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ON CALCULATION OF ZETA FUNCTION OF INTEGRAL MATRIX

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Abstract. Values of the Epstein zeta function of a positive definite matrix and the knowledge of matrices with minimal values of the Epstein zeta function are important in various mathematical disciplines. Analytic expressions for the matrix theta functions of integral matrices can be used for evaluation of the Epstein zeta function of matrices. As an example, principal coefficients in asymptotic expansions of variance of the lattice point count in the random ball are calculated for some lattices.

Keywords: Epstein zeta function, Riemann theta function, variance of volume estimate, Rankin-Sobolev problem

MSC 2000: 33F05, 60D05

1. Introduction

Let $M \in \mathbb{R}^{d\times d}$ be a positive definite matrix and let $\mathbf{T} = M^{-1/2}\mathbb{Z}^d$ be the corresponding point lattice. The sum of the powers of the lattice point norms

$$Z(M,s) = \sum_{0\neq x\in\mathbb{Z}^d} (x'Mx)^{-s/2},$$

convergent for $\operatorname{Re} s > d$, is the Epstein zeta function of the matrix, which is of considerable importance in various fields of mathematics. For example, the error of the d-dimensional numerical integration with the point lattice $\mathbf{T}$ for functions in the unit ball of the Sobolev space of $\mathbf{T}$-periodic functions is proportional to the zeta function of the matrix M [6]. The volume of a d-dimensional random hypersphere B_d may be estimated by counting the lattice points in B_d [4]. The matrix zeta function is then included in the asymptotic term of the estimate variance, where the asymptotics relates to homothetic transforms of the lattice by a scaling factor $u \to 0+$. A similar asymptotic expansion applies also to many other randomly positioned bounded sets in $\mathbb{R}^d$ [3].

Especially interesting are matrices that have the least value of $Z(M,s)$ (are optimal) for real values $s > d$ among all matrices with the same determinant. Rankin [5] proved such optimality for triangular matrices. It follows from [2] that matrices critical for $s > d$ are (proportional to) integral matrices and that there is only a finite number of critical matrices in any dimension. The optimality of the zeta function changes to the sphere packing problem [1] when $s \to +\infty$.

Many properties of integral matrices can be deduced from their theta series. Analytic expressions for the theta function of many integral matrices are known [1]. The aim of this paper is the calculation of the Epstein zeta function of such integral matrices.

2. An expression for the matrix zeta function

The Riemann theta function of a positive definite matrix $M \in \mathbb{R}^{d\times d}$ is

$$\Theta(M,\tau) = \sum_{x\in\mathbb{Z}}^{d} \mathrm{e}^{\pi\mathrm{i}\tau x' M x}.$$

From $\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) r^{-s} = \int_0^\infty \mathrm{e}^{-r^2\pi t} t^{s/2-1}\,\mathrm{d}t$, where Γ is the Euler gamma function, we have for $\operatorname{Re} s > d$ (by the Mellin transform)

$$\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big) Z(M,s) = \int_0^\infty \left(\Theta\left(M,\mathrm{i}t\right) - 1\right) t^{s/2-1}\,\mathrm{d}t$$

and the Poisson summation formula (or the Jacobi identity for Θ)

$$\Theta\left(M,\mathrm{i}t\right) = |M|^{-1/2} t^{-d/2}\Theta\left(M,\mathrm{i}t^{-1}\right)$$

then yields

$$\begin{aligned}
&\int_0^1 (\Theta(M,\mathrm{i}t) - 1) t^{s/2-1}\,\mathrm{d}t \\
&\quad = |M|^{-1/2}\int_0^1 \left(\Theta\left(M,\mathrm{i}t^{-1}\right) - 1\right) t^{(s-d)/2-1}\,\mathrm{d}t - \frac{2}{s} + \frac{2}{s-d}|M|^{-1/2} \\
&\quad = \frac{2}{s-d}|M|^{-1/2} - \frac{2}{s} + |M|^{-1/2}\int_1^\infty \left(\Theta\left(M,\mathrm{i}t\right) - 1\right) t^{(d-s)/2-1}\,\mathrm{d}t.
\end{aligned}$$

As $\int_1^\infty \mathrm{e}^{-bt} t^{a-1}\,\mathrm{d}t = b^{-a}\Gamma(a,b)$, the Riemann transform of the zeta function follows by the theorem of Lebesgue on dominated convergence (the sums are dominated by integrable functions $(\Theta(M,it)-1)t^{a-1}$, where a is $\operatorname{Re} s/2$ or $\operatorname{Re}(d-s)/2$ for the first and second sum, respectively):

$$\tag{2.1} \begin{aligned} &\Gamma\Big(\frac{s}{2}\Big)\pi^{-s/2}Z(M,s) \\ &\quad= \frac{2}{s-d}|M|^{-1/2} - \frac{2}{s} + \sum_{0\neq x\in\mathbb{Z}^d} \Gamma\Big(\frac{s}{2}, \pi x'Mx\Big)(\pi x'Mx)^{-s/2} \\ &\quad\quad + |M|^{-1/2} \sum_{0\neq x\in\mathbb{Z}^d} \Gamma\Big(\frac{d-s}{2}, \pi x'M^{-1}x\Big)\left(\pi x'M^{-1}x\right)^{(s-d)/2}. \end{aligned}$$

We can group the terms with the same value of $x'Mx$ together in the first sum in (2.1) and analogously we can transform also the second sum. Let $N(M,\alpha) = \#\{x \in \mathbb{Z}^d\colon\ x'Mx = \alpha\}$, $\alpha\in\mathbb{R}$, let K be a positive definite integral matrix and let $\beta > 0$ be such that $M = \beta K$. Then we can express (2.1) as

$$\tag{2.2} \begin{aligned} &\Gamma\Big(\frac{s}{2}\Big)\pi^{-s/2}Z(M,s) \\ &\quad= \frac{2}{s-d}|M|^{-1/2} - \frac{2}{s} + \sum_{n=1}^{\infty} N(M,\beta n)\Gamma\Big(\frac{s}{2}, \pi\beta n\Big)(\pi\beta n)^{-s/2} \\ &\quad\quad + |M|^{-1/2} \sum_{n=1}^{\infty} N\Big(M^{-1}, \frac{n}{\beta|M|}\Big)\Gamma\Big(\frac{d-s}{2}, \frac{\pi n}{\beta|M|}\Big)\Big(\frac{\pi n}{\beta|M|}\Big)^{(s-d)/2}. \end{aligned}$$

3. An approximation of the zeta function and its precision

If the summations in (2.1) are restricted only to the lattice points with moduli $((x'Mx)^{-1/2}$, $(x'M^{-1}x)^{-1/2})$ bounded by R the resulting error will be equal to the sums over the lattice points with moduli greater than R. Let $|M| = 1$. Approximating the latter sums by integrals (i.e. replacing $\sum_{x\in\mathbb{Z}^d, |M^{\pm1/2}x|>R} f(M^{\pm1/2}x)$ by $\int_{x\in\mathbb{R}^d, |x|>R} f(x)\,\mathrm{d}x$) we obtain the following approximate expression for the error:

(3.1)

$$\begin{aligned} &d\kappa_d\Big(\int_R^\infty \Gamma\Big(\frac{s}{2}, \pi r^2\Big)\left(\pi r^2\right)^{-s/2} r^{d-1}\,\mathrm{d}r + \int_R^\infty \Gamma\Big(\frac{d-s}{2}, \pi r^2\Big)\left(\pi r^2\right)^{-(d-s)/2} r^{d-1}\,\mathrm{d}r\Big) \\ &\quad= \frac{d}{2\Gamma\left(\frac{d}{2}+1\right)}\Big(\frac{2}{d-s}\Big(\Gamma\Big(\frac{d}{2}, \pi R^2\Big) - \left(\pi R^2\right)^{(d-s)/2}\Gamma\Big(\frac{s}{2}, \pi R^2\Big)\Big) \\ &\quad\quad + \frac{2}{s}\Big(\Gamma\Big(\frac{d}{2}, \pi R^2\Big) - \left(\pi R^2\right)^{s/2}\Gamma\Big(\frac{d-s}{2}, \pi R^2\Big)\Big)\Big), \end{aligned}$$

where $\kappa_d = \pi^{d/2}\Gamma(d/2+1)^{-1}$ is the volume of the unit ball $B_d(1)$ in $\mathbb{R}^d$.

The functions under the integral signs in (3.1) are subharmonic in the neighborhood of infinity. Hence the approximate expression with argument $R - D/2$, where D is the maximum of the diameters of the Voronoi cells of the lattices $M^{-1/2}\mathbb{Z}^d$ and $M^{1/2}\mathbb{Z}^d$, is an upper bound of the error for R sufficiently large.

4. Theta function identities

For the calculation of $N(M, \beta n)$ we can use the identity

$$\Theta(M,\tau) - 1 = \sum_{n=1}^{\infty} N\left(M, \beta n\right) \mathrm{e}^{\pi \mathrm{i} \tau n}$$

and the known relations for Θ for particular M's with Jacobi theta functions with zero argument defined by

$$\theta_2(0|\tau) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\pi \mathrm{i}\tau\left(n+\frac{1}{2}\right)^2},\ \theta_3(0|\tau) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\pi \mathrm{i}\tau n^2},\ \theta_4(0|\tau) = \sum_{n=-\infty}^{\infty} (-1)^n \mathrm{e}^{\pi \mathrm{i}\tau n^2}.$$

The relations (those following immediately and those listed in Appendix A) are adopted from [1]. We shall use the notation $\theta_i(\tau)$ for $\theta_i(0|\tau)$, $i = 2, 3, 4$. For the identity matrix I_d generating the cubic lattice $\mathbb{Z}^d$ consisting of points in $\mathbb{R}^d$ with integral coordinates we have

$$\Theta(I_d,\tau) = \theta_3(\tau)^d.$$

Let $p \in \mathbb{N}$. The matrix T_p generating the triangular lattice with the quadratic form $x'T_p x = x_1^2 + x_1x_2 + \frac{1}{4}(p+1)x_2^2$ in $\mathbb{R}^2$ satisfies

$$\Theta(T_p,\tau) = \theta_3(\tau)\theta_3(p\tau) + \theta_2(\tau)\theta_2(p\tau).$$

The integral matrices corresponding to lattices generated by root systems of Lie algebras are of interest, namely the zero sum d-dimensional matrices A_d, d-dimensional checkerboard matrices D_d, Gosset matrices E_6, E_7 and E_8 and their duals [1].

For the d-dimensional checkerboard root matrix D_d generating the lattice consisting of points of the cubic lattice $\mathbb{Z}^d$ with an even sum of coordinates, the determinant $|D_d| = 4$,

$$\Theta(D_d,\tau) = \frac{1}{2}(\theta_3(\tau)^d + \theta_4(\tau)^d)$$

and for its dual,

$$\Theta(D_d^{-1},\tau) = \theta_3(\tau)^d + \theta_2(\tau)^d.$$

Other important integral lattices in dimensions $12, 16, 24, 32$ are generated by the Coxeter-Todd matrix K_{12}, the Barnes-Wall matrix Λ_{16}, the Leech matrix Λ_{24} and the Quebbemann's lattice Q_{32} [1] (Appendix A).

The Leech matrix is an example of an extremal even unimodular ($|M| = 1$) matrix. The extremality means that the theta series have the maximum number of zero leading coefficients. Theta functions of such matrices are very strongly constrained as they are modular forms invariant to transformations $z \to z + 1$, $z \to -1/z$, and it follows from the Hecke theorem that they can be expressed using special modular functions $\Theta(E_8, \tau)$ and $\Delta_{24}(\tau)$ [1] (Appendix A). Thus it is easy to calculate the theta functions for small dimensions d. Extremal even unimodular matrices corresponding to the theta functions $24 \leqslant d \leqslant 80$, $8 \mid d$, except $d = 72$, are already known. For large values of d (about $d > 41000$) extremal even unimodular matrices do not exist, as the constraints on their theta functions would imply negative coefficients in the theta series [1]. The extremal theta functions can be either adopted from [1] or calculated from the above constraints (Appendix A).

5. Variance of lattice points number in random ball

If $\mathbf{T} = M^{-1/2}\mathbb{Z}^d$ is a d-periodical point lattice in the d-dimensional Euclidean space $\mathbb{R}^d$ with spatial intensity $\alpha = |M|^{-1/2}$ then the mean value of

$$(\mathbf{card}((B_d(r) + x) \cap \mathbf{T}) - \alpha\lambda^d(B_d(r)))^2,$$

i.e., the variance of the lattice point count in the ball with radius r with uniform random position, is

$$C_{\mathbf{T}} H^{d-1}(\partial B_d(r))\Phi(r),$$

where

$$C_{\mathbf{T}} = \frac{1}{2\pi^2 d\kappa_d} Z(M, d+1) \tag{5.1}$$

is a lattice constant; $\kappa_d = \pi^{d/2}\Gamma\left(\frac{1}{2}d + 1\right)^{-1}$ is the volume of the unit ball $B_d(1)$ in $\mathbb{R}^d$, H^{d-1} is the surface measure, and Φ defined by the above equality fulfils $\lim\limits_{x\to\infty} x^{-1}\int_0^x \Phi(x)\,\mathrm{d}x = 1$. Hence the variance of the lattice point count in the ball has asymptotics "in the mean" $C_{\mathbf{T}} H^{d-1}(\partial B_d(1)) r^{d-1}$ and is $O(r^{d-1})$, $r \to \infty$. This result holds also for bounded sets with smooth isotropic covariogram (i.e., such that has the fractional derivation of order $\frac{1}{2}(d+1)$ with bounded variation) and sufficiently regular boundary (full-dimensional locally finite union of sets of finite reach [3]).

The values of coefficients (5.1) for cubic lattices are in Table 1. Coefficients (5.1) of various integral lattices are in Table 2. The values of the zeta functions for some integral matrices with known analytic expressions of the Riemann theta functions rescaled to unit density ($|M| = 1$) were calculated from (2.2) using the values of $N(M, \beta n)$ from the Taylor series of the theta functions and a sufficient number of coefficients, according to (3.1). The numerical values were rounded.

d	$C_{\mathbb{Z}^d}$	d	$C_{\mathbb{Z}^d}$
1	0.083333333	7	0.061828449
2	0.072837040	8	0.064852630
3	0.066649070	12	0.116517998
4	0.062959415	16	0.480456123
5	0.061045829	24	52.76720063
6	0.060656899	48	$7.35885{\cdot}10^{10}$

Table 1. Coefficients (5.1) of cubic point lattices in $\mathbb{R}^d$.

d	M	$C_{\mathbf{T}}$	d	M	$C_{\mathbf{T}}$
2	A_2	0.071701169	12	K_{12}	0.039608249
3	D_3	0.064350404	12	D_{12}	0.050000883
3	D_3^{-1}	0.064389706	16	Λ_{16}	0.035067857
4	D_4	0.058670401	24	Λ_{24}	0.028950578
5	D_5	0.054722140	24	D_{24}	0.463897082
5	D_5^{-1}	0.054818805	32	Q_{32}	0.026945374
6	E_6	0.051197993	32	Λ_{32}	0.028838712
6	E_6^{-1}	0.051262375	40	Λ_{40}	0.028873965
6	D_6	0.051932950	48	Λ_{48}	0.022561504
7	E_7	0.048337049	48	D_{48}	$2.957919{\cdot}10^5$
7	E_7^{-1}	0.048541494	56	Λ_{56}	0.022535527
7	D_7	0.049997583	64	Λ_{64}	0.022541824
8	E_8	0.045596961	72	Λ_{72}	0.019164428
8	D_8	0.048752920	80	Λ_{80}	0.019095025

Table 2. Coefficients (5.1) of point lattices (rescaled to unit density) in $\mathbb{R}^d$. Λ_d, $d = 32, 40, 48, 56, 64, 72, 80$, are extremal even unimodular lattices (hypothetic for $d = 72$). Notation of the other lattices was defined in Section 4.

The errors of the approximate values of coefficients of lattices in 24 dimensions calculated using approximate values of the zeta function obtained by summing the first n terms of the series in (2.2) can be estimated either from (3.1) or by summing the terms in (2.2) from $n + 1$ to $2n$. The estimates are shown in Table 3.

n	$est.$	$\mathbb{Z}^{24}$	D_{24}	Λ_{24}
5	−3.45	−3.82	−3.41	−3.61
10	−7.61	−8.18	−7.89	−8.91
15	−12.8	−13.4	−12.4	−13.1
20	−18.4	−19.1	−18.4	−20.0
25	−24.3	−25.0	−24.4	−24.7

Table 3. Estimated precision (decadic logarithm of error, est.—using (3.1) and the values obtained by summing the terms in (2.2) from $n+1$ to $2n$) of coefficients (5.1) for cubic, checkerboard and Leech lattices in $\mathbb{R}^{24}$.

6. Discussion

The formulas enable one to calculate matrix zeta functions using theta series expansions in symbolic algebra packages that allow feasible manipulations with coefficients of formal series (in our case it was the program Mathematica—see Appendix B).

Table 3 shows that our simplistic estimate of the error of calculation of the matrix zeta function by finite series gives reasonable results for $d = 24$. According to our calculations (data not shown) this holds also for d up to 80.

The comparison of Tables 1 and 2 shows that differences between the best lattice and cubic (and also checkerboard) lattices grow rapidly with the dimension. This is caused by holes in the cubic lattice the size of which grows with dimension. The big difference between the cubic and optimal lattices can also be expected in other criteria, e.g. in performance of numerical quadrature.

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Appendix A

The matrix A_2 coincides with the matrix T_p with $p=3$.

For the 6-dimensional Gosset matrix E_6, $|E_6|=3$, we have

$$\Theta(E_6,\tau)=\Theta(T_3,2\tau)^3+\frac{1}{4}\Big(\Theta\Big(T_3,\frac{2}{3}\tau\Big)-\Theta(T_3,2\tau)\Big)^3$$

and for its dual

$$\Theta\left(E_6^{-1},\tau\right)=\frac{1}{3}\Big(\Theta\Big(T_3,\frac{2}{3}\tau\Big)^3+\frac{1}{4}\Big(3\Theta(T_3,2\tau)-\Theta\Big(T_3,\frac{2}{3}\tau\Big)\Big)^3\Big).$$

Further, for the 7-dimensional Gosset matrix E_7, $|E_7|=2$, we have

$$\Theta\left(E_7,\tau\right)=\theta_3(2\tau)^7+7\theta_3(2\tau)^3\theta_2(2\tau)^4$$

and for its dual

$$\Theta\left(E_7^{-1},\tau\right)=\theta_3(2\tau)^7+7\theta_3(2\tau)^3\theta_2(2\tau)^4+\theta_2(2\tau)^7+7\theta_2(2\tau)^3\theta_3(2\tau)^4.$$

For the 8-dimensional Gosset matrix E_8, $|E_8|=1$,

$$\Theta(E_8,\tau)=\frac{1}{2}(\theta_2(\tau)^8+\theta_3(\tau)^8+\theta_4(\tau)^8).$$

For the Coxeter-Todd matrix K_{12}, $|K_{12}|=3^6$, we have

$$\begin{aligned}\Theta\left(K_{12},\tau\right)&=\left(\theta_2(4\tau)\theta_2(12\tau)+\theta_3(4\tau)\theta_3(12\tau)\right)^6\\&\quad+45\left(\theta_2(4\tau)\theta_2(12\tau)+\theta_3(4\tau)\theta_3(12\tau)\right)^2\left(\theta_2(4\tau)\theta_3(12\tau)+\theta_3(4\tau)\theta_2(12\tau)\right)^4\\&\quad+18\left(\theta_2(4\tau)\theta_3(12\tau)+\theta_3(4\tau)\theta_2(12\tau)\right)^6.\end{aligned}$$

For the Barnes-Wall matrix Λ_{16}, $|\Lambda_{16}|=2^8$,

$$\Theta\left(\Lambda_{16},\tau\right)=\frac{1}{2}(\theta_2\left(2\tau\right)^{16}+\theta_3\left(2\tau\right)^{16}+\theta_4\left(2\tau\right)^{16}+30\theta_2\left(2\tau\right)^8\theta_3\left(2\tau\right)^8).$$

Let $\Delta_{24}(\tau)=\frac{1}{256}(\theta_2(\tau)\theta_3(\tau)\theta_4(\tau))^8$, then for the Leech matrix Λ_{24} we have

$$\Theta(\Lambda_{24},\tau)=\Theta(E_8,\tau)^3-720\Delta_{24}(\tau).$$

Let $\Delta_{16}(\tau)=\frac{1}{96}(\Theta(D_4,\tau)^4-\Theta(\Lambda_{16},\tau))$, then for Quebbemann's matrix Q_{32}, $|Q_{32}|=2^{16}$, we have

$$\Theta(Q_{32},\tau)=\Theta(D_4,\tau)^8-192\Theta(D_4,\tau)^4\Delta_{16}(\tau)+576\Delta_{16}(\tau)^2.$$

For extremal theta functions $\Theta(\Lambda_d, \tau)$ either adopted from [1] or calculated from the modular functions constraints we have:

For $d = 32$,

$$\Theta(E_8, \tau)^4 - 960\Delta_{24}(\tau)\Theta(E_8, \tau);$$

$d = 40$,

$$\Theta(E_8, \tau)^5 - 1200\Delta_{24}(\tau)\Theta(E_8, \tau)^2;$$

$d = 48$,

$$\Theta(E_8, \tau)^6 - 1440\Delta_{24}(\tau)\Theta(E_8, \tau)^3 + 125280\Delta_{24}(\tau)^2;$$

$d = 56, 64, 72, 80$:

$$\begin{aligned}
&\Theta(E_8, \tau)^7 - 1680\Delta_{24}(\tau)\Theta(E_8, \tau)^4 + 347760\Delta_{24}(\tau)^2\Theta(E_8, \tau), \\
&\Theta(E_8, \tau)^8 - 1920\Delta_{24}(\tau)\Theta(E_8, \tau)^5 + 627840\Delta_{24}(\tau)^2\Theta(E_8, \tau)^2, \\
&\Theta(E_8)^9 - 2160\Delta_{24}\Theta(E_8)^6 + 965520\Delta_{24}^2\Theta(E_8)^3 - 27302400\Delta_{24}^3, \\
&\Theta(E_8)^{10} - 2400\Delta_{24}\Theta(E_8)^7 + 1360800\Delta_{24}^2\Theta(E_8)^4 - 103488000\Delta_{24}^3\Theta(E_8).
\end{aligned}$$

Appendix B

Calculation of the coefficient (5.1) of the bcc (D_3) lattice in Mathematica (Wolfram Research, Inc., USA).

Determinants of D_3^{-1} and D_3 lattices:

$\texttt{DdDet} := 4$
$\texttt{DdSDet} := \frac{1}{4}$

Theta series of D_3^{-1} and D_3 lattices:

$\texttt{DdTheta}[d\_, x\_] := \frac{1}{2}(\texttt{EllipticTheta}[3, 0, x]^d + \texttt{EllipticTheta}[4, 0, x]^d)$
$\texttt{DdSTheta}[d\_, x\_] := \texttt{EllipticTheta}[3, 0, x^4]^d + \texttt{EllipticTheta}[2, 0, x^4]^d$

Coefficients of the theta series:

$\texttt{DdThetaN}[d\_, m\_]$
$\quad := \texttt{CoefficientList}[\texttt{Series}[\texttt{DdTheta}[d, x], \{x, 0, \texttt{Ceiling}[m] + 2\}], x]$
$\texttt{DdStep} := 1$
$\texttt{DdSThetaN}[d\_, m\_]$
$\quad := \texttt{CoefficientList}[\texttt{Series}[\texttt{DdSTheta}[d, x], \{x, 0, \texttt{Ceiling}[m] + 2\}], x]$

$\texttt{DdSStep} := 4$

Formula (2.2) for the Zeta function:

$$\texttt{ZTheta}[d\_, Num\_, mult\_, NumS\_, multS\_, s\_, m\_]$$
$$:= \texttt{Re}\left[\frac{\pi^{s/2}}{\texttt{Gamma}[\frac{s}{2}]}\left(\frac{2}{s-d} - \frac{2}{s} + \sum_{\texttt{n}=1}^{mult\ m} Num[[\texttt{n}+1]]\frac{\texttt{Gamma}[\frac{s}{2}, \frac{\pi\texttt{n}}{mult}]}{(\frac{\pi\texttt{n}}{mult})^{s/2}}\right.\right.$$
$$\left.\left. + \sum_{\texttt{n}=1}^{multSm} NumS[[\texttt{n}+1]]\frac{\texttt{Gamma}[\frac{d-s}{2}, \frac{\pi\texttt{n}}{multS}]}{(\frac{\pi\texttt{n}}{multS})^{(d-s)/2}}\right)\right]$$

Formula (5.1) for the coefficient:

$\texttt{Coeff}[d\_, Num\_, mult\_, NumS\_, multS\_, m\_] := \frac{1}{2\pi^2}\frac{\texttt{Gamma}[\frac{d}{2}+1]}{d\pi^{d/2}}\texttt{ZTheta}$
$[d, NumS[d, multS\ m], mulS, Num[d, mult\ m], mult, d+1, m]$

The coefficient of the checkerboard lattice:

$\texttt{DdSCoeff}[d\_, m\_]$
$:= \texttt{Coeff}[d, \texttt{DdSThetaN}, \texttt{DdSStep DdSDet}^{1/2}, \texttt{DdThetaN}, \texttt{DdStep DdDet}^{1/2}, m]$

Finally, the coefficient of the bcc (D_3) lattice:

$\texttt{DdSCoeff}[3, 10] \sim \texttt{N} \sim 12$
0.0643504041372

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