SOME NEW ERROR ESTIMATES FOR FINITE ELEMENT METHODS FOR SECOND ORDER HYPERBOLIC EQUATIONS USING THE NEWMARK METHOD

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(Received June 30, 2013)

Abstract. We consider a family of conforming finite element schemes with piecewise polynomial space of degree k in space for solving the wave equation, as a model for second order hyperbolic equations. The discretization in time is performed using the Newmark method. A new a priori estimate is proved. Thanks to this new a priori estimate, it is proved that the convergence order of the error is $h^k+\tau^2$ in the discrete norms of $\mathcal{L}^\infty(0,T;\mathcal{H}^1(\Omega))$ and $\mathcal{W}^{1,\infty}(0,T;\mathcal{L}^2(\Omega))$, where h and τ are the mesh size of the spatial and temporal discretization, respectively.

These error estimates are useful since they allow us to get second order time accurate approximations for not only the exact solution of the wave equation but also for its first derivatives (both spatial and temporal).

Even though the proof presented in this note is in some sense standard, the stated error estimates seem not to be present in the existing literature on the finite element methods which use the Newmark method for the wave equation (or general second order hyperbolic equations).

Keywords: acoustic wave equation; finite element method; Newmark method; new error estimate

MSC 2010: 65N30, 65N15, 65M15, 35L15, 35L05

1. Preliminaries and a brief description of the main results

Let us consider the wave equation, as a model for second order hyperbolic equations:

$$u_{tt}(x,t)-\Delta u(x,t)=f(x,t),\quad (x,t)\in\Omega\times(0,T), \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^d$ ($d=1$, 2 or 3) with a polyhedral boundary $\partial\Omega$, $T>0$, and f is a given function.

Initial conditions are given by:

$$u(x,0) = u^0(x) \text{ and } u_t(x,0) = u^1(x),\ x \in \Omega. \tag{1.2}$$

For the sake of simplicity, we consider homogeneous Dirichlet boundary conditions, that is

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{1.3}$$

The wave equation is an important model which appears in several areas of applications like acoustics, electromagnetics, and fluid dynamics, see for instance [2], page 213.

Let $\{\mathcal{T}_h\colon\ h > 0\}$ be a family of shape regular and quasi-uniform triangulations of the domain Ω. The elements of $\mathcal{T}_h$ will be denoted by K. For each triangulation $\mathcal{T}_h$, the subscript h refers to the level of refinement of the triangulation, which is defined by $h = \max\limits_{K \in \mathcal{T}_h} h_K$, where h_K denotes the diameter of the element K.

Let $\mathcal{V}_0^h$ be the standard finite element space of continuous, piecewise polynomial functions of degree $k \geqslant 1$ which vanish on $\partial\Omega$

$$\mathcal{V}_0^h = \{v \in \mathcal{C}(\overline{\Omega})\colon\ \ v|_K \in \mathcal{P}_k,\ \forall\, K \in \mathcal{T}_h\} \cap \mathcal{H}_0^1(\Omega). \tag{1.4}$$

The time discretization is performed using a constant time step $\tau = T/(M+1)$, where $M \in \mathbb{N} \setminus \{0\}$, and we shall denote $t_n = n\tau$ for $n \in [\![0, M+1]\!]$.

Throughout this paper, the notations C_i, where $i \in \mathbb{N} \setminus \{0\}$, stand for positive constants independent of the parameters of the discretization.

The discretization scheme we want to consider is implicit and it is based on the use of the Newmark method (see for instance [10]) as discretization in time and on the use of the finite element mesh described above.

In order to define the finite element approximation for our problem (1.1)–(1.3), we need to define the following discrete first and second time derivatives:

$$\partial^1 v^n = \frac{v^n - v^{n-1}}{\tau}, \quad \forall\, n \in [\![1, M+1]\!], \tag{1.5}$$

and

$$\partial^2 v^n = \frac{v^n - 2v^{n-1} + v^{n-2}}{\tau^2}, \quad \forall\, n \in [\![2, M+1]\!]. \tag{1.6}$$

The following rules will be useful for our analysis, for any *smooth* function ψ defined on $[0,T]$

$$\partial^1\psi(t_{n+1}) = \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \psi_t(t)\,\mathrm{d}t \quad \text{and} \quad \partial^2\psi(t_{n+1}) = \frac{1}{\tau^2}\int_{t_n}^{t_{n+1}}\int_{t-\tau}^{t} \psi_{tt}(s)\,\mathrm{d}s\,\mathrm{d}t. \tag{1.7}$$

The family of finite element schemes approximating (1.1)–(1.3) we want to study in this work is based on the use of Newmark's method as discretization in time, see for instance [8], [9], [10], [11] and the references therein. For a parameter $\gamma \in]1/2, 1]$, we define the finite element approximate solution $(u_h^n)_{n=0}^{M+1} \in (\mathcal{V}_0^h)^{M+2}$ (see (1.4)) such that

$$\mathbf{a}(u_h^0, v) = -(\Delta u^0, v)_{\mathcal{L}^2(\Omega)} = \mathbf{a}(u^0, v),\ \forall\, v \in \mathcal{V}_0^h, \tag{1.8}$$

$$\mathbf{a}(\partial^1 u_h^1, v) = -(\Delta \bar{u}^1, v)_{\mathcal{L}^2(\Omega)} = \mathbf{a}(\bar{u}^1, v),\ \forall\, v \in \mathcal{V}_0^h, \tag{1.9}$$

and for any $n \in [\![1, M]\!]$, find $u_h^{n+1} \in \mathcal{V}_0^h$ such that, for all $v \in \mathcal{V}_0^h$

$$\begin{aligned}(\partial^2 u_h^{n+1}, v)_{\mathcal{L}^2(\Omega)} &+ \frac{1}{2}\mathbf{a}(\gamma u_h^{n+1} + 2(1-\gamma)u_h^n + \gamma u_h^{n-1}, v) \\ &= \frac{1}{2}(\gamma f(t_{n+1}) + 2(1-\gamma) f(t_n) + \gamma f(t_{n-1}), v)_{\mathcal{L}^2(\Omega)},\end{aligned} \tag{1.10}$$

where $\mathbf{a}(\cdot,\cdot)$ denotes the bilinear form defined for all $(u, v) \in \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)$ by

$$\mathbf{a}(u, v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, \mathrm{d}x, \tag{1.11}$$

$$\bar{u}^1 = u^1 + \frac{\tau}{2}(\Delta u^0 + f(0)), \tag{1.12}$$

and $(\cdot,\cdot)_{\mathcal{L}^2(\Omega)}$ denotes the $\mathcal{L}^2$ inner product.

To compute the solution of the finite element scheme (1.8)–(1.12) (see either [9], pages 500–501, or [10], pages 205–206), we first compute the initial solution u_h^0 using (1.8) and then we use u_h^0 to compute u_h^1 using (1.9). Equations (1.8) and (1.9) can, respectively, be written in the following matrix forms

$$AU^0 = \eta^0 \quad \text{and} \quad AU^1 = \eta^1, \tag{1.13}$$

where A is a symmetric and positive definite matrix, η^0 is known, η^1 defined in terms of U^0, and $U^n = (U_1^n, \dots, U_N^n)^T$ with $u_h^n = \sum_{i=1}^{N} U_i^n \varphi_i$ where φ_i are the basis functions of $\mathcal{V}^h$.

Equation (1.10) leads to the following linear systems, for each time step $n \in [\![2, M+1]\!]$

$$\Big(M + \frac{\tau^2}{2}\gamma A\Big) U^n = \eta^n, \tag{1.14}$$

where $M + \tau^2\gamma/2A$ is a symmetric, positive definite matrix and η^n is known from the previous steps.

The following $\mathcal{L}^\infty(\mathcal{L}^2)$-error estimate is the subject of [8], Theorem 2, page 200 (see also [10], Theorem 8.7–2, page 214–215): under the regularity assumptions $u \in \mathcal{C}^2([0,T];\mathcal{H}^{k+1}(\Omega))$, $u_{ttt} \in \mathcal{C}([0,T];\mathcal{L}^2(\Omega))$, and $u_{tttt} \in \mathcal{L}^1(0,T;\mathcal{L}^2(\Omega))$,

$$\max_{n\in[\![0,M+1]\!]} \|u_h^n - u(t_n)\|_{\mathcal{L}^2(\Omega)} \leqslant C_1(h^{k+1}+\tau^2), \tag{1.15}$$

where C_1 is depending on the exact solution u.

However, we are not aware with of the existence of any error estimate in the discrete norm of $\mathcal{W}^{1,\infty}(\mathcal{L}^2)$ or $\mathcal{L}^\infty(\mathcal{H}^1)$. We aim in this contribution to prove that the error is of order $h^k+\tau^2$ in the discrete norms of $\mathcal{L}^\infty(\mathcal{H}^1)$ and $\mathcal{W}^{1,\infty}(\mathcal{L}^2)$.

It is clear that deriving error estimate of order $h^k+\tau^2$, in the discrete norms of $\mathcal{L}^\infty(\mathcal{H}^1)$ and $\mathcal{W}^{1,\infty}(\mathcal{L}^2)$ yields approximations of order $h^k+\tau^2$ in the discrete norm $\mathcal{L}^\infty(\mathcal{L}^2)$ for the first derivatives (both temporal and spatial) of the exact solution. Such results are important from the mathematical point of view.

From the practical point of view, the approximation of the spatial and temporal derivatives of the exact solution of the wave equation is important when we are interested for instance in the total electric charge density ϱ (including both free and bound charge) given by $\varrho = \varepsilon_0 \nabla \cdot \mathbf{E}$, where the electric field $\mathbf{E}$ satisfies the wave equation $\Delta\mathbf{E} - \mu\varepsilon\mathbf{E}_{tt} = 0$ with ε_0 is the electric constant and $c = 1/\sqrt{\mu\varepsilon}$ is the speed of light in the medium, or simply when we are interested in the particle velocity in the one-dimensional equation of motion for a linear elastic continuum, assuming small strains $\sigma_x = \varrho(x)u_{tt} = \varrho(x)v_t$, where $\sigma = E(x)u_x$ (according to Hooke's law for elastic media, with E, the modulus of elasticity), ϱ, u, and $v \equiv u_t$ are stress, density, and particle displacement and particle velocity, respectively, (x and t denote spatial position and time), see [3].

We assume that $f \in \mathcal{L}^2((0,T)\times\Omega))$, $u^0 \in \mathcal{H}^1(\Omega)$ and $u^1 \in \mathcal{L}^2(\Omega)$. Then, cf. [5], Theorems 3–4, pages 384–385, there exists a unique weak solution for (1.1)–(1.3).

The following *coercivity* will be useful, for all $v \in \mathcal{H}_0^1(\Omega)$

$$\mathbf{a}(v,v) = \int_\Omega |\nabla v|^2(x)\,\mathrm{d}x = |v|^2_{1,\Omega}. \tag{1.16}$$

The convergence of the finite element schemes is analyzed thanks to the use of the spaces $\mathcal{C}^m([0,T];\mathcal{H}^l(\Omega))$, where m and l are integers, of m-times continuously differentiable mappings of the interval $[0,T]$ with values in the Sobolev space $\mathcal{H}^l(\Omega)$, see [6], pages 47–48, and [10], page 156.

2. Statement of the main results

We first begin by a regularity assumption for the Problem (2.1) below. For any $r \in \mathcal{L}^2(\Omega)$, let $\varphi(r) \in \mathcal{H}_0^1(\Omega)$ be the exact solution of the following problem (the existence and uniqueness are ensured by the Lax-Milgram lemma)

$$\mathbf{a}(\varphi(r), v) = \langle r, v\rangle, \quad \forall\, v \in \mathcal{H}_0^1(\Omega). \tag{2.1}$$

Assumption 2.1 (Regularity assumption, see [9], Remark 6.2.1, page 173)**.** For any $r \in \mathcal{L}^2(\Omega)$, we assume that the solution $\varphi(r)$ of (2.1) belongs to $\mathcal{H}^2(\Omega)$ and there exists a constant $C_{\mathrm{reg}} > 0$ such that $\|\varphi(r)\|_{\mathcal{H}^2(\Omega)} \leqslant C_{\mathrm{reg}}\|r\|_{\mathcal{L}^2(\Omega)}$, for all $r \in \mathcal{L}^2(\Omega)$.

Among the the main results of the present contribution is the following theorem:

Theorem 2.2 (New error estimates)**.** *Under Assumption* 2.1, *let* $u \in \mathcal{L}^2(0, T; \mathcal{H}_0^1(\Omega))$ *be the weak solution of* (1.1)–(1.3). *Let* $\{\mathcal{T}_h; h > 0\}$ *be a family of shape regular and quasi-uniform triangulations of the domain* Ω *and* $h = \max_{K\in\mathcal{T}_h} h_K$, *where* h_K *denotes the diameter of the element* K. *Let* $\mathcal{V}_0^h$ *be the standard finite element space defined by* (1.4) *where* $k \in \mathbb{N} \setminus \{0\}$. *We assume that the time discretization is performed using a constant time step* $\tau = T/(M + 1)$, *where* $M \in \mathbb{N} \setminus \{0\}$, *and we define* $t_n = n\tau$, *for* $n \in [\![0, M + 1]\!]$.

Let $\gamma \in\,]1/2, 1]$. *Then, there exists a unique solution* $(u_h^n)_{n=0}^{M+1} \in (\mathcal{V}_0^h)^{M+2}$ *for* (1.8)–(1.12). *Assume that the exact solution* u *belongs to* $\mathcal{C}^4([0, T]; \mathcal{H}^{k+1}(\Omega))$. *Then, the following error estimates hold:*

▷ *Discrete* $\mathcal{L}^\infty(0, T; \mathcal{H}_0^1(\Omega))$*-estimate: for all* $n \in [\![0, M + 1]\!]$

$$|u_h^n - u(t_n)|_{1,\Omega} \leqslant C_2(h^k + \tau^2)\|u\|_{\mathcal{C}^4([0,T];\ \mathcal{H}^{k+1}(\Omega))}. \tag{2.2}$$

▷ *Discrete* $\mathcal{W}^{1,\infty}(0, T; \mathcal{L}^2(\Omega))$*-estimate: for all* $n \in [\![1, M + 1]\!]$

$$\|\partial^1(u_h^n - u(t_n))\|_{\mathcal{L}^2(\Omega)} \leqslant C_3(h^k + \tau^2)\|u\|_{\mathcal{C}^4([0,T];\mathcal{H}^{k+1}(\Omega))}, \tag{2.3}$$

where ∂^1 *denotes the discrete temporal derivative* (1.5).

Remark 2.3 (The case of two parameters scheme). The scheme (1.10) contains only one parameter, namely γ. It is possible to define a similar scheme to that of (1.10) which contains two parameters as in [10], (8.6)–(8.9), page 206. The scheme (1.10) is a particular case of the two parameters scheme [10], (8.6)–(8.9), page 206. We are concerned with the particular case (1.10) in this study because it yields order two in time, whereas the general scheme [10], (8.6)–(8.9), page 206, yields generally order one in time.

R e m a r k 2.4 (Applications of Theorem 2.2). Theorem 2.2 is useful thanks to the following facts.

1. Approximation of first spatial derivatives. Thanks to the error estimate (2.2), the partial derivative $\frac{\partial u_h^n}{\partial x_i}$ of u_h^n with respect to x_i approximates the corresponding spatial derivative $\frac{\partial u(t_n)}{\partial x_i}$ of $u(t_n)$ by order $h^k+\tau^2$ in $\mathcal{L}^\infty(\mathcal{L}^2)$, for all $i \in [\![1,d]\!]$ (recall that d is the space dimension), uniformly in n.
2. Approximation of first temporal derivative. Thanks to the error estimate (2.3) and the triangle inequality (since $u_t(t_{n-1/2}) - \partial^1 u(t_n)$ is of order τ^2), $\partial^1 u_h^n$ approximates $u_t(t_{n-1/2})$ with $t_{n-1/2} = (t_n + t_{n-1})/2$, by order $h^k+\tau^2$ in $\mathcal{L}^\infty(\mathcal{L}^2)$, uniformly in n.

To prove Theorem 2.2, we need to use the following new *a priori estimate* which may also be found in [1] but without Newmark's scheme. The estimate on the *time derivative* stated below in Lemma 2.5 may also be found in [4] but with a proof different from the one we present here.

Lemma 2.5 (A new a priori estimate). *We consider the time and space discretizations as in Theorem 2.2. Under Assumption* 2.1, *let* $\gamma \in]1/2,1]$. *Assume that there exits* $(\eta_h^n)_{n=0}^{M+1} \in (\mathcal{V}_0^h)^{M+2}$ *such that for all* $n \in [\![1,M]\!]$ *and for all* $v \in \mathcal{V}_0^h$

$$(\partial^2 \eta_h^{n+1}, v)_{\mathcal{L}^2(\Omega)} + \frac{1}{2}\mathbf{a}(\gamma\eta_h^{n+1} + 2(1-\gamma)\eta_h^n + \gamma\eta_h^{n-1}, v) = (\mathcal{S}^n, v)_{\mathcal{L}^2(\Omega)}, \tag{2.4}$$

where $\mathcal{S}^n \in \mathcal{L}^2(\Omega)$, *for all* $n \in [\![1,M]\!]$.

Then the following estimate holds, for all $j \in [\![1,M]\!]$:

$$\|\partial^1 \eta_h^{j+1}\|^2_{\mathcal{L}^2(\Omega)} + (2\gamma-1)|\eta_h^{j+1}|^2_{1,\Omega} \leqslant C_4(\|\partial^1\eta_h^1\|^2_{\mathcal{L}^2(\Omega)} + |\eta_h^1|^2_{1,\Omega} + |\eta_h^0|^2_{1,\Omega} + (\mathcal{S})^2), \tag{2.5}$$

where

$$\mathcal{S} = \max_{1\leqslant n\leqslant M} \|\mathcal{S}^n\|_{\mathcal{L}^2(\Omega)}. \tag{2.6}$$

P r o o f. The following simple equality will be useful

$$\eta_h^{n+1} - \eta_h^{n-1} = \tau(\partial^1\eta_h^{n+1} + \partial^1\eta_h^n). \tag{2.7}$$

Taking $v = \eta_h^{n+1} - \eta_h^{n-1}$ in (2.4) and using (2.7) we get

$$\begin{aligned} &\|\partial^1\eta_h^{n+1}\|^2_{\mathcal{L}^2(\Omega)} - \|\partial^1\eta_h^n\|^2_{\mathcal{L}^2(\Omega)} + (1-\gamma)(\mathbf{a}(\eta_h^n,\eta_h^{n+1}) - \mathbf{a}(\eta_h^n,\eta_h^{n-1})) \\ &\quad + \frac{\gamma}{2}(\mathbf{a}(\eta_h^{n+1},\eta_h^{n+1}) - \mathbf{a}(\eta_h^{n-1},\eta_h^{n-1})) = (\mathcal{S}^n, \eta_h^{n+1} - \eta_h^{n-1})_{\mathcal{L}^2(\Omega)}. \end{aligned} \tag{2.8}$$

On the other hand

$$(2.9)\qquad (1-\gamma)(\mathbf{a}(\eta_h^n,\eta_h^{n+1})-\mathbf{a}(\eta_h^n,\eta_h^{n-1}))+\frac{\gamma}{2}(\mathbf{a}(\eta_h^{n+1},\eta_h^{n+1})-\mathbf{a}(\eta_h^{n-1},\eta_h^{n-1}))$$
$$=\mathbf{a}(\eta_h^{n+1},\eta_h^n)-\mathbf{a}(\eta_h^n,\eta_h^{n-1})+\frac{\gamma}{2}\mathbf{a}(\eta_h^{n+1}-\eta_h^n,\eta_h^{n+1}-\eta_h^n)$$
$$-\frac{\gamma}{2}\mathbf{a}(\eta_h^n-\eta_h^{n-1},\eta_h^n-\eta_h^{n-1}).$$

Thanks to (2.9), (2.8) can be written as

$$(2.10)\qquad \mathcal{E}_h^{n+1}-\mathcal{E}_h^n=(\mathcal{S}^n,\eta_h^{n+1}-\eta_h^{n-1})_{\mathcal{L}^2(\Omega)},$$

where

$$(2.11)\qquad \mathcal{E}_h^{n+1}=\|\partial^1\eta_h^{n+1}\|^2_{\mathcal{L}^2(\Omega)}+\mathbf{a}(\eta_h^{n+1},\eta_h^n)+\frac{\gamma}{2}\mathbf{a}(\eta_h^{n+1}-\eta_h^n,\eta_h^{n+1}-\eta_h^n).$$

Summing (2.10) over $n\in[\![1,j]\!]$, where $j\in[\![1,M]\!]$, we get

$$(2.12)\qquad \mathcal{E}_h^{j+1}=\sum_{n=1}^{j}(\mathcal{S}^n,\eta_h^{n+1}-\eta_h^{n-1})_{\mathcal{L}^2(\Omega)}+\mathcal{E}_h^1.$$

We have, using (2.11)

$$(2.13)\quad \mathcal{E}_h^{j+1}=\|\partial^1\eta_h^{j+1}\|^2_{\mathcal{L}^2(\Omega)}+\frac{\gamma}{2}(\mathbf{a}(\eta_h^{j+1},\eta_h^{j+1})+\mathbf{a}(\eta_h^j,\eta_h^j))+(1-\gamma)\mathbf{a}(\eta_h^{j+1},\eta_h^j).$$

On the other hand, we can justify easily that

$$(2.14)\qquad \mathbf{a}(\eta_h^{j+1},\eta_h^j)\geqslant-\frac{1}{2}(\mathbf{a}(\eta_h^{j+1},\eta_h^{j+1})+\mathbf{a}(\eta_h^j,\eta_h^j)).$$

This with (2.13) yields that (recall that $1\geqslant\gamma$ which means that $1-\gamma\geqslant0$)

$$(2.15)\qquad \mathcal{E}_h^{j+1}\geqslant\|\partial^1\eta_h^{j+1}\|^2_{\mathcal{L}^2(\Omega)}+\frac{2\gamma-1}{2}(\mathbf{a}(\eta_h^{j+1},\eta_h^{j+1})+\mathbf{a}(\eta_h^j,\eta_h^j)).$$

This with (2.12) leads to

$$(2.16)\qquad \|\partial^1\eta_h^{j+1}\|^2_{\mathcal{L}^2(\Omega)}+\frac{2\gamma-1}{2}(\mathbf{a}(\eta_h^{j+1},\eta_h^{j+1})+\mathbf{a}(\eta_h^j,\eta_h^j))$$
$$\leqslant\sum_{n=1}^{j}(\mathcal{S}^n,\eta_h^{n+1}-\eta_h^{n-1})_{\mathcal{L}^2(\Omega)}+\mathcal{E}_h^1.$$

Combining (2.16) with (2.7), the fact that $\mathbf{a}(\eta_h^j,\eta_h^j)\geqslant 0$ (which stems from (1.16)), $\gamma>1/2$, the triangle inequality and the Cauchy-Schwarz inequality leads to (recall that $\mathcal{S}$ is defined in (2.6))

$$\|\partial^1\eta_h^{j+1}\|^2_{\mathcal{L}^2(\Omega)}+\frac{2\gamma-1}{2}\mathbf{a}(\eta_h^{j+1},\eta_h^{j+1})\leqslant 2\tau\mathcal{S}\sum_{n=1}^{j+1}\|\partial^1\eta_h^n\|_{\mathcal{L}^2(\Omega)}+\mathcal{E}_h^1. \tag{2.17}$$

This with the inequality $ab\leqslant\varepsilon a^2+b^2/\varepsilon$, for all $\varepsilon>0$, implies that, for all $j\in[\![1,M]\!]$ (recall that $\tau(M+1)=T$ and $\tau/T=1/(M+1)\leqslant 1/2$)

$$\begin{aligned}\|\partial^1\eta_h^{j+1}\|^2_{\mathcal{L}^2(\Omega)}&+(2\gamma-1)|\eta_h^{j+1}|^2_{1,\Omega}\\&\leqslant\frac{2\tau}{T}\sum_{n=2}^{j}(\|\partial^1\eta_h^n\|^2_{\mathcal{L}^2(\Omega)}+(2\gamma-1)|\eta_h^n|^2_{1,\Omega})+2\mathcal{E}_h^1+8T^2(\mathcal{S})^2+\|\partial^1\eta_h^1\|^2_{\mathcal{L}^2(\Omega)}.\end{aligned} \tag{2.18}$$

On the other hand, using the fact that $\gamma\leqslant 1$ and $\mathbf{a}(\eta_h^1-\eta_h^0,\eta_h^1-\eta_h^0)\geqslant 0$, (2.11) implies

$$\mathcal{E}_h^1\leqslant\|\partial^1\eta_h^1\|^2_{\mathcal{L}^2(\Omega)}+\frac{1}{2}(|\eta_h^1|^2_{1,\Omega}+|\eta_h^0|^2_{1,\Omega}). \tag{2.19}$$

This with (2.18), the discrete Gronwall's lemma and the fact that $(N+1)\tau=T$ implies the estimate (2.5). □

Proof of Theorem 2.2. We use here some techniques from [7], [8] and we will prove Theorem 2.2 item by item.

1. Existence and uniqueness of the discrete solutions. The existence and uniqueness for schemes (1.8)–(1.12) stems from the fact that the matrices involved in the linear systems of these schemes are either the matrix A or the matrix $M+\tau^2\gamma/2A$ (see (1.13)–(1.14)) which are positive definite.

2. Proof of the estimates (2.2)–(2.3). The proof of the estimates (2.2)–(2.3) of Theorem 2.2 is based essentially on the comparison with the following finite element scheme: for each $n\in[\![0,M+1]\!]$, we compute $\bar{u}_h^n\in\mathcal{V}_0^h$ (see (1.4)) such that

$$\mathbf{a}(\bar{u}_h^n,v)=-(\Delta u(t_n),v)_{\mathcal{L}^2(\Omega)}=\mathbf{a}(u(t_n),v),\ \forall\, v\in\mathcal{V}_0^h. \tag{2.20}$$

The following convergence result in $\mathcal{H}^1$-norm is known, see for instance [9], Proposition 6.2.2, page 173,

$$|\bar{u}_h^n-u(t_n)|_{1,\Omega}\leqslant C_4h^k\|u\|_{\mathcal{C}([0,T];\ \mathcal{H}^{k+1}(\Omega))}. \tag{2.21}$$

Applying the discrete operator ∂^j (see (1.5) and (1.6)), $j \in \{1, 2\}$, on the both sides of (2.20) yields

$$\mathbf{a}(\partial^j \bar{u}_h^n, v) = -(\Delta \partial^j u(t_n), v)_{\mathcal{L}^2(\Omega)}, \quad \forall\, v \in \mathcal{V}_0^h. \tag{2.22}$$

This with the known convergence result in $\mathcal{L}^2$-norm, see [9], Proposition 6.2.2, page 173, and (1.7) implies that

$$\|\partial^j \bar{u}_h^n - \partial^j u(t_n)\|_{\mathcal{L}^2(\Omega)} \leqslant C_5 h^{k+1} \|u\|_{\mathcal{C}^j([0,T];\mathcal{H}^{k+1}(\Omega))}. \tag{2.23}$$

This with the fact that $\|u\|_{\mathcal{C}^j([0,T];\mathcal{H}^{k+1}(\Omega))} \leqslant \|u\|_{\mathcal{C}^2([0,T];\mathcal{H}^{k+1}(\Omega))}$ (since $j \leqslant 2$) implies that

$$\|\partial^j \bar{u}_h^n - \partial^j u(t_n)\|_{\mathcal{L}^2(\Omega)} \leqslant C_5 h^{k+1} \|u\|_{\mathcal{C}^2([0,T];\mathcal{H}^{k+1}(\Omega))}. \tag{2.24}$$

We consider the error given by

$$\bar{e}_h^n = u_h^n - \bar{u}_h^n. \tag{2.25}$$

Take $n = 0$ in (2.20), using (1.2), and compare the result with (1.8) to get the following nice property which will be used later

$$\bar{e}_h^0 = 0. \tag{2.26}$$

Writing (2.20) in the steps $n+1$ and $n-1$ yields, for all $n \in [\![1, M]\!]$ and for all $v \in \mathcal{V}_0^h$

$$\begin{aligned} &\frac{1}{2}\mathbf{a}(\gamma \bar{u}_h^{n+1} + 2(1-\gamma)\bar{u}_h^n + \gamma \bar{u}_h^{n-1}, v) \\ &\qquad = -\frac{1}{2}(\gamma \Delta u(t_{n+1}) + 2(1-\gamma)\Delta u(t_n) + \gamma \Delta u(t_{n-1}), v)_{\mathcal{L}^2(\Omega)}. \end{aligned} \tag{2.27}$$

Subtracting (2.27) from (1.10), and adding $-\partial^2 \bar{u}_h^{n+1}$ to both sides of the result, we get

$$(\partial^1 \bar{e}_h^{n+1}, v)_{\mathcal{L}^2(\Omega)} + \frac{1}{2}\mathbf{a}(\gamma \bar{e}_h^{n+1} + 2(1-\gamma)\bar{e}_h^n + \gamma \bar{e}_h^{n-1}, v) = (\mathcal{S}^n, v)_{\mathcal{L}^2(\Omega)}, \tag{2.28}$$

where

$$\begin{aligned} \mathcal{S}^n = &\frac{1}{2}(\gamma f(t_{n+1}) + 2(1-\gamma)f(t_n) + \gamma f(t_{n-1})) \\ &+ \frac{1}{2}(\gamma \Delta u(t_{n+1}) + 2(1-\gamma)\Delta u(t_n) + \gamma \Delta u(t_{n-1})) - \partial^2 \bar{u}_h^{n+1}. \end{aligned} \tag{2.29}$$

Equation (2.28) with Lemma 2.5 and (2.26) implies that, for all $n \in [\![1, M]\!]$,

$$\|\partial^1 \bar{e}_h^{n+1}\|^2_{\mathcal{L}^2(\Omega)} + (2\gamma - 1)|\bar{e}_h^{n+1}|^2_{1,\Omega} \leqslant C_4(\|\partial^1 \bar{e}_h^1\|^2_{\mathcal{L}^2(\Omega)} + |\bar{e}_h^1|^2_{1,\Omega} + (\mathcal{S})^2), \tag{2.30}$$

with $\mathcal{S}$ defined by (2.6).

To estimate the terms on the right hand side of the previous inequality, we consider

$$\xi_h^n = \bar{u}_h^n - u(t_n), \quad \forall\, n \in [\![0, M+1]\!]. \tag{2.31}$$

It is useful to remark that ξ_h^n is estimated in (2.21) and (2.24) and the following relation holds

$$u_h^n - u(t_n) = \bar{e}_h^n + \xi_h^n. \tag{2.32}$$

1. Estimate of $\|\partial^1 \bar{e}_h^1\|_{\mathcal{L}^2(\Omega)}$. Using (2.32) and the triangle inequality, we get

$$\|\partial^1 \bar{e}_h^1\|_{\mathcal{L}^2(\Omega)} \leqslant \sum_{i=1}^{3} \mathbb{T}_i, \tag{2.33}$$

where

$$\mathbb{T}_1 = \|\partial^1 \xi_h^1\|_{\mathcal{L}^2(\Omega)}, \quad \mathbb{T}_2 = \|\partial^1 u_h^1 - \bar{u}^1\|_{\mathcal{L}^2(\Omega)} \text{ and } \mathbb{T}_3 = \|\bar{u}^1 - \partial^1 u(t_1)\|_{\mathcal{L}^2(\Omega)}. \tag{2.34}$$

Estimate (2.24), when $j = 1$, with (2.31) leads to

$$\mathbb{T}_1 \leqslant C_5 h^{k+1} \|u\|_{\mathcal{C}^2([0,T];\mathcal{H}^{k+1}(\Omega))}. \tag{2.35}$$

Equation (1.9) with $\mathcal{L}^2$-error estimate implies, see [9], Proposition 6.2.2, page 173

$$\mathbb{T}_2 \leqslant C_5 h^{k+1} \|\bar{u}^1\|_{\mathcal{C}([0,T];\mathcal{H}^{k+1}(\Omega))}. \tag{2.36}$$

This with (1.12) and the facts that $u_{tt}(0) = f(0) + \Delta u(0)$ and $\tau < T$ implies that

$$\mathbb{T}_2 \leqslant C_6 h^{k+1} \|u\|_{\mathcal{C}^2([0,T];\mathcal{H}^{k+1}(\Omega))}. \tag{2.37}$$

A convenient Taylor expansion implies that

$$\mathbb{T}_3 \leqslant C_7 \tau^2 \|u\|_{\mathcal{C}^3([0,T];\mathcal{L}^2(\Omega))}. \tag{2.38}$$

Gathering now (2.33), (2.35), (2.37), and (2.38) implies that

$$\|\partial^1 \bar{e}_h^1\|_{\mathcal{L}^2(\Omega)} \leqslant C_8 (h^{k+1} + \tau^2) \|u\|_{\mathcal{C}^3([0,T];\mathcal{C}(\overline{\Omega}))}. \tag{2.39}$$

2. Estimate of $|\bar{e}_h^1|_{1,\Omega}$. Let us first remark that thanks to (1.8)–(1.9), we have

$$\mathbf{a}(u_h^1, v) = -(\Delta(u^0 + \tau\bar{u}^1), v)_{\mathcal{L}^2(\Omega)}, \ \forall\, v \in \mathcal{V}_0^h. \tag{2.40}$$

In order to bound $|\bar{e}_h^1|_{1,\Omega} = |u_h^1 - \bar{u}_h^1|_{1,\Omega}$, we use the triangle inequality to get

$$|\bar{e}_h^1|_{1,\Omega} \leqslant |u_h^1 - \omega|_{1,\Omega} + |\omega - u(t_1)|_{1,\Omega} + |u(t_1) - \bar{u}_h^1|_{1,\Omega}, \tag{2.41}$$

where, using the expression (1.12) and (1.1)–(1.2),

$$\omega = u^0 + \tau\bar{u}^1 = u(0) + \tau u_t(0) + \frac{\tau^2}{2} u_{tt}(0). \tag{2.42}$$

Equation (2.40) with $\mathcal{H}^1$-error estimate implies

$$|u_h^1 - \omega|_{1,\Omega} \leqslant C_4 h^k \|\omega\|_{\mathcal{H}^{k+1}(\Omega)}. \tag{2.43}$$

This with (2.42) and the fact that $\tau < T$ leads to

$$|u_h^1 - \omega|_{1,\Omega} \leqslant C_9 h^k \|u\|_{\mathcal{C}^2([0,T];\ \mathcal{H}^{k+1}(\Omega))}. \tag{2.44}$$

A convenient Taylor expansion implies that

$$|\omega - u(t_1)|_{1,\Omega} \leqslant C_{10}\tau^3 \|u\|_{\mathcal{C}^3([0,T];\ \mathcal{H}^1(\Omega))}. \tag{2.45}$$

Gathering now (2.41), (2.44), (2.45), and (2.21) (when $n = 1$) implies that

$$|\bar{e}_h^1|_{1,\Omega} \leqslant C_{11}(h^k + \tau^3) \|u\|_{\mathcal{C}^3([0,T];\ \mathcal{H}^{k+1}(\Omega))}. \tag{2.46}$$

3. Estimate of $\mathcal{S}$: substituting f by $u_{tt} - \Delta u$, see (1.1), in the expansion of $\mathcal{S}^n$, we get

$$\mathcal{S}^n = \frac{1}{2}(\gamma u_{tt}(t_{n+1}) + 2(1-\gamma)u_{tt}(t_n) + \gamma u_{tt}(t_{n-1})) - \partial^2 \bar{u}_h^{n+1}. \tag{2.47}$$

Thanks to the Taylor expansion, (2.24) (when $j = 2$), and the triangle inequality, we have

$$\mathcal{S} \leqslant C_{12}(h^{k+1} + \tau^2) \|u\|_{\mathcal{C}^4([0,T];\mathcal{H}^{k+1}(\Omega))}. \tag{2.48}$$

Gathering now (2.30), (2.39), (2.46), (2.48), and the inequality $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$, for all $a, b \geqslant 0$, to get

$$\|\partial^1 \bar{e}_h^{n+1}\|_{\mathcal{L}^2(\Omega)} \leqslant C_{13}(h^k + \tau^2)\|u\|_{\mathcal{C}^4([0,T];\mathcal{H}^{k+1}(\Omega))} \text{ and } |\bar{e}_h^{n+1}|_{1,\Omega} \leqslant C_{14}(h^k + \tau^2)\|u\|_{\mathcal{C}^4([0,T];\mathcal{H}^{k+1}(\Omega))}.$$

The previous estimate with (2.21), (2.24), and (2.32) implies the desired estimates (2.2)–(2.3). □

3. Conclusion

We considered the wave equation, as a model of second order hyperbolic equations, and a family of finite element schemes with a parameter denoted by γ based on the use of the Newmark method as a discretization in time. We proved in the present contribution that the error is of order $h^k + \tau^2$ in the discrete norms of $\mathcal{L}^\infty(\mathcal{H}^1)$ and $\mathcal{W}^{1,\infty}(\mathcal{L}^2)$. These simple results seem not to be present in the existing literature. The stated results can be extended to second order hyperbolic equations with time independent variable coefficients.

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