ON AN EXTREMAL PROBLEM

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Summary. Let S denote the class of functions $f(z)=z+a_2z^2+a_3z^3+\ldots$ univalent and holomorphic in the unit disc $\Delta=\{z\colon|z|<1\}$. In the paper we obtain a sharp estimate of the functional $|a_3-\alpha a_2^2|+\alpha|a_2|^2$ in the class S for an arbitrary $\alpha\in\mathbb{R}$.

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1. Introduction

Let S stand for the well-known class of functions

$$f(z)=z+a_2z^2+a_3z^3+\ldots \tag{1.1}$$

holomorphic and univalent in the unit disc $\Delta=\{z\colon|z|<1\}$. It is well known that, for each function $f\in S$ ([1]),

$$|a_2|\leqslant 2 \tag{1.2}$$

with equality occurring only for the Koebe function

$$f_0(z)=\frac{z}{(1-\varepsilon z)^2},\qquad z\in\Delta,\quad|\varepsilon|=1. \tag{1.3}$$

In many papers, the functional $|a_3-\alpha a_2^2|$ was studied for different classes of univalent functions of the form (1.1). As is known, the maximum of this functional in the class S for $\alpha\in(0,1)$ is not attained for function (1.3) ([2]). Hence and from some applications an idea arises to consider the functional

$$\mathcal{F}(f)=|a_3-\alpha a_2^2|+\alpha|a_2|^n,\qquad\alpha\in\mathbb{R},\quad n\in\{1,2,3,\ldots\}, \tag{1.4}$$

in the class S. From (1.2) and the well-known result of Jenkins [4] we immediately get that the maximum of functional (1.4) for $\alpha \geqslant 1$, $n = 1, 2 \ldots$, is attained for function (1.3).

In paper [3], the Valiron-Landau lemma was applied to determinate the maximum of functional (1.4). The final estimate of this functional for some values of α, n is not sharp.

And so, in the case $n = 2$, the method applied did not give a sharp estimate for $\alpha \in (\frac{1}{2}, 1)$. In the present paper, using the variational method, we have succeeded in getting a complete result; namely, we have obtained the maximum of functional (1.4) in the case $n = 2$ for all $\alpha \in \mathbb{R}$. So, we shall consider the functional

$$H(f) = |a_3 - \alpha a_2^2| + \alpha |a_2|^2, \qquad f \in S, \quad \alpha \in \mathbb{R}. \tag{1.5}$$

In the cases $\alpha \leqslant \frac{1}{2}$ and $\alpha \geqslant 1$, the estimate of the maximum of functional (1.5) given in paper [3] is sharp and is attained for function (1.3). Thus it will be sufficient to limit our further considerations to the case $\alpha \in (\frac{1}{2}, 1)$. It will be seen that this restriction is not essential for the fundamental procedure.

The case $n \neq 2$ needs additional considerations and will be a subject matter of separate investigations.

2. Discussion of the form of the equation for extremal functions

Let us consider the functional

$$G(f) = \mathrm{Re}(a_3 - \alpha a_2^2) + \alpha |a_2|^2 \tag{2.1}$$

defined in the class S, where $\alpha \in (\frac{1}{2}, 1)$. The family S is compact, whereas functional (2.1) is continuous, thus, for each $\alpha \in \mathbb{R}$, there exists a function $f_\alpha \in S$ for which $G(f_\alpha) = \max\limits_{f \in S} G(f)$. In the sequel, the function $f = f_\alpha$ will be called extremal.

Functional (2.1) satisfies the assumptions of the Schaeffer-Spencer theorem ([6], pp. 36–37), hence each extremal function satisfies the following equation:

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1 + uf(z)}{f^2(z)} = \frac{z^4 + \bar{u}z^3 + 2B_0 z^2 + uz + 1}{z^2}, \qquad z \in \Delta, \tag{2.2}$$

where

$$B_0 = a_3 - \alpha a_2^2 + \alpha |a_2|^2, \tag{2.3}$$

$$u = 2[\mathrm{Re}\, a_2 + \mathrm{i}(1 - 2\alpha)\,\mathrm{Im}\, a_2]. \tag{2.4}$$

Besides, it is known ([6]) that $B_0 > 0$, and that the right-hand side of (2.2) is nonnegative on the circle $|z| = 1$ and possesses on it at least one double root.

Since (2.2) is a differential-functional equation, the determination of the upper bound of functional (2.1) for any fixed $\alpha \in (\frac{1}{2}, 1)$ is reduced to the finding of suitable functions which satisfy this equation. It is worth recalling that the fulfilment of equation (2.2) by a function is only a necessary condition for this function to be extremal for the functional being examined.

For $z \in \Delta$, $z \neq 0$, let us put

$$N(z) = \frac{z^4 + \bar{u}z^3 + 2B_0 z^2 + uz + 1}{z^2}. \tag{2.5}$$

It follows from the general properties of equation (2.2) that function (2.5) is factorized in the following way ([6]):

$$N(z) = \frac{(z - \mathrm{e}^{\mathrm{i}\psi})^2 (z^2 - t\mathrm{e}^{-\mathrm{i}\varphi} z + \mathrm{e}^{-2i\varphi})}{z^2}$$

where $\psi, \varphi \in (-\pi, \pi\rangle$, $t \geqslant 2$.

Note that if the function $f(z)$ is extremal with respect to the functional considered, then also the functions $-f(-z)$ and $\overline{f(\bar{z})}$ are extremal. Hence it appears that, in our further considerations, it is enough to assume that $\psi \in \langle 0, \frac{\pi}{2}\rangle$.

Taking into account all the factorizations of function (2.5) and the above remarks, it is easy to prove that equation (2.2) can only be of the form

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1 + uf(z)}{f^2(z)} = \frac{(z - z_0)^2 (z - z_1)(z - z_2)}{z^2}, \qquad u \neq 0, \tag{a}$$

or

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1 + uf(z)}{f^2(z)} = \frac{(z - z_0)^2 (z - z_3)^2}{z^2}, \qquad u \neq 0, \tag{b}$$

or

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1 + uf(z)}{f^2(z)} = \frac{(z - z_0)^4}{z^2}, \qquad u \neq 0, \tag{c}$$

or

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1}{f^2(z)} = \frac{(z - z_0)^2 (z - z_3)^2}{z^2}, \qquad u = 0, \tag{d}$$

where $z_0 = \mathrm{e}^{\mathrm{i}\psi}$, $z_1 = \varrho \mathrm{e}^{\mathrm{i}\varphi}$, $z_2 = \frac{1}{\bar{z}_1}$, $\varrho \in (0, 1)$, $\psi \in \langle 0, \frac{\pi}{2}\rangle$, $\varphi \in (-\pi, \pi\rangle$, $|z_3| = |z_0| = 1$, $z_3 \neq z_0$.

The next sections of the paper will be devoted to the investigation of solutions of equations (a), (b), (c), (d), respectively. A part of detailed computations will be omitted because they are similar to the reasonings in many other papers.

3. Equation of the form (a)

Let us first consider the case when equation (2.2) is of the form (a). Comparing the right-sides of (2.2) and (a), we get

$$z_1 = \varrho \bar{z}_0, \qquad z_2 = \frac{1}{\varrho}\bar{z}_0, \tag{3.1}$$

$$u = -2\mathrm{e}^{-\mathrm{i}\psi} - \Big(\varrho + \frac{1}{\varrho}\Big)\mathrm{e}^{\mathrm{i}\psi}, \tag{3.2}$$

$$B_0 = \varrho + \frac{1}{\varrho} + \cos 2\psi, \tag{3.3}$$

$z_0 = \mathrm{e}^{\mathrm{i}\psi}$, $\psi \in \langle 0, \frac{\pi}{2}\rangle$, $\varrho \in (0,1)$.

From formulae (2.3) and (3.3) it can be seen that the value of the expression $a_3 - \alpha a_2^2 + \alpha |a_2|^2$ for an extremal function satisfying an equation of form (a) is determined by two real parameters ψ and ϱ. Hence it appears that, in order to determine the upper bound of functional (2.1), one has to find some relationships between ψ, ϱ and α.

In virtue of (3.1), equation (a) is equivalent to

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1+uf(z)}{f^2(z)} = \frac{(1-\bar{z}_0 z)^2(1-\varrho z_0 z)^2 \frac{1-\frac{1}{\varrho}z_0 z}{1-\varrho z_0 z}}{z^2}, \qquad z \in \varDelta. \tag{3.4}$$

Integrating (3.4) and, next, expanding both sides of the equation obtained in a Laurent series with centre $z=0$ and comparing the coefficients at the corresponding powers of z, we get

$$\log \frac{2+(\varrho+\frac{1}{\varrho})z_0^2}{(\frac{1}{\varrho}-\varrho)z_0^2} + \frac{\varrho+\frac{1}{\varrho}+2z_0^2}{2+(\varrho+\frac{1}{\varrho})z_0^2}\log\frac{1-\varrho}{1+\varrho} = \frac{2(a_2 z_0+2)}{2+(\varrho+\frac{1}{\varrho})z_0^2}. \tag{3.5}$$

From (2.4) and (3.2) we have

$$a_2 = -\frac{1}{2}\Big(\varrho+\frac{1}{\varrho}+2\Big)\cos\psi + \mathrm{i}\frac{1}{2(2\alpha-1)}\Big(\varrho+\frac{1}{\varrho}-2\Big)\sin\psi. \tag{3.6}$$

By isolating the real and the imaginary part in (3.5), in view of (3.6) we obtain the following system of equations:

(3.7)
$$\Big(\varrho+\frac{1}{\varrho}+2\Big)\cos\psi\log\frac{\sqrt{(\varrho+\frac{1}{\varrho})^2+4(\varrho+\frac{1}{\varrho})\cos 2\psi+4}}{\varrho+\frac{1}{\varrho}+2} + \Big(\varrho+\frac{1}{\varrho}-2\Big)\sin\psi\arctan\frac{2\sin 2\psi}{\varrho+\frac{1}{\varrho}+2\cos 2\psi}+\Big(\varrho+\frac{1}{\varrho}-2\Big)\cos\psi=0,$$

$$\begin{aligned}&\Big(\varrho+\frac{1}{\varrho}-2\Big)\sin\psi\log\frac{\sqrt{(\varrho+\frac{1}{\varrho})^2+4(\varrho+\frac{1}{\varrho})\cos 2\psi+4}}{\varrho+\frac{1}{\varrho}-2}\\ &\quad-\Big(\varrho+\frac{1}{\varrho}+2\Big)\cos\psi\arctan\frac{2\sin 2\psi}{\varrho+\frac{1}{\varrho}+2\cos 2\psi}+4\sin\psi\\ &\quad=\frac{1}{2\alpha-1}\Big(\varrho+\frac{1}{\varrho}-2\Big)\sin\psi,\end{aligned}\tag{3.8}$$

where $\psi\in\langle 0,\frac{\pi}{2}\rangle$, $\varrho\in(0,1)$.

Let us first observe that if $\psi=0$, then case (a) does not hold; this follows from estimate (1.2) and equality (3.6) for $\psi=0$ and $\varrho\in(0,1)$.

Next, putting $\psi=\frac{\pi}{2}$ in system (3.7)–(3.8), we obtain $\varrho+\frac{1}{\varrho}=8\alpha-2$. Then from (3.3) we get

Lemma 1. *If, for $\psi=\frac{\pi}{2}$, $\alpha\in(\frac{1}{2},1)$, the extremal function satisfies the equation of the form* (a) *with $\psi=\frac{\pi}{2}$, then*

$$B_0=8\alpha-3.\tag{3.9}$$

Equality (3.9) *holds only for Koebe function* (1.3) *with $\varepsilon=\mathrm{i}$. For $\psi=0$, the extremal function does not satisfy the equation of form* (a).

Consequently, we shall consider the system of equations (3.7)–(3.8) for $\psi\in(0,\frac{\pi}{2})$, $\varrho\in(0,1)$. Of course, the following questions arise: for what α's does the system of equations (3.7)–(3.8) possess a solution, and is this solution the only one?

Let us consider the first equation of this system. For $\psi\in(0,\frac{\pi}{2})$, equation (3.7) will take the form

$$\Phi_1(\psi,\varrho)=0\tag{3.7'}$$

where

$$\Phi_1(\psi,\varrho)=\Big(\varrho+\frac{1}{\varrho}+2\Big)\log\frac{\sqrt{(\varrho+\frac{1}{\varrho})^2+4(\varrho+\frac{1}{\varrho})\cos 2\psi+4}}{\varrho+\frac{1}{\varrho}+2} \tag{3.10}$$
$$+\Big(\varrho+\frac{1}{\varrho}-2\Big)\tan\psi\cdot\arctan\frac{2\sin 2\psi}{\varrho+\frac{1}{\varrho}+2\cos 2\psi}+\varrho+\frac{1}{\varrho}-2.$$

From the investigation of the function Φ_1 as a function of ϱ, $\varrho\in(0,1)$, we conclude that equation (3.7′) has the only solution ϱ for any fixed $\psi\in(0,\frac{\pi}{2})$. Hence equation (3.7′) defines a function $\varrho=\varrho(\psi)$, $\psi\in(0,\frac{\pi}{2})$. Moreover, from (3.7′) and (3.10) we have

$$\lim_{\psi\to 0^+}\varrho(\psi)=1 \tag{3.11}$$

and

$$\lim_{\psi\to\frac{\pi}{2}^-}\varrho(\psi)=\frac{\sqrt{\mathrm{e}}-1}{\sqrt{\mathrm{e}}+1}. \tag{3.12}$$

Since $\Phi'_{1\varrho}\neq 0$, therefore from (3.7′) we have

$$\varrho'(\psi)=-\frac{\Phi'_{1\psi}(\psi,\varrho)}{\Phi'_{1\varrho}(\psi,\varrho)},\qquad \varrho=\varrho(\psi),\quad \psi\in\Big(0,\frac{\pi}{2}\Big).$$

Hence and from (3.10), after some transformations, we obtain

$$\varrho'(\psi)=\frac{1}{4}\frac{(\varrho+\frac{1}{\varrho}-2)^2}{(1-\frac{1}{\varrho^2})\cos^2\psi}\cdot\frac{\arctan\frac{2\sin 2\psi}{\varrho+\frac{1}{\varrho}+2\cos 2\psi}-\frac{2\sin 2\psi}{\varrho+\frac{1}{\varrho}-2}}{\log\frac{\sqrt{(\varrho+\frac{1}{\varrho})^2+4(\varrho+\frac{1}{\varrho})\cos 2\psi+4}}{\varrho+\frac{1}{\varrho}+2}} \tag{3.13}$$

where $\varrho=\varrho(\psi)$, $\psi\in(0,\frac{\pi}{2})$. In view of (3.13), it can be demonstrated that

$$\varrho'(\psi)<0\qquad\text{for}\quad\Big(0,\frac{\pi}{2}\Big). \tag{3.14}$$

It follows from the above that equation (3.7′) defines one function $\varrho=\varrho(\psi)$, $\psi\in(0,\frac{\pi}{2})$, and, moreover, this function is decreasing from the value 1 to the value $\frac{\sqrt{\mathrm{e}}-1}{\sqrt{\mathrm{e}}+1}$.

Let next

$$D=\Big\{(\psi,\varrho)\colon\Big(0,\frac{\pi}{2}\Big)\wedge\varrho=\varrho(\psi)\Big\} \tag{3.15}$$

where $\varrho(\psi)$ is defined by equation (3.7'). Let us consider the second equation of system (3.7)–(3.8). From (3.8) we have

$$\frac{1}{2\alpha-1}=\Phi_2(\psi,\varrho) \tag{3.8'}$$

where

$$\begin{aligned}\Phi_2(\psi,\varrho)= {}&\log\frac{\sqrt{(\varrho+\frac{1}{\varrho})^2+4(\varrho+\frac{1}{\varrho})\cos 2\psi+4}}{\varrho+\frac{1}{\varrho}-2}\\ &-\frac{\varrho+\frac{1}{\varrho}+2}{\varrho+\frac{1}{\varrho}-2}\operatorname{cotan}\psi\cdot\arctan\frac{2\sin 2\psi}{\varrho+\frac{1}{\varrho}+2\cos 2\psi}+\frac{4}{\varrho+\frac{1}{\varrho}-2}.\end{aligned} \tag{3.16}$$

From the investigation of equation (3.7') and the form of equation (3.8') it follows that if $(\psi,\varrho)\in D$ where D is defined by (3.15), then there exists exactly one α as the function of variable $\psi\in(0,\frac{\pi}{2})$; so, from (3.8') we have

$$\frac{1}{2\alpha(\psi)-1}=\Phi_2(\psi,\varrho(\psi)),\qquad \Big(0,\frac{\pi}{2}\Big). \tag{3.17}$$

Differentiating both sides of equation (3.17) and taking account of (3.16) and (3.13) in it, after suitable transformations we get

$$\begin{aligned}&\frac{2}{[2\alpha(\psi)-1]^2}\alpha'(\psi)\\ &=\frac{(\varrho+\frac{1}{\varrho}-2\cos 2\psi)(\varrho+\frac{1}{\varrho}+2\cos 2\psi)}{\log\frac{\sqrt{(\varrho+\frac{1}{\varrho})^2+4(\varrho+\frac{1}{\varrho})\cos 2\psi+4}}{\varrho+\frac{1}{\varrho}+2}}\,\frac{\arctan\frac{2\sin 2\psi}{\varrho+\frac{1}{\varrho}+2\cos 2\psi}-\frac{2\sin 2\psi}{\varrho+\frac{1}{\varrho}+2\cos 2\psi}}{(\varrho+\frac{1}{\varrho}-2)(\varrho+\frac{1}{\varrho}+2)\sin^2\psi\cos^2\psi}\end{aligned} \tag{3.18}$$

where $(\psi,\varrho)\in D$. Hence it is easy to check that $\alpha'(\psi)>0$ for $\psi\in(0,\frac{\pi}{2})$.

From (3.17), (3.16) and (3.11), (3.12) it can be verified that

$$\lim_{\psi\to 0^+}\alpha(\psi)=\frac{1}{2},\qquad \lim_{\psi\to\frac{\pi}{2}^-}\alpha(\psi)=\alpha_0$$

where

$$\alpha_0=\frac{\mathrm{e}}{2(\mathrm{e}-1)},\qquad \alpha_0<1. \tag{3.19}$$

In view of the above, we infer that $\alpha(\psi)$ is an increasing function of the variable $\psi\in(0,\frac{\pi}{2})$; besides, it increases from $\frac{1}{2}$ to α_0. Hence, for the function $\alpha(\psi)$, we have the inverse function $\psi=\psi(\alpha)$ defined for $\alpha\in(\frac{1}{2},\alpha_0)$ where α_0 is given by (3.19).

To sum up, we have proved that, for $\alpha \in (\frac{1}{2}, \alpha_0)$, we have a single solution to the system of equations (3.7)–(3.8). Consequently, from (3.3) we obtain

Lemma 2. *If, for $\alpha \in (\frac{1}{2}, \alpha_0)$, $\alpha_0 = \frac{\mathrm{e}}{2(\mathrm{e}-1)}$, the extremal function satisfies the equation of form (a), then*

$$B_0 = B_0(\alpha) = \varrho + \frac{1}{\varrho} + \cos 2\psi$$

where ψ, ϱ are the only solutions to the system of equations (3.7)–(3.8), $(\psi, \varrho) \in D$, *and D is defined by* (3.15). *Moreover,*

$$\lim_{\alpha \to \frac{1}{2}^-} B_0(\alpha) = 3, \qquad \lim_{\alpha \to \alpha_0^-} B_0(\alpha) = \frac{\mathrm{e}+3}{\mathrm{e}-1}.$$

For $\alpha \in \langle \alpha_0, 1)$, the extremal function does not satisfy equation (a) with $\psi \in (0, \frac{\pi}{2})$, $\varrho \in (0,1)$.

4. Equation of the form (b)

Let us next consider the case when equation (2.2) is of form (b), that is,

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1+uf(z)}{f^2(z)} = \frac{(z-z_0)^2(z-z_3)^2}{z^2}, \qquad z \in \Delta, \tag{4.1}$$

where $z_0 = \mathrm{e}^{\mathrm{i}\psi}$, $\psi \in \langle 0, \frac{\pi}{2} \rangle$, $|z_3| = 1$, $z_3 \neq z_0$, $u \neq 0$.

From the comparison of the right-hand sides of equations (2.2) and (4.1) and from the fact that $B_0 > 0$ it follows that $z_3 = \bar{z}_0 = \mathrm{e}^{-\mathrm{i}\psi}$ and, in consequence,

$$u = -4\cos\psi, \tag{4.2}$$

$$B_0 = 2 + \cos 2\psi. \tag{4.3}$$

After integrating (4.1) and making use of the fact that there exists $x \in \mathbb{R}$ such that $f(\mathrm{e}^{\mathrm{i}x}) = -\frac{1}{u}$, we obtain $\psi = 0$ or $\psi = \frac{\pi}{2}$. Since $u \neq 0$, therefore from (4.2) we have $\psi = 0$. Comparing (2.4) and (4.2), we get $\operatorname{Re} a_2 = -2$, $\operatorname{Im} a_2 = 0$, $\operatorname{Re} a_3 = 3$, $\operatorname{Im} a_3 = 0$.

Consequently, from (4.3) we obtain

Lemma 3. *If, for $\alpha \in (\frac{1}{2}, 1)$, the extremal function satisfies the equation of form* (b), *then*

$$B_0 = 3. \tag{4.4}$$

Equality (4.4) *holds for Koebe function* (1.3) *with* $\varepsilon = -1$.

5. Equation of the form (c)

In this case, equation (2.2) is equivalent to

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1+uf(z)}{f^2(z)} = \frac{(z-z_0)^4}{z^2}, \qquad z \in \Delta, \tag{5.1}$$

where $z_0 = \mathrm{e}^{\mathrm{i}\psi}$, $\psi \in \langle 0, \frac{\pi}{2}\rangle$, $u \neq 0$.

From the comparison of the right-hand sides of equations (2.2) and (5.1) and from the fact that $B_0 > 0$ it follows that $z_0 = 1$. In consequence, $u = -4$ and $B_0 = 3$, thus $\operatorname{Re} a_2 = -2$, $\operatorname{Im} a_2 = 0$, $\operatorname{Re} a_3 = 3$, $\operatorname{Im} a_3 = 0$. So, we have

Lemma 4. *If, for* $\alpha \in (\frac{1}{2}, 1)$, *the extremal function satisfies the equation of the form* (c), *then*

$$B_0 = 3$$

and this equality holds for function (1.3) *with* $\varepsilon = -1$.

6. Equation of the form (d)

Let us consider the last case when equation (2.2) is of form (d), that is,

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1}{f^2(z)} = \frac{(z-z_0)^2(z-z_3)^2}{z^2}, \qquad z \in \Delta, \tag{6.1}$$

where $z_0 = \mathrm{e}^{\mathrm{i}\psi}$, $\psi \in \langle 0, \frac{\pi}{2}\rangle$, $z_3 \neq z_0$, $|z_3| = 1$.

Putting $u = 0$ in equation (2.2) and comparing the right-hand side of this equation with that of (6.1), we get $z_3 = \bar{z}_0$, $\psi = \frac{\pi}{2}$ and $B_0 = 1$. So, we have

Lemma 5. *If, for* $\alpha \in (\frac{1}{2}, 1)$, *the extremal function satisfies the equation of the form* (d), *then*

$$B_0 = 1$$

and this equality holds for the function $f(z) = \frac{z}{1-z^2}$, $z \in \Delta$.

7. The main theorem

Note first that if $f \in S$, then, for all $\Theta \in \langle 0, 2\pi)$, the function $e^{i\Theta} f(e^{-i\Theta} z)$, $z \in \Delta$, belongs to S, too. In consequence, the determination of the maximum of functional (1.5) is equivalent to the determination of the maximum of functional (2.1) in the class S. For this purpose, we make use of the lemmas just proved. From Lemmas 1–5 and (2.1), (2.3) it follows that

$$G(f) \leqslant \begin{cases} \max\{8\alpha - 3, \varrho + \frac{1}{\varrho} + \cos 2\psi, 3, 1\} & \text{when } \alpha \in (\frac{1}{2}, \alpha_0), \\ \max\{8\alpha - 3, 3, 1\} & \text{when } \alpha \in \langle \alpha_0, 1), \end{cases} \tag{7.1}$$

where ψ, ϱ are defined in Lemma 2, and $\alpha_0 = \frac{e}{2(e-1)}$.

Since $\alpha_0 > \frac{3}{4}$, therefore $\max\{8\alpha - 3, 3, 1\} = 8\alpha - 3$ when $\alpha \in \langle \alpha_0, 1)$.

It follows from the results of Section 3 that if $\alpha \in (\frac{1}{2}, \alpha_0)$, then

$$B_0 = B_0(\alpha) = \varrho + \frac{1}{\varrho} + \cos 2\psi \tag{7.2}$$

where $\psi = \psi(\alpha)$, $\varrho = \varrho(\psi(\alpha))$ (cf. Lemma 2).

It is easily noticed that $B_0(\alpha) > 3$ for $\alpha \in (\frac{1}{2}, \alpha_0)$. So, it remains to compare the values $8\alpha - 3$ and $B_0(\alpha)$ given by (7.2) for $\alpha \in (\frac{1}{2}, \alpha_0)$.

Let us put

$$\tilde{B}(\alpha) = B_0(\alpha) - (8\alpha - 3), \qquad \alpha \in \left(\frac{1}{2}, \alpha_0\right) \tag{7.3}$$

where B_0 is defined by (7.2).

Making use of the results obtained in Section 3, it can be verified that

$$\lim_{\alpha \to \frac{1}{2}^+} \tilde{B}(\alpha) = 2 \qquad \text{and} \qquad \lim_{\alpha \to \alpha_0^-} \tilde{B}(\alpha) = 0.$$

Moreover, from (7.2) and (7.3) we have

$$\tilde{B}'(\alpha) = \left[\left(1 - \frac{1}{\varrho^2(\psi(\alpha))}\right) \varrho'(\psi(\alpha)) - 2\sin(2\psi(\alpha))\right] \psi'(\alpha) - 8 \tag{7.4}$$

where $\psi(\alpha)$ is the inverse function of the function $\alpha(\psi)$ defined by formula (3.17). Taking account of formulae (3.13), (3.18) in (7.4), after suitable transformations we obtain $\tilde{B}'(\alpha) < 0$ for $\alpha \in (\frac{1}{2}, \alpha_0)$. Hence and from (7.3) it follows that

$$B_0(\alpha) > 8\alpha - 3 \qquad \text{for} \quad \left(\frac{1}{2}, \alpha_0\right).$$

Consequently, for $\alpha \in (\frac{1}{2}, \alpha_0)$ we have

$$\max\left\{8\alpha - 3, \varrho + \frac{1}{\varrho} + \cos 2\psi, 3, 1\right\} = \varrho + \frac{1}{\varrho} + \cos 2\psi$$

where $(\psi, \varrho) \in D$ and D is defined by (3.13).

To sum up, we have proved

Theorem. *For any function $f \in S$ we have*

$$|a_3 - \alpha a_2^2| + \alpha |a_2|^2 \leqslant \varrho + \frac{1}{\varrho} + \cos 2\psi \qquad \textit{for} \quad \frac{1}{2} < \alpha < \frac{\mathrm{e}}{2(\mathrm{e}-1)}, \tag{7.5}$$

$$|a_3 - \alpha a_2^2| + \alpha |a_2|^2 \leqslant 8\alpha - 3 \qquad \textit{for} \quad \frac{\mathrm{e}}{2(\mathrm{e}-1)} \leqslant \alpha < 1, \tag{7.6}$$

where ψ is the inverse function to the function $\alpha(\psi)$ of form (3.17) *and $\varrho = \varrho(\psi)$ is defined by equation* (3.7′). *Estimates* (7.5)–(7.6) *are sharp. In case* (7.6), *the equality holds for Koebe function* (1.3).

Remark 1. As was mentioned before, by modifying the procedure presented, it can be proved that

$$|a_3 - \alpha a_2^2| + \alpha |a_2|^2 \leqslant 3 \qquad \text{for} \quad \alpha \leqslant \frac{1}{2}, \tag{7.7}$$

and that estimate (7.6) holds also for $\alpha \geqslant 1$. In paper [3], these results are also presented.

Remark 2. The result we have obtained also proves that, in the case of functional (1.5), the variational method turned out to be more effective though tiresome in calculations.

Remark 3. Similarly as in paper [3] (Section 4) one can use some applications of inequalities (7.5)–(7.7) to obtain estimates of suitable functionals considered in other classes of univalent functions.

In addition, one can find some considerations of functionals of type (1.5) in other classes of functions, for instance, in paper [5].

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