \tag{2.6}$$

$$x(a) = x(b). \tag{2.7}$$

Solving step-by-step on every of the intervals $[a,t_1), [t_1,t_2), \ldots, [t_k,b)$ the initial value problem $x'(t) = z(t)$, $x(a) = \alpha$, $t \in [a,b]$, with condition (2.2), where $z \in L_\infty$, $\alpha \in \mathbb{R}$ we obtain

$$x(t) = \int_a^t \Omega(t,s)z(s)\,\mathrm{d}s + \omega(t)\alpha, \tag{2.8}$$

where

$$\omega(t)=\sum_{i=1}^{k+1}\chi_{[t_{i-1},t_i)}(t)\prod_{j=1}^{i}\beta_{i-j},$$

$$\Omega(t,s)=\sum_{i=1}^{k+1}\chi_{[t_{i-1},t_i)}(t)\chi_{[t_{i-1},t_i)}(s)\beta_0+\sum_{i=2}^{k+1}\sum_{r=1}^{i-1}\chi_{[t_{i-1},t_i)}(t)\chi_{[t_{i-1},t_i)}(s)\prod_{j=1}^{i-r}\beta_{i-j};$$

here $\beta_0=1$.

It is clear that $x(t)$ is absolutely continuous in (t_{i-1},t_i), $i=1,\dots,k+1$, satisfying the equality $x(t_i)=\beta_i x(t_i-0)$. We see that $x(t)$ has only discontinuities of the first kind and is continuous from the right at the points t_i, $i=1,\dots,k$.

We can consider the equality (2.8) as a definition of the space $D(t_1,\dots,t_k)$ of piecewise continuous functions $x\colon [a,b]\to\mathbb{R}$. It is clear that this space is isomorphic to the topological product $L_\infty\times\mathbb{R}$. Actually for every pair (z,α) where $z\in L_\infty$, $\alpha\in\mathbb{R}$ we obtain by (2.8) the unique $x\in D(t_1,\dots,t_k)$, and every solution of equation (2.1)–(2.3) can be written in the form (2.8).

Let us consider the auxiliary equation

$$\begin{aligned}x'(t)&=z(t),\quad t\in[a,b],\\ x(t_j)&=\beta_j x(t_j-0),\quad j=1,\dots,k.\end{aligned}\tag{2.9}$$

In [7] it was proved that the general solution of equations (2.1)–(2.3) can be represented in the form

$$x(t)=\int_a^t C(t,s)f(s)\,\mathrm{d}s+C(t,a)x(a),\tag{2.10}$$

where $C(t,s)$ is called the Cauchy function of equation (2.1)–(2.3). For each s fixed, the function $x(t)=C(t,s)$ satisfies equations (2.1), (2.2) and $x(\zeta)=0$ for $\zeta<s$. In particular, $C(t,s)=0$ for $t<s$. In the case when problem (2.1)–(2.4) is uniquely solvable, its Green's function $G(t,s)$ is of the form

$$G(t,s)=C(t,s)-C(t,a)\frac{\int_s^b\varphi(w)C'_w(w,s)\,\mathrm{d}w+\varphi(s)}{\theta+\int_a^b\varphi(w)C'_w(w,a)\,\mathrm{d}w}.\tag{2.11}$$

This can be proved when we insert the solution representation (2.10) into the condition (2.4) and determine the proper value of the parameter α.

Green's functions for various impulsive boundary value problems were constructed and conditions ensuring their positivity were discussed in [12]. Existence of Green's function $G_0(t,s)$ for problem (2.9), (2.4) was discussed in [12]. On the basis of estimates of Green's function $G_0(t,s)$ of problem (2.9), (2.4), sufficient conditions of positivity of Green's function $G(t,s)$ of nonlocal boundary value problems of the type (2.1)–(2.4) were obtained in [11]. In this paper we propose theorems about differential inequalities that allow us to obtain results about positivity/negativity of Green's function of the nonlocal boundary value problem (2.1)–(2.4) based only on sign-constancy of $G_0(t,s)$ and without knowledge of the explicit formula for $G_0(t,s)$.

Define the operator $K\colon L_\infty \to L_\infty$ by the equality

$$(Kz)(t) = -\sum_{i=1}^{m} p_i(t)\chi(t-\tau_i(t),0)\int_a^b G_0(t-\tau_i(t),s)z(s)\,\mathrm{d}s, \tag{2.12}$$

where

$$\chi(t,s) = \begin{cases} 1 & \text{for } t \geqslant s,\\ 0 & \text{for } t < s. \end{cases}$$

It is clear that K is a positive operator whenever $G_0(t,s) \leqslant 0$ and $p_i(t) \geqslant 0$ for $t,s \in [a,b]$ and $i = 1,2,\ldots,m$.

Theorem 2.1. *Let $G_0(t,s) \leqslant 0$ for $t,s \in [a,b]$, while $G_0(t,s) < 0$ if $a \leqslant t < s \leqslant b$, $p_i(t) \geqslant 0$ for $i = 1,\ldots,m$ and $\theta \neq 0$. Then the following assertions are equivalent:*

(1) *There exists a positive function $v \in D(t_1,\ldots,t_k)$ such that $v'(t) \leqslant \varepsilon < 0$, $v(a) + \theta^{-1}\int_a^b \varphi(s)v'(s)\,\mathrm{d}s \geqslant 0$, and $(Lv)(t) \leqslant -\varepsilon < 0$ for $t \in [a,b]$.*
(2) *The spectral radius of the operator K is less than one.*
(3) *$G(t,s) \leqslant 0$ for $t,s \in [a,b]$, and $G(t,s) < 0$ for $a \leqslant t < s \leqslant b$.*
(4) *There exists a positive function $z \in L_\infty$ such that $z(t) - Kz(t) \geqslant \varepsilon > 0$.*

P r o o f. (1) $\Rightarrow$ (4) Let $v(t)$ be a function satisfying assertion (1). We can set $u(t) = -v(t)$ and choose $u'(t) = z(t)$. We have

$$u(t) = \int_a^b G_0(t,s)z(s)\,\mathrm{d}s + \left[u(a) + \frac{1}{\theta}\int_a^b \varphi(s)u'(s)\,\mathrm{d}s\right]\prod_{i=0}^{j}\beta_i,$$

where $t_j \leqslant t < t_{j+1},\ j = 0,1,\ldots,k$.

It is clear that $z(t) = u'(t) \geqslant \varepsilon > 0$ and $z(t) - (Kz)(t) = \psi(t)$, where

$$\psi(t) = -\left\{\sum_{i=1}^{m} p_i(t)\chi(t-\tau_i(t),0)\left[u(a) + \frac{1}{\theta}\int_a^b \varphi(s)u'(s)\,\mathrm{d}s\right]\prod_{i=0}^{j}\beta_i\right\} + (Lu)(t),$$

where $t_j \leqslant t < t_{j+1}$, $j = 0,1,\ldots,k$.

According to our condition, $\psi(t) \geqslant \varepsilon > 0$.

The implication (1) $\Rightarrow$ (4) is proved.

(2) $\Rightarrow$ (3) Assuming in (2.4) $c = 0$ and substituting $x(t) = \int_a^b G_0(t,s)z(s)\,\mathrm{d}s$, we get that the solution of the problem (2.1)–(2.4), where $c = 0$, can be represented in the form

$$x(t) = \int_a^b G_0(t,s)(I-K)^{-1}f(s)\,\mathrm{d}s.$$

The operator K is positive and by (2) its spectral radius is less than 1. Hence, using the condition about the spectral radius, we get

$$x(t) = \int_a^b G_0(t,s)\{f(s) + Kf(s) + K^2 f(s) + \ldots\}\,\mathrm{d}s. \tag{2.13}$$

It is clear that for every nonpositive f we get $x(t) - \int_a^b G_0(t,s)f(s)\,\mathrm{d}s \geqslant 0$ and consequently

$$0 \leqslant \int_a^b G(t,s)f(s)\,\mathrm{d}s - \int_a^b G_0(t,s)f(s)\,\mathrm{d}s = \int_a^b [G(t,s) - G_0(t,s)]f(s)\,\mathrm{d}s$$

and $G(t,s) - G_0(t,s) \leqslant 0$ for $t, s \in (a,b)$.

The implication (2) $\Rightarrow$ (3) is proved.

(3) $\Rightarrow$ (1) Let us set $v(t) = -\int_a^b G(t,s)\,\mathrm{d}s$. It is clear that $(Lv)(t) = -1,\ t \in [a,b]$.

The implication (3) $\Rightarrow$ (1) is proved.

(4) $\Rightarrow$ (2) This implication follows theorem in the paragraph 5.7 of the book [15], page 87.

Theorem 2.1 is proved. □

Definition 2.1. *We say that problem* (2.1)–(2.4) *satisfies the condition* Θ *if*

$$\frac{\int_s^b \varphi(\xi)C'_\xi(\xi,s)\,\mathrm{d}\xi + \varphi(s)}{\theta + \int_a^b \varphi(s)C'_s(s,a)\,\mathrm{d}s} < 0. \tag{2.14}$$

Let us assume existence of Green's function $P_0(t,s)$ of the problem (2.9), (2.6), Green's function $P(t,s)$ of the problem (2.1)–(2.3), (2.6), and Green's function $P_1(t,s)$ of the problem (2.1)–(2.3), (2.7).

Define the operator $M\colon L_\infty \to L_\infty$ by the equality

$$(Mx)(t) = -\sum_{i=1}^m p_i(t)\chi(t-\tau_i(t),0)\int_a^b P_0(t-\tau_i(t),s)z(s)\,\mathrm{d}s.$$

Theorem 2.2. *Let $p_i(t) \geqslant 0$ for $i = 1, \dots, m$. Then the following assertions are equivalent*

(1) *The Cauchy function of* (2.1)–(2.3) *is positive for $a \leqslant s \leqslant t \leqslant b$.*

(2) *A nontrivial solution of the homogeneous equation $(Lx)(t) = 0$,* (2.2), (2.3) *has no zeros on $[a, b]$.*

(3) *The spectral radius of the operator M is less than one.*

(4) *Problem* (2.1)–(2.3), (2.6) *is uniquely solvable for every $f \in L$ and its Green's function $P(t, s)$ is negative for $a \leqslant t < s \leqslant b$ and nonpositive for $a \leqslant s \leqslant t \leqslant b$.*

(5) *If in addition $\beta_1 < 1, \dots, \beta_k < 1$, then periodic problem* (2.1)–(2.3), (2.7) *is uniquely solvable and its Green's function $P_1(t, s)$ is positive for $t, s \in [a, b]$.*

(6) *There exists a nonnegative function $v \in D(t_1, \dots, t_k)$ such that $(Lv)(t) \leqslant 0$, $v(b) - \int_t^b (Lv)(s)\,\mathrm{d}s > 0$, $t \in [a, b]$.*

(7) *If in addition $\beta_1 < 1, \dots, \beta_k < 1$, and the Θ condition is fulfilled, then Green's function of the problem* (2.1)–(2.4) *satisfies $G(t, s) > 0$, for $t, s \in [a, b]$.*

The equivalence of assertions (1)–(6) was shown in [7]. To prove equivalence of the remaining assertions we prove the implications (1) $\Rightarrow$ (7) and (7) $\Rightarrow$ (2).

P r o o f. (1) $\Rightarrow$ (7) It is clear from conditions (2.11) and (2.14) that $G(t, s) > 0$ for $t, s \in [a, b]$.

(7) $\Rightarrow$ (2) For $t < s$ we have $C(t, s) = 0$ and consequently

$$G(t, s) = -C(t, a) \frac{\int_s^b \varphi(w) C'_w(w, s)\,\mathrm{d}w + \varphi(s)}{\theta + \int_a^b \varphi(w) C'_w(w, a)\,\mathrm{d}w}.$$

Now, due to the assumption Θ and since an arbitrary nontrivial solution x of the homogeneous problem $Lx = 0$, (2.2), (2.3) is of the form $x(t) = C(t, a)x(a)$ with $x(a) \neq 0$, it follows that assertion (2) is true.

Theorem 2.2 is proved. $\square$

Denote

$$d_-(t) = \min\{j\colon\ t_j \in (t - \tau_1(t), t)\},$$
$$d_+(t) = \max\{j\colon\ t_j \in (t - \tau_1(t), t)\},$$
$$B(t) = \prod_{j=d_-(t)}^{d_+(t)} \beta_j.$$

Corollary 2.3. *Let $m = 1$, $p_1(t) \geqslant 0$, $\int_{t-\tau_1(t)}^{t} p_1(s)\,\mathrm{d}s \leqslant (1+\ln B(t))/\mathrm{e}$ and the condition Θ be fulfilled. Then the Green's function $G(t,s)$ of the problem* (2.1)–(2.4) *is positive for $t, s \in [a,b]$.*

P r o o f. To prove the corollary we set

$$v(t) = \begin{cases} \exp\Bigl(-\mathrm{e}\displaystyle\int_a^t p_1(s)\,\mathrm{d}s\Bigr), & a \leqslant t \leqslant t_1, \\ \beta_1 \exp\Bigl(-\mathrm{e}\displaystyle\int_a^t p_1(s)\,\mathrm{d}s\Bigr), & t_1 \leqslant t \leqslant t_2, \\ \quad\vdots & \\ \beta_1\beta_2\ldots\beta_k \exp\Bigl(-\mathrm{e}\displaystyle\int_a^t p_1(s)\,\mathrm{d}s\Bigr), & t_k \leqslant t \leqslant b, \end{cases}$$

in assertion (6) of Theorem 2.2. □

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Authors' address: *Alexander Domoshnitsky*, *Irina Volinsky*, Department of Mathematics and Computer Science, Ariel University, The Ariel University Center of Samaria, 448 37 Ariel, Israel, e-mail: `adom@ariel.ac.il`, `irinav@ariel.ac.il`.