ON EXISTENCE OF KNESER SOLUTIONS OF A CERTAIN CLASS OF n-TH ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. The paper deals with existence of Kneser solutions of n-th order nonlinear differential equations with quasi-derivatives.

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1. Introduction

The aim of our paper is to give some conditions for existence of Kneser solutions of the differential equation

$$L(y) \equiv 0, \tag{L}$$

where

$$\begin{aligned}
&L(y) \equiv L_n y + \sum_{k=1}^{n-1} P_k(t) L_k y + P_0(t) f(y),\\
&L_0 y(t) = y(t),\\
&L_1 y(t) = p_1(t)(L_0 y(t))' = p_1(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t},\\
&L_k y(t) = p_k(t)(L_{k-1} y(t))' \quad \text{for} \quad k = 2, 3, \ldots, n-1,\\
&L_n y(t) = (L_{n-1} y(t))',
\end{aligned}$$

n is an arbitrary positive integer, $n \geqslant 2$, $P_k(t)$, $k = 0, 1, \ldots, n-1$, $p_i(t)$, $i = 1, 2, \ldots, n-1$ are real-valued continuous functions on the interval $I_a = [a, \infty)$, $-\infty < a < \infty$; $f(t)$ is a real-valued function continuous on $E_1 = (-\infty, \infty)$.

If $n = 1$, then $L(y) \equiv L_1 y + P_0(t) f(y) = y' + P_0(t) f(y)$, $P_0(t)$ and $f(t)$ are real-valued continuous functions on I_a and on E_1, respectively.

It is assumed throughout that

(A) $P_k(t) \leqslant 0$, $p_i(t) > 0$ for all $t \in I_a$, $k = 0, 1, \ldots, n-1$, $i = 1, 2 \ldots, n-1$; $f(0) \neq 0$, $f(t) \geqslant 0$ for all $t \in E_1$; $P_0(t)$ is not identically zero in any subinterval of I_a; n is an arbitrary positive integer, $n \geqslant 2$. If $n = 1$, then $P_0(t) \leqslant 0$ and $f(t) \geqslant 0$ for all $t \in I_a$ and E_1, respectively.

The problems of existence of monotone or Kneser solutions for third order ordinary differential equations with quasi-derivatives were studied in several papers ([5], [7], [8], [10]). The equation (L), where $p_i(t) \equiv 1$, $i = 1, 2, 3$ $(n = 4)$ was studied, for example, in ([6], [9], [12]). Equations of the fourth order with quasi-derivatives were also studied, for instance, in ([1], [3], [13]).

Existence of monotone solutions for n-th order equations with quasi-derivatives was studied in [4].

In our paper, Theorem 1 and Theorem 2 give sufficient conditions for existence of a Kneser solution of (L) on $[a, \infty)$ for n an even number or for an odd one, respectively.

Now we explain the concept of a Kneser solution, and other useful ones:

Definition 1. A nontrivial solution $y(t)$ of a differential equation of the n-th order is called a Kneser solution on $I_a = [a, \infty)$ iff $(y(t) > 0,\ (-1)^k L_k y(t) \geqslant 0)$ or $(y(t) < 0,\ (-1)^k L_k y(t) \leqslant 0)$ for all $t \in I_a$, $k = 1, 2, \ldots, n-1$.

Definition 2. Let J be an arbitrary type of an interval with endpoints t_1, t_2, where $-\infty \leqslant t_1 < t_2 \leqslant \infty$. The interval J is called the maximum interval of existence of $u\colon J \to E_1^n$, where $u(t)$ is a solution of the differential system $u' = F(t, u)$ iff $u(t)$ can be continued neither to the right nor to the left of J.

Definition 3. Let $y' = U(t, y)$ be a scalar differential equation. Then $y_0(t)$ is called the maximum solution of the Cauchy problem

$$(*) \qquad y' = U(t, y),\ y(t_0) = y_0$$

iff $y_0(t)$ is a solution of $(*)$ on the maximum interval of existence and if $y(t)$ is another solution of $(*)$, then $y(t) \leqslant y_0(t)$ for all t belonging to the common interval of existence of $y(t)$ and $y_0(t)$.

We give some preliminary results.

Lemma 1. *Let $A(t,s)$ be a nonpositive and continuous function for $a \leqslant t \leqslant s \leqslant t_0$. If $g(t)$, $\psi(t)$ are continuous functions in the interval $[a,t_0]$ and*

$$\psi(t) \geqslant g(t) + \int_{t_0}^{t} A(t,s)\psi(s)\,\mathrm{d}s \quad \text{for } t \in [a,t_0],$$

then every solution $y(t)$ of the integral equation

$$y(t) = g(t) + \int_{t_0}^{t} A(t,s)y(s)\,\mathrm{d}s$$

satisfies the inequality $y(t) \leqslant \psi(t)$ in $[a,t_0]$.

P r o o f. See [6], page 331. □

Lemma 2. (Wintner) *Let $U(t,u)$ be a continuous function on a domain $t_0 \leqslant t \leqslant t_0+\alpha$, $\alpha > 0$, $u \geqslant 0$, let $u(t)$ be a maximum solution of the Cauchy problem $u' = U(t,u)$, $u(t_0) = u_0 \geqslant 0$ ($u' = U(t,u)$ is a scalar differential equation) existing on $[t_0,t_0+\alpha]$; for example, let $U(t,u) = \psi(u)$, where $\psi(u)$ is a continuous and positive function for $u \geqslant 0$ such that*

$$\int^{\infty} \frac{\mathrm{d}u}{\psi(u)} = \infty.$$

Let us assume $f(t,y)$ to be continuous on $t_0 \leqslant t \leqslant t_0+\alpha$, $y \in E_1^n$, y arbitrary, and to satisfy the condition

$$|f(t,y)| \leqslant U(t,|y|).$$

Then the maximum interval of existence of a solution of the Cauchy problem

$$y' = f(t,y), \quad y(t_0) = y_0,$$

where $|y_0| \leqslant u_0$, is $[t_0,t_0+\alpha]$.

P r o o f. See [2], Theorem III.5.1. □

Lemma 3. *Let* (A) *hold, and let there exist real nonnegative constants a_1, a_2 such that $f(t) \leqslant a_1|t| + a_2$ for all $t \in E_1$. Let initial values $L_k y(a) = b_k$ be given for $k = 0,1,\ldots,n-1$. Then there exists a solution $y(t)$ of* (L) *on $[a,\infty)$, which fulfils these initial conditions.*

P r o o f. See [4], Lemma 3. □

Lemma 4. *Let us assume $g(t,z)$ to be continuous on $t_0-\alpha \leqslant t \leqslant t_0$, α a positive constant, $z \in E_1^n$, z is arbitrary and satisfies a condition*

$$|g(t,z)| \leqslant \psi(|z|),$$

where $\psi(t)$ is a continuous and positive function for $t \geqslant 0$ such that

$$\int^{\infty} \frac{\mathrm{d}t}{\psi(t)} = \infty.$$

Then the maximum interval of existence of a solution of the Cauchy problem

$$z' = g(t,z),\ z(t_0) = z_0,$$

is $[t_0-\alpha, t_0]$.

P r o o f. Let us consider the Cauchy problem

$$\text{(u)} \qquad u' = \psi(u),\ u(-t_0) = u_0 = |z_0|.$$

According to the assumptions, the problem (u) admits a single solution $u_0(t)$ on $[-t_0, \infty)$, where

$$u_0(t) = R_{-1}(t+t_0)$$

and $R\colon [u_0,\infty) \to [0,\infty)$, $R(u) = \int_{u_0}^{u} \frac{1}{\psi(t)}\,\mathrm{d}t$, $R_{-1}(R(u)) = u$, $u \in [u_0,\infty)$. Let us consider the Cauchy problems

$$\text{(U)} \qquad u' = U(t,u) = \psi(u),\ u(-t_0) = u_0 = |z_0|,\ (t,u) \in [-t_0, -t_0+\alpha] \times [0,\infty),$$

$$\text{(y)} \quad y'(t) = g(-t,-y),\ y(-t_0) = -z_0,\ (t,y) \in [-t_0, -t_0+\alpha] \times E_1^n,$$

$$\text{(z)} \quad z'(t) = g(t,z),\ z(t_0) = z_0,\ (t,z) \in [t_0-\alpha, t_0] \times E_1^n.$$

Then $u_0(t) = R_{-1}(t+t_0)$ is the maximum solution of (U) on the maximum interval of existence $[-t_0, -t_0+\alpha]$. According to Lemma 2 there exists a solution $y_0(t)$ of (y) on $[-t_0, -t_0+\alpha]$. Then the Cauchy problem (z) admits the solution $z_0(t) = -y_0(-t)$ on $[t_0-\alpha, t_0]$ because of

$$z_0'(t) = y_0'(-t) = g(t, -y_0(-t)) = g(t, z_0(t))$$

on $[t_0-\alpha, t_0]$. So the maximum interval of existence of (z) is $[t_0-\alpha, t_0]$. □

Lemma 5. *Let* (A) *hold, and let there exist nonnegative real constants* a_1, a_2 *such that* $f(t) \leqslant a_1|t| + a_2$ *for all* $t \in E_1$. *Let initial values* $L_k y(t_0) = b_k$ *be given for* $k = 0, 1, \ldots, n-1$, $t_0 > a$. *Then there exists a solution* $y(t)$ *of* (L) *on* $[a, \infty)$, *which fulfils these initial conditions.*

Proof. According to Lemma 3 there exists a solution of (L) on $[t_0, \infty)$ such that the initial conditions hold. To prove our lemma we need to prove existence of a solution $y(t)$ of (L) on $[a, t_0]$ satisfying the given initial conditions. Consider now the following system (S), which corresponds to the equation (L):

$$\text{(S)}\qquad \begin{aligned} u_k'(t) &= \frac{u_{k+1}(t)}{p_k(t)}, \quad k = 1, 2, \ldots, n-1, \\ u_n'(t) &= -\sum_{k=1}^{n-1} P_k(t) u_{k+1}(t) - P_0(t) f(u_1(t)), \end{aligned}$$

where $u_k(t) = L_{k-1} y(t)$, $k = 1, 2, \ldots, n$, $f_k = u_{k+1}/p_k$, $k = 1, \ldots, n-1$, $f_n = -\sum P_k u_{k+1} - P_0 f(u_1)$, $F = (f_1, f_2, \ldots, f_n)$, $u = (u_1, u_2, \ldots, u_n)$, $u' = (u_1', u_2', \ldots, u_n')$, $|u| = \sum\limits_{k=1}^{n} |u_k|$, $|F| = \sum\limits_{k=1}^{n} |f_k|$, $(t, u) \in [a, t_0] \times E_1^n$. Then

$$\begin{aligned} |F(t,u)| &= \sum_{k=1}^{n-1} \left|\frac{u_{k+1}}{p_k}\right| + \left|-\sum_{k=1}^{n-1} P_k u_{k+1} - P_0 f(u_1)\right| \\ &\leqslant \sum_{k=1}^{n-1} (-P_k + \frac{1}{p_k}) |u_{k+1}| - P_0(a_1|u_1| + a_2) \leqslant K_1|u| + K_2 = \psi(|u|), \end{aligned}$$

where K_1, K_2 are appropriate positive real constants. It is obvious that

$$\int^{\infty} \frac{\mathrm{d}s}{\psi(s)} = \infty$$

for $s \in E_1$, $s > 0$. Lemma 4 yields existence of a solution of (S) on $[a, t_0]$. This fact implies existence of a solution $y(t)$ of the equation (L) on $[a, t_0]$ which satisfies the given initial conditions. The lemma is proved. □

Lemma 6. *Let* (A) *hold, and let* $y(t)$ *be a solution of* (L) *on* $[t_1, \infty)$, *where* $t_1 \geqslant a$. *Let* (B) *hold, where* $(s_0 = s)$

$$\text{(B)}\qquad \sum_{k=1}^{n-1} (-1)^{k-1} M_k(t, s) \leqslant 0, \quad N_n(t) \leqslant 0, \quad n \geqslant 2$$

and

$$M_k(t,s) = \int\limits_t^s \frac{\mathrm{d}s_1}{p_{n-2}(s_1)} \int\limits_t^{s_1} \frac{\mathrm{d}s_2}{p_{n-3}(s_2)} \dots \int\limits_t^{s_{k-2}} \frac{-P_{n-k}(s_{k-1})}{p_{n-1}(s)} \,\mathrm{d}s_{k-1},$$

$$M_1(t,s) = -P_{n-1}(s), \quad N_n(t) = \int\limits_{t_2}^{t} \sum_{k=1}^{n-1} (-P_{n-k}(s)G_k(s))\,\mathrm{d}s,$$

$$\begin{aligned} G_k(s) = L_{n-k}y(t_2) &+ (-1)^1 L_{n-k+1}y(t_2) \int\limits_s^{t_2} \frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)} + (-1)^2 L_{n-k+2}y(t_2) \\ &\times \int\limits_s^{t_2} \frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)} \int\limits_{s_1}^{t_2} \frac{\mathrm{d}s_2}{p_{n-k+2}(s_2)} + \dots + (-1)^{k-2} L_{n-2}y(t_2) \\ &\times \int\limits_s^{t_2} \frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)} \int\limits_{s_1}^{t_2} \frac{\mathrm{d}s_2}{p_{n-k+2}(s_2)} \dots \int\limits_{s_{k-3}}^{t_2} \frac{\mathrm{d}s_{k-2}}{p_{n-2}(s_{k-2})} \end{aligned}$$

for $k = 2, 3, \dots, n-1$, $G_1(s) = 0$.

a) *Let* n *be an even number and* $t_2 \in (t_1, \infty)$ *such that* $(-1)^k L_k y(t_2) \geqslant 0$ *for* $k = 0, 1, \dots, n-1$. *Then* $(-1)^k L_k y(t) \geqslant 0$ *for* $t \in [t_1, t_2]$, $k = 0, 1, \dots, n-1$.

b) *Let* n *be an odd number and* $t_2 \in (t_1, \infty)$ *such that* $(-1)^k L_k y(t_2) \leqslant 0$ *for* $k = 0, 1, \dots, n-1$. *Then* $(-1)^k L_k y(t) \leqslant 0$ *for* $t \in [t_1, t_2]$, $k = 0, 1, \dots, n-1$.

P r o o f. Let $n \geqslant 2$. Integration of the identity $L_n y = (L_{n-1}y)'$ over $[t_2, t]$, where $t_1 \leqslant t \leqslant t_2$ (n can be an even number as well as an odd one) yields

$$\begin{aligned} &L_{n-1}y(t) \\ &\quad = L_{n-1}y(t_2) - \int_{t_2}^{t} \sum_{k=1}^{n-1} P_k(s) L_k y(s)\,\mathrm{d}s - \int_{t_2}^{t} P_0(s) f(y(s))\,\mathrm{d}s \\ &\quad = L_{n-1}y(t_2) + \int_{t_2}^{t} (-P_0(s) f(y(s)))\,\mathrm{d}s + \int_{t_2}^{t} \sum_{k=1}^{n-1} (-P_{n-k}(s) L_{n-k} y(s))\,\mathrm{d}s. \end{aligned}$$

Let us denote the expression $L_{n-1}y(t_2) + \int\limits_{t_2}^{t} (-P_0(s) f(y(s)))\,\mathrm{d}s$ by $K_n(t)$. It is obvious that $K_n(t) \leqslant 0$ for all $t \in [t_1, t_2]$. We have

$$L_{n-1}y(t) = K_n(t) + \int_{t_2}^{t} \sum_{k=1}^{n-1} (-P_{n-k}(s) L_{n-k} y(s))\,\mathrm{d}s.$$

It can be proved that

$$
\begin{aligned}
&L_{n-k}y(s)\\
&= L_{n-k}y(t_2) + L_{n-k+1}y(t_2)\int_{t_2}^{s}\frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)}\\
&\quad + L_{n-k+2}y(t_2)\int_{t_2}^{s}\frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)}\int_{t_2}^{s_1}\frac{\mathrm{d}s_2}{p_{n-k+2}(s_2)} + \dots\\
&\quad + L_{n-2}y(t_2)\int_{t_2}^{s}\frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)}\int_{t_2}^{s_1}\frac{\mathrm{d}s_2}{p_{n-k+2}(s_2)}\dots\int_{t_2}^{s_{k-3}}\frac{\mathrm{d}s_{k-2}}{p_{n-2}(s_{k-2})}\\
&\quad + \int_{t_2}^{s}\frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)}\int_{t_2}^{s_1}\frac{\mathrm{d}s_2}{p_{n-k+2}(s_2)}\int_{t_2}^{s_2}\frac{\mathrm{d}s_3}{p_{n-k+3}(s_3)}\dots\int_{t_2}^{s_{k-2}}\frac{L_{n-1}y(s_{k-1})\,\mathrm{d}s_{k-1}}{p_{n-1}(s_{k-1})}
\end{aligned}
$$

for $k = 2, 3, \dots, n-1$. By interchanging the upper and the lower bounds in the previous integrals, we have

$$
\begin{aligned}
&L_{n-k}y(s)\\
&= L_{n-k}y(t_2) + (-1)^1 L_{n-k+1}y(t_2)\int_{s}^{t_2}\frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)}\\
&\quad + (-1)^2 L_{n-k+2}y(t_2)\int_{s}^{t_2}\frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)}\int_{s_1}^{t_2}\frac{\mathrm{d}s_2}{p_{n-k+2}(s_2)} + \dots\\
&\quad + (-1)^{k-2} L_{n-2}y(t_2)\int_{s}^{t_2}\frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)}\int_{s_1}^{t_2}\frac{\mathrm{d}s_2}{p_{n-k+2}(s_2)}\dots\int_{s_{k-3}}^{t_2}\frac{\mathrm{d}s_{k-2}}{p_{n-2}(s_{k-2})}\\
&\quad + (-1)^{k-1}\int_{s}^{t_2}\frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)}\int_{s_1}^{t_2}\frac{\mathrm{d}s_2}{p_{n-k+2}(s_2)}\dots\int_{s_{k-2}}^{t_2}\frac{L_{n-1}y(s_{k-1})\,\mathrm{d}s_{k-1}}{p_{n-1}(s_{k-1})}.
\end{aligned}
$$

Denoting the last $(k-1)$-dimensional integral by $I_k(s)$, the previous sum by $G_k(s)$, $I_1(s) = L_{n-1}y(s)$, $G_1(s) = 0$ for $k = 1, 2, \dots, n-1$ $(s_0 = s)$ we obtain

$$L_{n-k}y(s) = G_k(s) + (-1)^{k-1} I_k(s).$$

Hence

$$
\begin{aligned}
&L_{n-1}y(t)\\
&= K_n(t) + \int_{t_2}^{t}\sum_{k=1}^{n-1}(-P_{n-k}(s)[G_k(s) + (-1)^{k-1}I_k(s)])\,\mathrm{d}s\\
&= K_n(t) + \int_{t_2}^{t}\sum_{k=1}^{n-1}(-P_{n-k}(s)G_k(s))\,\mathrm{d}s + \int_{t_2}^{t}\sum_{k=1}^{n-1}(-P_{n-k}(s)(-1)^{k-1}I_k(s))\,\mathrm{d}s.
\end{aligned}
$$

Denoting $K_n(t) + \int_{t_2}^{t} \sum\limits_{k=1}^{n-1} (-P_{n-k}(s)G_k(s))\,\mathrm{d}s$ by $g_n(t)$ and denoting $\int_{t_2}^{t}(-P_{n-k}(s) \times (-1)^{k-1} I_k(s))\,\mathrm{d}s$ by $(-1)^{k-1}J_k(t)$ we have

$$L_{n-1}y(t) = g_n(t) + \sum_{k=1}^{n-1} (-1)^{k-1} J_k(t),$$

where $J_k(t)$ is the k-dimensional integral

$$\begin{aligned} J_k(t) = \; & -\int_t^{t_2} (-P_{n-k}(s))\,\mathrm{d}s \int_s^{t_2} \frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)} \int_{s_1}^{t_2} \frac{\mathrm{d}s_2}{p_{n-k+2}(s_2)} \cdots \\ & \cdots \int_{s_{k-2}}^{t_2} \frac{L_{n-1}y(s_{k-1})\,\mathrm{d}s_{k-1}}{p_{n-1}(s_{k-1})} \end{aligned}$$

for $k = 2, 3, \ldots, n-1$ and $J_1(t) = -\int_t^{t_2}(-P_{n-1}(s)L_{n-1}y(s))\,\mathrm{d}s$.
By changing the notation of the variables we have

$$\begin{aligned} J_k(t) = \; & -\int_t^{t_2} (-P_{n-k}(s_{k-1}))\,\mathrm{d}s_{k-1} \int_{s_{k-1}}^{t_2} \frac{\mathrm{d}s_{k-2}}{p_{n-k+1}(s_{k-2})} \int_{s_{k-2}}^{t_2} \frac{\mathrm{d}s_{k-3}}{p_{n-k+2}(s_{k-3})} \cdots \\ & \cdots \int_{s_1}^{t_2} \frac{L_{n-1}y(s)\,\mathrm{d}s}{p_{n-1}(s)}. \end{aligned}$$

$J_k(t)$ is a k-dimensional integral on a k-dimensional domain. This domain can be described as an elementary domain in the following way:

$$\begin{aligned} t \leqslant s_{k-1} &\leqslant t_2 \\ s_{k-1} \leqslant s_{k-2} &\leqslant t_2 \\ s_{k-2} \leqslant s_{k-3} &\leqslant t_2 \\ &\vdots \\ s_2 \leqslant s_1 &\leqslant t_2 \\ s_1 \leqslant s &\leqslant t_2, \end{aligned}$$

as well as like

$$\begin{aligned} & t \leqslant s \leqslant t_2 \\ & t \leqslant s_1 \leqslant s \\ & t \leqslant s_2 \leqslant s_1 \\ & \vdots \\ & t \leqslant s_{k-2} \leqslant s_{k-3} \\ & t \leqslant s_{k-1} \leqslant s_{k-2} \end{aligned}$$

for $k = 2, 3, \dots, n-1$. Hence

$$J_k(t) = -\int_t^{t_2} L_{n-1}y(s)\,\mathrm{d}s \int_t^s \frac{\mathrm{d}s_1}{p_{n-2}(s_1)} \int_t^{s_1} \frac{\mathrm{d}s_2}{p_{n-3}(s_2)} \dots \int_t^{s_{k-2}} \frac{-P_{n-k}(s_{k-1})}{p_{n-1}(s)}\,\mathrm{d}s_{k-1}.$$

The last integral can be rewritten into the form

$$J_k(t) = -\int_t^{t_2} M_k(t,s)L_{n-1}y(s)\,\mathrm{d}s = \int_{t_2}^t M_k(t,s)L_{n-1}y(s)\,\mathrm{d}s,$$

where

$$M_k(t,s) = \int_t^s \frac{\mathrm{d}s_1}{p_{n-2}(s_1)} \int_t^{s_1} \frac{\mathrm{d}s_2}{p_{n-3}(s_2)} \dots \int_t^{s_{k-2}} \frac{-P_{n-k}(s_{k-1})}{p_{n-1}(s)}\,\mathrm{d}s_{k-1}$$

for $k = 2, 3, \dots, n-1$, $M_1(t,s) = -P_{n-1}(s)$. Hence

$$\begin{aligned} &L_{n-1}y(t) \\ &= g_n(t) + \sum_{k=1}^{n-1}(-1)^{k-1}J_k(t) = g_n(t) + \sum_{k=1}^{n-1}(-1)^{k-1}\int_{t_2}^t M_k(t,s)L_{n-1}y(s)\,\mathrm{d}s \\ &= g_n(t) + \int_{t_2}^t \left[\sum_{k=1}^{n-1}(-1)^{k-1}M_k(t,s)\right]L_{n-1}y(s)\,\mathrm{d}s = g_n(t) + \int_{t_2}^t A_n(t,s)L_{n-1}y(s)\,\mathrm{d}s, \end{aligned}$$

where $A_n(t,s) = \sum\limits_{k=1}^{n-1}(-1)^{k-1}M_k(t,s)$. We note that $s \leqslant t_2$, $s_i \leqslant t_2$, $t \leqslant s$, $t \leqslant s_i$ for $i = 1, 2, \dots, n-3$. According to the assumptions of the lemma, we have $g_n(t) = K_n(t) + N_n(t)$ and $g_n(t) \leqslant 0$, $A_n(t,s) \leqslant 0$. According to Lemma 1 we have $L_{n-1}y(t) \leqslant 0$ for all $t \in [t_1, t_2]$. By virtue of

$$L_{n-2}y(t) = L_{n-2}y(t_2) + \int\limits_{t_2}^{t} \frac{L_{n-1}y(s)}{p_{n-1}(s)}\,\mathrm{d}s \geqslant L_{n-2}y(t_2) \geqslant 0,$$

we have $L_{n-2}y(t) \geqslant 0$ on $[t_1, t_2]$. By using of a similar procedure (n can be an even number or an odd one), we get for $n \geqslant 2$:

a) $(-1)^k L_k y(t) \geqslant 0$ on $[t_1, t_2]$ for $k = 0, 1, \dots, n-1$, for n an even number,

b) $(-1)^k L_k y(t) \leqslant 0$ on $[t_1, t_2]$ for $k = 0, 1, \dots, n-1$, for n an odd number.

If $n = 1$, then the assertion of the lemma is obvious. □

Lemma 7. *Consider a solution $y(t)$ of* (L) *on $[t_1,\infty)$, $t_1 \geqslant a$ such that* (A) *holds, let n be an even number and $t_2 \in (t_1,\infty)$ such that $(-1)^k L_k y(t_2) \geqslant 0$ for $k=0,1,\dots,n-1$. Let $P_k(t)\equiv 0$ on $[t_1,t_2]$ for all odd integers $k\in[1,n]$. Then* (B) *holds.*

P r o o f. We have $G_k(s)\geqslant 0$ for all even numbers $k\in[1,n]$, and $G_k(s)\leqslant 0$ for all odd ones. If k is an odd number, then $n-k$ is an odd number too, and $P_{n-k}(t)\equiv 0$ on $[t_1,t_2]$. Therefore $N_n(t)=\int\limits_{t_2}^{t}\sum\limits_{k=1}^{n-1}(-P_{n-k}(s)G_k(s))\,\mathrm{d}s\leqslant 0$. Similarly, $M_k(t,s)=0$ for all odd $k\leqslant n$. So $A_n(t,s)=\sum\limits_{k=1}^{n-1}(-1)^{k-1}M_k(t,s)\leqslant 0$ because $M_k(t,s)\geqslant 0$ for all $k=1,2,\dots,n-1$. □

Lemma 8. *Consider a solution $y(t)$ of* (L) *on $[t_1,\infty)$, $t_1\geqslant a$ such that* (A) *holds, let $n>1$ be an odd number and $t_2\in(t_1,\infty)$ such that $(-1)^k L_k y(t_2)\leqslant 0$ for $k=0,1,\dots,n-1$. Let $P_k(t)\equiv 0$ on $[t_1,t_2]$ for all even integers $k\in[1,n]$. Then* (B) *holds.*

P r o o f. The proof is similar to the proof of the previous lemma, so it is omitted. □

Lemma 9. *Let $\{y_m(t)\}_{m=n_0}^{\infty}$ be a sequence of solutions of* (L) *on $[t_0,\infty)$, where $a<t_0<n_0$, n is an even number, and $L_k y_m(m)=(-1)^k$ for all $m\geqslant n_0$, $k=0,1,\dots,n-1$. Let* (A) *hold, and let $P_k(t)\equiv 0$ on $[a,\infty)$ for all odd integer numbers $k\in[1,n]$. Let $-\infty<\int\limits_{t_0}^{\infty}P_0(s)\,\mathrm{d}s=P<0$, $\int\limits_{t_0}^{\infty}P_k(s)\,\mathrm{d}s\geqslant -\frac12$ for $k=1,2,\dots,n-1$, let P_k be nondecreasing functions for $k=0,1,\dots,n-1$, $\int\limits_{t_0}^{\infty}1/p_r(s)\,\mathrm{d}s\leqslant\frac12$ for $r=1,2,\dots,n-1$, and let K be a real positive constant such that $0\leqslant f(t)\leqslant K$ for $t\in(-\infty,\infty)$. Then there exists a subsequence of $\{y_m(t)\}_{m=n_0}^{\infty}$ which converges to $\varphi_0(t)$. This function $\varphi_0(t)$ is a solution of* (L) *on $[t_0,\infty)$, and $(-1)^k L_k\varphi_0(t)\geqslant 0$ on $[t_0,\infty)$ for $k=0,1,\dots,n-1$.*

P r o o f. Because $L_n y_m(t)\geqslant 0$ on $[t_0,m]$ for $m=n_0,n_0+1,\dots$ (this follows from Lemma 7 and Lemma 6, part a)), we have that $L_{n-1}y_m(t)$ is nondecreasing and negative on $[t_0,n_0]$ for $m>n_0$. If we prove that $L_{n-1}y_m(t_0)$ is bounded from below, it means $L_{n-1}y_m(t)$ is uniformly bounded on $[t_0,n_0]$. Using the expression (C) several times, where

$$\text{(C)}\qquad L_k y_m(s)=L_k y_m(m)+\int_m^s\Big(L_{k+1}\frac{y_m(s)}{p_{k+1}(s)}\Big)\,\mathrm{d}s \text{ for } k=0,1,\dots,n-2,$$

we obtain for $n > 3$, $2 \leqslant k < n-1$ ($s_0 = s$):

$$
\text{(D)} \qquad
\begin{aligned}
L_k y_m(s) &= L_k y_m(m) + L_{k+1} y_m(m) \int\limits_m^s \frac{\mathrm{d}s_1}{p_{k+1}(s_1)} \\
&\quad + L_{k+2} y_m(m) \int\limits_m^s \frac{\mathrm{d}s_1}{p_{k+1}(s_1)} \int\limits_m^{s_1} \frac{\mathrm{d}s_2}{p_{k+2}(s_2)} + \ldots \\
&\quad + L_{n-2} y_m(m) \int\limits_m^s \frac{\mathrm{d}s_1}{p_{k+1}(s_1)} \int\limits_m^{s_1} \frac{\mathrm{d}s_2}{p_{k+2}(s_2)} \ldots \int\limits_m^{s_{n-k-3}} \frac{\mathrm{d}s_{n-k-2}}{p_{n-2}(s_{n-k-2})} \\
&\quad + \int\limits_m^s \frac{\mathrm{d}s_1}{p_{k+1}(s_1)} \int\limits_m^{s_1} \frac{\mathrm{d}s_2}{p_{k+2}(s_2)} \ldots \int\limits_m^{s_{n-k-2}} \frac{L_{n-1} y_m(s_{n-k-1})}{p_{n-1}(s_{n-k-1})} \, \mathrm{d}s_{n-k-1}.
\end{aligned}
$$

Integration of (L) over $[t_0, m]$ yields

$$
\begin{aligned}
&L_{n-1} y_m(t_0) \\
&= L_{n-1} y_m(m) + \int_{t_0}^m P_0(s) f(y_m(s)) \, \mathrm{d}s + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^m P_{2k}(s) L_{2k} y_m(s) \, \mathrm{d}s \\
&= L_{n-1} y_m(m) + \int_{t_0}^m P_0(s) f(y_m(s)) \, \mathrm{d}s + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^m P_{2k}(s) [B_{2k}(s) + C_{2k}(s)] \, \mathrm{d}s,
\end{aligned}
$$

where $C_k(s)$ is the last integral in (D) and $B_k(s)$ is the rest of the right-hand side of (D). Let us denote the expression $L_{n-1} y_m(m) + \int\limits_{t_0}^m P_0(s) f(y_m(s)) \, \mathrm{d}s$ by F_m. Then

$$
\begin{aligned}
&L_{n-1} y_m(t_0) \\
&= F_m + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^m P_{2k}(s) B_{2k}(s) \, \mathrm{d}s + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^m P_{2k}(s) C_{2k}(s) \, \mathrm{d}s \\
&\geqslant F_m + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^m P_{2k}(s) B_{2k}(s) \, \mathrm{d}s + L_{n-1} y_m(t_0) \\
&\quad \times \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^m P_{2k}(s) \left[\int_m^s \frac{\mathrm{d}s_1}{p_{2k+1}(s_1)} \int_m^{s_1} \frac{\mathrm{d}s_2}{p_{2k+2}(s_2)} \ldots \int_m^{s_{n-2k-2}} \frac{\mathrm{d}s_{n-2k-1}}{p_{n-1}(s_{n-2k-1})} \right] \mathrm{d}s \\
&\geqslant F_m + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^m P_{2k}(s) B_{2k}(s) \, \mathrm{d}s + L_{n-1} y_m(t_0) \\
&\quad \times \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^\infty \Big[- P_{2k}(s) \Big[\int_{t_0}^\infty \frac{\mathrm{d}s_1}{p_{2k+1}(s_1)} \int_{t_0}^\infty \frac{\mathrm{d}s_2}{p_{2k+2}(s_2)} \ldots \int_{t_0}^\infty \frac{\mathrm{d}s_{n-2k-1}}{p_{n-1}(s_{n-2k-1})} \Big] \Big] \, \mathrm{d}s.
\end{aligned}
$$

(We have used the fact that the last integral has the dimension $n-2k$, which is an even number, and $t_0 \leqslant s_i \leqslant m < \infty$ for $i = 1, 2, \dots, n-2k-2$, $t_0 \leqslant s \leqslant m < \infty$). An easy arrangement yields

$$L_{n-1} y_m(t_0) \Big[1 + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^{\infty} P_{2k}(s)\,\mathrm{d}s \int_{t_0}^{\infty} \frac{\mathrm{d}s_1}{p_{2k+1}(s_1)} \int_{t_0}^{\infty} \frac{\mathrm{d}s_2}{p_{2k+2}(s_2)} \dots$$
$$\dots \int_{t_0}^{\infty} \frac{\mathrm{d}s_{n-2k-1}}{p_{n-1}(s_{n-2k-1})} \Big] \geqslant F_m + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^{m} P_{2k}(s) B_{2k}(s)\,\mathrm{d}s.$$

According to the assumptions, the expression in the parentheses above is a positive number because of $\sum\limits_{k=1}^{\frac{n}{2}-1} \int\limits_{t_0}^{\infty} [-P_{2k}(s)]\,\mathrm{d}s \dots \int\limits_{t_0}^{\infty} \frac{\mathrm{d}s_{n-2k-1}}{p_{n-1}(s_{n-2k-1})} \leqslant \sum\limits_{k=1}^{\frac{n}{2}-1} (\frac{1}{2})^{n-2k} < 1$. Therefore

$$L_{n-1} y_m(t_0) \geqslant \frac{F_m + \sum\limits_{k=1}^{\frac{n}{2}-1} \int\limits_{k_0}^{m} P_{2k}(s) B_{2k}(s)\,\mathrm{d}s}{1 + \sum\limits_{k=1}^{\frac{n}{2}-1} \int\limits_{t_0}^{\infty} P_{2k}(s)\,\mathrm{d}s \int\limits_{t_0}^{\infty} \frac{\mathrm{d}s_1}{p_{2k+1}(s_1)} \dots \int\limits_{t_0}^{\infty} \frac{\mathrm{d}s_{n-2k-1}}{p_{n-1}(s_{n-2k-1})}}.$$

We have

$$F_m = L_{n-1} y_m(m) + \int\limits_{t_0}^{m} P_0(s) f(y_m(s))\,\mathrm{d}s \geqslant -1 + \int\limits_{t_0}^{\infty} P_0(s) f(y_m(s))\,\mathrm{d}s$$
$$\geqslant -1 + K \int\limits_{t_0}^{\infty} P_0(s)\,\mathrm{d}s = -1 + KP,$$

$$B_{2k}(s) = L_{2k} y_m(m) + L_{2k+1} y_m(m) \int\limits_{m}^{s} \frac{\mathrm{d}s_1}{p_{2k+1}(s_1)} + \dots + L_{n-2} y_m(m) \int\limits_{m}^{s} \frac{\mathrm{d}s_1}{p_{2k+1}(s_1)} \dots$$
$$\dots \int\limits_{m}^{s_{n-2k-3}} \frac{\mathrm{d}s_{n-2k-2}}{p_{n-2}(s_{n-2k-2})} = 1 + 1 \int\limits_{s}^{m} \frac{\mathrm{d}s_1}{p_{2k+1}(s_1)} + \dots + 1 \int\limits_{s}^{m} \frac{\mathrm{d}s_1}{p_{2k+1}(s_1)} \dots$$
$$\dots \int\limits_{s_{n-2k-3}}^{m} \frac{\mathrm{d}s_{n-2k-2}}{p_{n-2k-2}(s_{n-2k-2})} \leqslant 1 + (n-2k-2)\frac{1}{2} \leqslant n$$

because of $s \leqslant m$, $s_i \leqslant m$ for $i = 1, 2, \dots, n-2k-3$. So we have

$$\sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^{m} P_{2k}(s) B_{2k}(s)\,\mathrm{d}s \geqslant n \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^{m} P_{2k}(s)\,\mathrm{d}s$$
$$\geqslant n \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^{\infty} P_{2k}(s)\,\mathrm{d}s \geqslant -n(\frac{n}{2}-1)\frac{1}{2}.$$

Hence

$$L_{n-1}y_m(t_0) \geqslant \frac{-1+KP-\frac{n}{2}(\frac{n}{2}-1)}{1+\sum\limits_{k=1}^{\frac{n}{2}-1}\int\limits_{t_0}^{\infty} P_{2k}(s)\,\mathrm{d}s\int\limits_{t_0}^{\infty}\frac{\mathrm{d}s_1}{p_{2k+1}(s_1)}\dots\int\limits_{t_0}^{\infty}\frac{\mathrm{d}s_{n-2k-1}}{p_{n-1}(s_{n-2k-1})}}$$
$$= S_{n-1} \in (-\infty, 0)$$

for $n > 3$. If $n = 2$, then $L_{n-1}y_m(t_0) = F_m \geqslant -1 + KP \in (-\infty, 0)$. It implies that $\{L_{n-1}y_m(t_0)\}_{m=n_0}^{\infty}$ is bounded from below for any fixed even number $n \geqslant 2$. So we have

$$0 \leqslant L_{n-2}y_m(t_0) = L_{n-2}y_m(m) + \int\limits_{t_0}^{m}\frac{-L_{n-1}y_m(s)}{p_{n-1}(s)}\,\mathrm{d}s \leqslant 1 - L_{n-1}y_m(t_0)\int\limits_{t_0}^{\infty}\frac{\mathrm{d}s}{p_{n-1}(s)}$$
$$\leqslant 1 - S_{n-1}\int\limits_{t_0}^{\infty}\frac{\mathrm{d}s}{p_{n-1}(s)} = S_{n-2} \in (0, \infty),$$
$$0 \geqslant L_{n-3}y_m(t_0) = L_{n-3}y_m(m) + \int\limits_{t_0}^{m}\frac{-L_{n-2}y_m(s)}{p_{n-2}(s)}\,\mathrm{d}s \geqslant -1 - L_{n-2}y_m(t_0)\int\limits_{t_0}^{\infty}\frac{\mathrm{d}s}{p_{n-2}(s)}$$
$$\geqslant -1 - S_{n-2}\int\limits_{t_0}^{\infty}\frac{\mathrm{d}s}{p_{n-2}(s)} = S_{n-3} \in (-\infty, 0).$$

Similarly, it can be proved that $\{L_k y_m(t_0)\}_{m=n_0}^{\infty}$ is bounded for $k = 0, 1, \dots, n-1$. However,

$$0 \leqslant L_n y_m(t) = -\sum_{k=1}^{\frac{n}{2}-1} P_{2k}(t)L_{2k}y_m(t) - P_0(t)f(y_m(t))$$
$$\leqslant -\sum_{k=1}^{\frac{n}{2}-1} P_{2k}(t_0)L_{2k}y_m(t_0) - P_0(t_0)K$$
$$\leqslant -\sum_{k=1}^{\frac{n}{2}-1} P_{2k}(t_0)S_{2k} - P_0(t_0)K = S_n \in (0, \infty),$$

and this implies that $\{L_n y_m(t)\}_{m=n_0}^{\infty}$ is uniformly bounded on $[t_0, n_0]$ for $m \geqslant n_0$ and so $L_{n-1}y_m(t)$ are uniformly equicontinuous on $[t_0, n_0]$ for $m \geqslant n_0$. According to Arzelà-Ascoli theorem, there exists a subsequence $\{L_{n-1}y_{k_m}\}_{m=n_0}^{\infty}$ of $\{L_{n-1}y_m\}_{m=n_0}^{\infty}$ such that $\{L_{n-1}y_{k_m}\}_{m=n_0}^{\infty}$ converges uniformly on $[t_0, n_0]$ to, for example, a function $\varphi_{n-1}(t)$.

To ensure uniform convergence of $\{L_{n-2}y_{k_m}\}_{m=n_0}^{\infty}$ on $[t_0,n_0]$ to, for instance, a function $\varphi_{n-2}(t)$, it suffices to show convergence of $\{L_{n-2}y_{k_m}\}_{m=n_0}^{\infty}$ at an inner point of $[t_0,n_0]$. This follows from the fact that $L_{n-2}y_{k_m}(t_0+\varepsilon)\leqslant L_{n-2}y_{k_m}(t_0)\leqslant S_{n-2}$ for $\varepsilon>0$, $\varepsilon<n_0-t_0$. Then there exists a convergent subsequence $\{L_{n-2}y_{k_{l_m}}(t_0+\varepsilon)\}_{m=n_0}^{\infty}$ of $\{L_{n-2}y_{k_m}(t_0+\varepsilon)\}_{m=n_0}^{\infty}$ and therefore $\{L_{n-2}y_{k_{l_m}}\}_{m=n_0}^{\infty}$ converges uniformly to $\varphi_{n-2}(t)$ on $[t_0,n_0]$. It is obvious that $L_{n-1}y_{k_{l_m}}\rightrightarrows\varphi_{n-1}$ on $[t_0,n_0]$, too. In a similar way we can prove uniform convergence of a subsequence $\{y_{r_m}\}_{m=n_0}^{\infty}$ of $\{y_m\}_{m=n_0}^{\infty}$ such that $L_ky_{r_m}(t)\rightrightarrows\varphi_k(t)$ on $[t_0,n_0]$ for $k=0,1,\ldots,n$. Due to the fact that uniform convergence makes changing of the order of limit processes possible (a quasi-derivative is a certain kind of limit), we have

$$\begin{aligned}0&=\lim_{m\to\infty}L(y_{r_m}(t))\\&=\lim_{m\to\infty}L_ny_{r_m}(t)+\sum_{k=1}^{\frac{n}{2}-1}P_{2k}(t)\lim_{m\to\infty}L_{2k}y_{r_m}(t)+P_0(t)f(\lim_{m\to\infty}y_{r_m}(t))\\&=\varphi_n(t)+\sum_{k=1}^{\frac{n}{2}-1}P_{2k}(t)\varphi_{2k}(t)+P_0(t)f(\varphi_0(t))\end{aligned}$$

for all $t\in[t_0,n_0]$.
But $\varphi_k(t)=\lim\limits_{m\to\infty}L_ky_{r_m}(t)=L_k(\lim\limits_{m\to\infty}y_{r_m}(t))=L_k(\lim\limits_{m\to\infty}L_0y_{r_m}(t))=L_k\varphi_0(t)$, so $\varphi_0(t)$ fulfils (L) on $[t_0,n_0]$. It is important that we are able to continue $\varphi_0(t)$ on $[t_0,n_0+1]$ in such a way that $\varphi_0(t)$ be a solution of (L) on $[t_0,n_0+1]$. Indeed, it suffices to repeat the whole previous part of the proof with the sequence y_{r_m} for $m\geqslant n_0+1$ instead of y_m for $m\geqslant n_0$. Now it is obvious that $\varphi_0(t)$ can be continued on $[t_0,n_0+v]$ (v is an arbitrary integer greater than 1) and therefore $\varphi_0(t)$ fulfils (L) on $[t_0,\infty)$. Now let us take an arbitrary point $t_1\in[t_0,\infty)$. Then there exists $m_0\in\{1,2,\ldots\}$, $t_1<m_0$ and a subsequence $\{y_{s_m}\}_{m=n_0}^{\infty}$ of $\{y_m\}_{m=n_0}^{\infty}$ such that $L_ky_{s_m}\rightrightarrows L_k\varphi_0(t)$ on $[t_0,m_0]$. But $(-1)^kL_ky_{s_m}(t)\geqslant 0$ on $[t_0,m_0]$. Therefore $(-1)^kL_k\varphi_0(t_1)\geqslant 0$. It implies that $(-1)^kL_k\varphi_0(t)\geqslant 0$ for all $t\geqslant t_0$, $k=0,1,\ldots,n-1$. □

Lemma 10. *Let $\{y_m(t)\}_{m=n_0}^{\infty}$ be a sequence of solutions of* (L) *on $[t_0,\infty)$, where $a<t_0<n_0$, n is an odd number, and $L_ky_m(m)=(-1)^{k-1}$ for all $m\geqslant n_0$, $k=0,1,\ldots,n-1$. Let* (A) *hold, and let $P_k(t)\equiv 0$ on $[a,\infty)$ for all even integers $k\in[1,n]$. Let $-\infty<\int_{t_0}^{\infty}P_0(s)\,\mathrm{d}s=P<0$, $\int_{t_0}^{\infty}P_k(s)\,\mathrm{d}s\geqslant-\frac{1}{2}$ for $k=1,2,\ldots,n-1$, let P_k be nondecreasing functions for $k=0,1,\ldots,n-1$, $\int_{t_0}^{\infty}1/p_r(s)\,\mathrm{d}s\leqslant\frac{1}{2}$ for $r=1,2,\ldots,n-1$, and let K be a real positive constant such that $0\leqslant f(t)\leqslant K$ for $t\in(-\infty,\infty)$. Then there exists a subsequence of $\{y_m(t)\}_{m=n_0}^{\infty}$ which converges to*

$\varphi_0(t)$. This function $\varphi_0(t)$ is a solution of (L) *on* $[t_0,\infty)$, *and* $(-1)^k L_k\varphi_0(t) \leqslant 0$ *on* $[t_0,\infty)$ *for* $k=0,1,\ldots,n-1$.

P r o o f. The proof is similar to the proof of Lemma 9 (instead of Lemma 6, part a), and Lemma 7 we use Lemma 6, part b) and Lemma 8, respectively), so it is omitted. □

Theorem 1. *Let n be an even number. Let* (A) *hold, and let $P_k(t)\equiv 0$ on $[a,\infty)$ for all odd integers $k\in[1,n]$. Let $P_k(t)$ be nondecreasing functions on $[a,\infty)$ such that $\int_a^\infty P_k(s)\,\mathrm{d}s > -\infty$ for $k=0,1,\ldots,n-1$, $\int_a^\infty 1/p_r(s)\,\mathrm{d}s<\infty$ for $r=1,2,\ldots,n-1$, and let K be a real positive constant such that $0\leqslant f(t)\leqslant K$ for all $t\in(-\infty,\infty)$. Then* (L) *admits a Kneser solution $y(t)$ on $[a,\infty)$, i.e. $y(t)>0$, $(-1)^k L_k y(t)\geqslant 0$ on $[a,\infty)$ for $k=1,2,\ldots,n-1$.*

P r o o f. Let us take $t_0\in(a,\infty)$ such that $\int_{t_0}^\infty P_k(s)\,\mathrm{d}s\geqslant -\frac{1}{2}$, $\int_{t_0}^\infty 1/p_r(s)\,\mathrm{d}s\leqslant\frac{1}{2}$ for $k=1,2,\ldots,n-1$; $r=1,2,\ldots,n-1$. According to Lemma 5, there exists a sequence $\{y_m(t)\}_{m=n_0}^\infty$ of solutions of (L) on $[t_0,\infty)$ such that $L_k y_m(m)=(-1)^k$ for all $m\geqslant n_0>t_0$, $k=0,1,\ldots,n-1$. Lemma 7 ensures validity of (B), and Lemma 6, part a), yields that $\{y_m(t)\}_{m=n_0}^\infty$ has the required properties from Lemma 9. According to the last-mentioned lemma, there exists a function $y(t)$ such that $L(y(t))\equiv 0$ on $[t_0,\infty)$, $(-1)^k L_k y(t)\geqslant 0$ on $[t_0,\infty)$ for $k=0,1,\ldots,n-1$. This solution $y(t)$ of (L) on $[t_0,\infty)$ can be continued onto $[a,\infty)$ by Lemma 5. According to Lemma 6, part a), $y(t)$ is a Kneser solution of (L) on $[a,\infty)$ because $y(t)>0$ on $[a,\infty)$ (this follows from $f(0)\neq 0$). □

Theorem 2. *Let n be an odd number. Let* (A) *hold, and let $P_k(t)\equiv 0$ on $[a,\infty)$ for all even integers $k\in[1,n]$. Let $P_k(t)$ be nondecreasing functions on $[a,\infty)$ such that $\int_a^\infty P_k(s)\,\mathrm{d}s > -\infty$ for $k=0,1,\ldots,n-1$, $\int_a^\infty 1/p_r(s)\,\mathrm{d}s<\infty$ for $r=1,2,\ldots,n-1$ and let K be a real positive constant such that $0\leqslant f(t)\leqslant K$ for all $t\in(-\infty,\infty)$. Then* (L) *admits a Kneser solution $y(t)$ on $[a,\infty)$, i.e. $y(t)<0$, $(-1)^k L_k y(t)\leqslant 0$ on $[a,\infty)$ for $k=1,2,\ldots,n-1$.*

P r o o f. The proof is similar to that of the previous theorem (instead of Lemma 6, part a) and Lemma 9 we will use Lemma 6, part b) and Lemma 10, respectively) and so it is omitted. □

3. Examples

Example 1. The equation

$$(t^4(t^3(t^2y')')')' - \frac{1}{t^2}(t^3(t^2y')') + [(\frac{72}{t^8} - \frac{1296}{t^4})\sqrt{1+t^{-18}}]\frac{1}{\sqrt{1+y^2}} \equiv 0$$

admits a Kneser solution $y(t) = t^{-9}$ on $[1,\infty)$ according to Theorem 1 because $\int_1^\infty (1/p_r(t))\,\mathrm{d}t < \infty$ for $r = 1,2,3$, $P_0(t)$ is nonpositive and nondecreasing on $[1,\infty)$, $\int_1^\infty P_k(t)\,\mathrm{d}t > -\infty$ for $k = 0,1,2,3$, $0 \leqslant 1/\sqrt{1+y^2} \leqslant 1$, $f(0) \neq 0$.

Example 2. The equation of the n-th order (n is an even number)

$$L_n y + \sum_{k=1}^{\frac{n}{2}-1} P_{2k}(t)L_{2k}y + P_0(t)f(y) \equiv 0,$$

where $P_{2k}(t) = -t^{-2k-2}$ for $k = 0,1,\dots,\frac{n}{2}-1$, $p_r(t) = t^{3r}$ for $r = 1,2,\dots,n-1$, $f(t) = e^{-t^2}$ admits a Kneser solution on $[1,\infty)$ according to Theorem 1 because $\int_1^\infty (1/p_r(t))\,\mathrm{d}t < \infty$ for $r = 1,2\dots,n-1$, $\int_1^\infty P_{2k}(t)\,\mathrm{d}t > -\infty$ for $k = 0,1,\dots,\frac{n}{2}-1$, $0 \leqslant e^{-t^2} \leqslant 1$, $f(0) \neq 0$.

Example 3. The equation

$$L_5 y - \frac{1}{t^6}L_3 y - \frac{1}{t^2}L_1 y + (12t^{-13} + 1188t^{-12} - 14256t^{-3})\frac{\sqrt{1+t^{-48}}}{\sqrt{1+y^4}} \equiv 0,$$

where $p_r(t) = t^{r+1}$ for $r = 1,2,3,4$ admits a Kneser solution $y(t) = -t^{-12} < 0$ on $[1,\infty)$ according to Theorem 2 because $\int_1^\infty (1/p_r(t))\,\mathrm{d}t < \infty$ for $r = 1,2,3,4$, $P_0(t)$ is nonpositive and nondecreasing on $[1,\infty)$, $\int_1^\infty P_k(t)\,\mathrm{d}t > -\infty$ for $k = 0,1,2,3,4$, $0 \leqslant \frac{1}{\sqrt{1+y^4}} \leqslant 1$, $f(0) \neq 0$.

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