ON SOME PROBLEMS CONNECTED WITH DIAGONAL MAP IN SOME SPACES OF ANALYTIC FUNCTIONS

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Abstract. For any holomorphic function f on the unit polydisk $\mathbb{D}^n$ we consider its restriction to the diagonal, i.e., the function in the unit disc $\mathbb{D}\subset\mathbb{C}$ defined by $\operatorname{Diag} f(z)=f(z,\ldots,z)$, and prove that the diagonal map Diag maps the space $Q_{p,q,s}(\mathbb{D}^n)$ of the polydisk onto the space $\widehat{Q}^q_{p,s,n}(\mathbb{D})$ of the unit disk.

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1. Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$, $T=\partial\mathbb{D}$ its boundary. Let $\mathbb{D}^n$ be the unit polydisk in $\mathbb{C}^n$ and T^n its Shilov boundary. Let $\mathrm{d}m_2(z)=\pi^{-1}r\,\mathrm{d}r\,\mathrm{d}\theta$ denote the normalized area measure on $\mathbb{D}$, $\mathrm{d}m_{2n}(z)=\prod_{j=1}^{n}\mathrm{d}m_2(z_j)$, where $z_j\in\mathbb{D}$ for each $j\in\{1,\ldots,n\}$, $\mathrm{d}m_n$ being the normalized surface measure on T^n. When $n=1$, we use $\mathrm{d}m(\xi)$ to denote the normalized Lebesgue measure on T. Let $H(\mathbb{D}^n)$ be the space of all holomorphic functions on $\mathbb{D}^n$. The integral mean of f is defined as

$$M_p^p(f,r)=\int_{T^n}|f(r\xi)|^p\,\mathrm{d}m_n(\xi),\ \ M_\infty(f,r)=\max_{\xi\in T^n}|f(r\xi)|, r\in(0,1),\ \ f\in H(\mathbb{D}^n).$$

Here $r\xi=(r_1\xi_1,\ldots,r_n\xi_n)$.

Let $X\subset H(\mathbb{D})$ and $F\in X$. If $Y=Y(\mathbb{D}^n)$ is a subspace of $H(\mathbb{D}^n)$ and

$$X=\operatorname{Diag} Y=\{f(z,\ldots,z);\ f\in Y, Y\subset H(\mathbb{D}^n)\}$$

then in some cases, that is, for some X and Y (for example, for the Bergman spaces), the relation

$$\|F\|_X \asymp \inf_{\varphi} \|\varphi(F)\|_Y \tag{1}$$

holds, where $\varphi(z_1, \dots, z_n)$ is an arbitrary extension of the function F from the diagonal $(z, \dots, z)$ onto the unit polydisk. The notation $A \asymp B$ means that there is a positive constant C such that $A/C \leqslant B \leqslant CA$.

With any holomorphic function f on the unit polydisk $\mathbb{D}^n$ we associate the function

$$\operatorname{Diag} f(z) = f(z, \dots, z).$$

This operator is called the diagonal mapping. In [9] Rudin suggested the study of this mapping. Recently, the diagonal mapping has been investigated by many authors, see, for example, [1], [8], [11] and the related references therein.

For example, it is well known ([11]) that for the weighted Bergman space $Y = \mathscr{A}^p_{\alpha}(\mathbb{D}^n)$, i.e.,

$$\mathscr{A}^p_{\alpha}(\mathbb{D}^n) = \left\{ f \in H(\mathbb{D}^n) \colon \int_{\mathbb{D}^n} |f(z)|^p \prod_{j=1}^{n} (1 - |z_j|^2)^{\alpha_j} \, \mathrm{d}m_2(z_j) < \infty \right\},$$

where $p \in (0, \infty)$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1$, $j = 1, \dots, n$, its diagonal is the weighted Bergman space on the unit disk $X = \mathscr{A}^p_{|\alpha|+2n-2}(\mathbb{D})$, where $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$ and $0 < p < \infty$.

As we can see from the above discussion, the problem of describing the diagonal of a space $Y(\mathbb{D}^n)$ which is a subspace of $H(\mathbb{D}^n)$ is equivalent (in some sense) to the problem of finding equivalent quasinorms $\|\cdot\|_X$ on $X = \operatorname{Diag} Y$. In Section 3 we give some new results in this direction.

The so called BMOA type spaces have been investigated recently by many authors, see, for example, [2] and [6]. Hence, the problem of describing the diagonal of the multidimensional BMOA type spaces appears naturally. In this paper we give complete solutions of this problem with some restrictions on parameters.

Throughout this paper, absolute constants will be denoted by C, which need not be the same from line to line.

2. Auxiliary results

In order to prove the main results of the paper we need some known auxiliary results which are incorporated in the following lemmas.

Lemma A. *Suppose* $0 < p < \infty$ *and* $\alpha > -1$. *Then*

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|)^\alpha \,\mathrm{d}m_2(z) \asymp \left(|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p+\alpha} \,\mathrm{d}m_2(z) \right)$$

for all $f \in H(\mathbb{D})$.

For the proof of Lemma A and its generalizations, see, for example, [12], [13], [15], [17], [18] and the references therein.

The following inequality can be found, for example, in [11].

Lemma B. *Suppose* $0 < p \leqslant 1$ *and* $\alpha > 1/p - 2$. *Then*

$$\left(\int_{\mathbb{D}} |f(z)|(1-|z|)^\alpha \,\mathrm{d}m_2(z) \right)^p \leqslant C \int_{\mathbb{D}} |f(z)|^p (1-|z|)^{\alpha p + 2p - 2} \,\mathrm{d}m_2(z) \tag{2}$$

for all $f \in H(\mathbb{D})$.

The next lemma is folklore.

Lemma C. *Suppose that* $f \in \mathscr{A}^p_\alpha(\mathbb{D})$, $p > 0$ *and* $\alpha > -1$. *Then there is a constant* $C = C(p, \alpha)$ *such that*

$$|f(z)| \leqslant \frac{C}{(1-|z|^2)^{(\alpha+2)/p}} \|f\|_{\mathscr{A}^p_\alpha}.$$

By a slight modification of the main result in [16], or of the proof of Theorem 7 in [14] the following result can be proved.

Lemma D. *Assume that* $p \geqslant 2$ *and* $\alpha > -1$. *Then*

$$|f(0)|^p + \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 (1-|z|)^{\alpha+2} \,\mathrm{d}m_2(z) \asymp \int_{\mathbb{D}} |f(z)|^p (1-|z|)^\alpha \,\mathrm{d}m_2(z). \tag{3}$$

The following inequality was proved in [3].

Lemma E. *Assume that $2 \leqslant s < p+2$ and $f \in H^p(\mathbb{D})$. Then*

$$\int_{\mathbb{D}} |f'(z)|^s |f(z)|^{p-s} (1-|z|)^{s-1}\, \mathrm{d}m_2(z) \leqslant C\|f\|_{H^p}^p$$

for some positive constant C independent of f.

Before we formulate the next lemma we introduce the following definitions and notation. The Hardy spaces, denoted by $H^p(\mathbb{D})(0<p<\infty)$, consist of all holomorphic functions on the unit disk such that

$$\sup_{0\leqslant r<1} \int_T |f(r\zeta)|^p \,\mathrm{d}m(\zeta) < \infty.$$

Let $\alpha > 1, \zeta \in T$. Define

$$\Gamma_\alpha(\zeta) = \{z \in \mathbb{D}\colon\ |1-\bar{\zeta}z| < \alpha(1-|z|)\}.$$

Assume that $f \in H(\mathbb{D})$ with the Taylor expansion $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$. Then $\mathscr{D}^t f(z)$ is defined as

$$\mathscr{D}^t f(z) = \sum_{k=0}^{\infty} (k+1)^t a_k z^k.$$

Lemma F ([4])**.** *Assume that $p, t \in (0,\infty)$. Then $f \in H^p(\mathbb{D})$ if and only if*

$$I := \int_T \left(\int_{\Gamma_\alpha(\zeta)} |\mathscr{D}^t f(z)|^2 (1-|z|)^{2t-2}\, \mathrm{d}m_2(z) \right)^{p/2} \mathrm{d}m(\zeta) < \infty,$$

moreover,

$$I \asymp \|f\|_{H^p}^p.$$

The following lemma was proved in [6], see also [11].

Lemma G. *Suppose that $s > -1$, $r, v \geqslant 0$, $v - s < 2 < r - s$ and $r + v - s > 2$. Then*

$$\int_{\mathbb{D}} \frac{(1-|\zeta|)^s \,\mathrm{d}m_2(\zeta)}{|1-\bar{\zeta}z|^r |1-\bar{\zeta}w|^v} \leqslant \frac{C}{|1-\bar{z}w|^v (1-|z|^2)^{r-s-2}}, \quad z, w \in \mathbb{D}.$$

3. Some inequalities for Bergman spaces and mixed norm spaces

In this section we present one-dimensional results of type (1).

Theorem 1. *The following statements hold true.*

(a) *Assume that* $\tau \in (-\alpha - 1, \alpha(2/p - 1))$, $p > 2$ *and* $\alpha > \max\{1, p/2\}$. *Then, for every* $f \in H(\mathbb{D})$,

$$\Big(\int_{\mathbb{D}} |f(z)|^p (1-|z|)^{\alpha-2}\,\mathrm{d}m_2(z)\Big)^{2/p} \leqslant C\Big(|f(0)|^2 + \inf_{w\in S}\int_{\mathbb{D}} |f'(z)|^2(1-|z|)^{\alpha+\tau}\frac{\mathrm{d}m_2(z)}{w(z)}\Big) \tag{4}$$

for some positive constant C *independent of* f, *where* S *is the set of all nonnegative measurable functions on* $\mathbb{D}$ *such that*

$$\|w\|_S = \sup_{z\in\mathbb{D}} w(z)(1-|z|)^{\alpha(2/p-1)-\tau} < 1.$$

(b) *Assume that* $\alpha > 1$ *and* $2 < p$. *Then, for every* $f \in H(\mathbb{D})$,

$$|f(0)|^p + \inf_{w\in S_1}\int_{\mathbb{D}} |f'(z)|^2(1-|z|)^{\alpha+\tau}\frac{\mathrm{d}m_2(z)}{w(z)} \leqslant C\int_{\mathbb{D}} |f(z)|^p(1-|z|)^{\alpha-2}\,\mathrm{d}m_2(z), \tag{5}$$

where S_1 *is the set of all nonnegative measurable functions on* $\mathbb{D}$ *such that*

$$\|w\|_{S_1} = \sup_{z\in\mathbb{D}}(w(z)(1-|z|)^{\alpha(2/p-1)-\tau})^{1/(2-p)} < C(\alpha,p).$$

P r o o f. (a) Without loss of generality we may assume that $f(0) = 0$. By inequality (2) we have

$$\Big(\int_{\mathbb{D}} |f(z)|(1-|z|)^{\alpha-2}\,\mathrm{d}m_2(z)\Big)^p \leqslant C\int_{\mathbb{D}} |f(z)|^p(1-|z|)^{\alpha p-2}\,\mathrm{d}m_2(z) \tag{6}$$

when $p \leqslant 1$, $\alpha > 1/p$ and $f \in H(\mathbb{D})$.

Employing inequality (6) with $p \to 2/p$, Lemma A and the definition of $\|\cdot\|_S$, we have

$$\begin{aligned}\Big(\int_{\mathbb{D}} |f(z)|^p(1-|z|)^{\alpha-2}\,\mathrm{d}m_2(z)\Big)^{2/p} &\leqslant C\int_{\mathbb{D}} |f(z)|^2(1-|z|)^{2\alpha/p-2}\,\mathrm{d}m_2(z)\\ &\leqslant C\int_{\mathbb{D}} |f'(z)|^2(1-|z|)^{2\alpha/p}\,\mathrm{d}m_2(z)\\ &\leqslant C\|w\|_S\int_{\mathbb{D}} |f'(z)|^2(1-|z|)^{\alpha+\tau}\frac{\mathrm{d}m_2(z)}{w(z)},\end{aligned} \tag{7}$$

from which the inequality in (4) follows.

(b) Set

$$w(z) = |f(z)|^{2-p}(1-|z|)^{\tau}.$$

We have

$$w(z)(1-|z|)^{\alpha\left(2/p-1\right)-\tau} = (|f(z)|(1-|z|)^{\alpha/p})^{2-p}.$$

Since $f \in \mathscr{A}^p_{\alpha-2}(\mathbb{D})$, in view of Lemma C it follows that $w \in S_1$.

Putting so defined $w(z)$ into the expression

$$I_w(f) = \int_{\mathbb{D}} |f'(z)|^2(1-|z|)^{\alpha+\tau} \frac{\mathrm{d}m_2(z)}{w(z)}$$

and using Lemma D, we obtain

$$I_w(f) = \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2(1-|z|)^{\alpha} \,\mathrm{d}m_2(z) \leqslant C \int_{\mathbb{D}} |f(z)|^p(1-|z|)^{\alpha-2} \,\mathrm{d}m_2(z).$$

Hence

$$\inf_{w\in S_1} \int_{\mathbb{D}} |f'(z)|^2(1-|z|)^{\alpha+\tau} \frac{\mathrm{d}m_2(z)}{w(z)} \leqslant C \int_{\mathbb{D}} |f(z)|^p(1-|z|)^{\alpha-2} \,\mathrm{d}m_2(z),$$

which completes the proof of the theorem. □

Theorem 2. *Assume that $s > 0$ and $p < 2$. Then*

$$\int_T \left(\int_{\Gamma_a(\zeta)} |\mathscr{D}f(z)|^2 \,\mathrm{d}m_2(z) \right)^{p/2} \mathrm{d}m(\zeta) \asymp \left(\inf_{w\in S_2} \int_{\mathbb{D}} |\mathscr{D}f(z)|^s (1-|z|)^{s-1} \frac{\mathrm{d}m_2(z)}{w(z)} \right)^{p/2}, \tag{8}$$

where S_2 is the set of all nonnegative measurable functions on $\mathbb{D}$ such that

$$I_1 := \Big\| \sup_{\Gamma_a(\zeta)} w(z)(1-|z|)^{2-s}|\mathscr{D}f(z)|^{2-s} \Big\|_{\mathscr{L}^{p/(2-p)}(T)} < 1,$$

where $\zeta \in T$.

P r o o f. Assume that $q \geqslant 2$ and let

$$I_2 = \Big\| \sup_{\Gamma_a(\zeta)} w(z)(1-|z|)^{q-s}|\mathscr{D}f(z)|^{q-s} \Big\|_{\mathscr{L}^{p/(q-p)}(T)}.$$

Then, by Hölder's inequality with exponents q/p and $q/(q-p)$, we obtain

$$\begin{aligned}
&\int_T \left(\int_{\Gamma_a(\zeta)} |\mathscr{D} f(z)|^q (1-|z|)^{q-2} \, \mathrm{d}m_2(z) \right)^{p/q} \mathrm{d}m(\zeta) \\
&\quad \leqslant C \int_T \left(\sup_{z \in \Gamma_a(\zeta)} w(z)(1-|z|)^{q-s} |\mathscr{D} f(z)|^{q-s} \right)^{p/q} \\
&\qquad \times \left(\int_{\Gamma_a(\zeta)} |\mathscr{D} f(z)|^s (1-|z|)^{s-2} \frac{\mathrm{d}m_2(z)}{w(z)} \right)^{p/q} \mathrm{d}m(\zeta) \\
&\quad \leqslant I_2^{1-p/q} \left(\int_T \int_{\Gamma_a(\zeta)} |\mathscr{D} f(z)|^s (1-|z|)^{s-2} \frac{\mathrm{d}m_2(z)}{w(z)} \, \mathrm{d}m(\zeta) \right)^{p/q} \\
&\quad \leqslant I_2^{1-p/q} \left(\int_{\mathbb{D}} |\mathscr{D} f(z)|^s (1-|z|)^{s-2} \int_T \chi_{A_z}(\zeta) \, \mathrm{d}m(\zeta) \frac{\mathrm{d}m_2(z)}{w(z)} \right)^{p/q} \\
&\quad \leqslant C I_2^{1-p/q} \left(\int_{\mathbb{D}} |\mathscr{D} f(z)|^s (1-|z|)^{s-1} \frac{\mathrm{d}m_2(z)}{w(z)} \right)^{p/q},
\end{aligned}$$

where we have used the fact that the linear measure of the set $A_z = \{\zeta \colon\ z \in \Gamma_\alpha(\zeta)\}$ behaves as $1-|z|$. For $q=2$ we obtain one direction of (8). □

Now we prove the reverse inequality. Let

$$w(z) = \frac{1}{\|f\|_{H^p}^{2-p}} \left((1-|z|)^{s-2} |\mathscr{D} f(z)|^{s-2} |f(z)|^{2-p} \right).$$

Then

$$\begin{aligned}
I_1 &= \| \sup_{\Gamma_a(\zeta)} w(z)(1-|z|)^{2-s} |\mathscr{D} f(z)|^{2-s} \|_{\mathscr{L}^{p/(2-p)}(T)} \\
&= \left(\int_T \sup_{\Gamma_a(\zeta)} |f(z)|^p / \|f\|_{H^p}^p \, \mathrm{d}m(\zeta) \right)^{(2-p)/p} < 1.
\end{aligned}$$

For so defined $w(z)$, by Lemma E we have

$$\begin{aligned}
&\int_{\mathbb{D}} |\mathscr{D} f(z)|^s (1-|z|)^{s-1} \frac{\mathrm{d}m_2(z)}{w(z)} \\
&\qquad = \|f\|_{H^p}^{2-p} \int_{\mathbb{D}} |\mathscr{D} f(z)|^2 |f(z)|^{p-2} (1-|z|) \, \mathrm{d}m_2(z) \leqslant C \|f\|_{H^p}^p.
\end{aligned}$$

Employing Lemma F with $t=1$, the result follows. Theorem 2 is proved.

Let $Q_{p,q,s} = Q_{p,q,s}(\mathbb{D}^n)$ be the subspace of all $f \in H(\mathbb{D}^n)$ such that

$$\|f\|^q_{Q_{p,q,s}} = \sup_{w\in\mathbb{D}^n} \prod_{j=1}^n (1-|w_j|)^p \int_{\mathbb{D}^n} \frac{|f(z)|^q \prod\limits_{k=1}^n (1-|z_k|)^s}{\prod\limits_{k=1}^n |1-\overline{w}_k z_k|^{2p}} \, \mathrm{d}m_{2n}(z) < \infty,$$

where $0 < p, q < \infty$ and $s > -1$. This is the so called BMOA-type space, see, for example, [2], [6].

Let $\widehat{Q}_{t,q,s,p}(\mathbb{D}^n)$ be the space of all $f \in H(\mathbb{D}^n)$ such that

$$\|f\|^q_{\widehat{Q}_{t,q,s,p}} = \sup_{w\in\mathbb{D}^n} (1-|w|)^t \int_{\mathbb{D}^n} \frac{|f(z)|^q \prod\limits_{k=1}^n (1-|z_k|)^s}{\prod\limits_{k=1}^n |1-|w|\zeta_k z_k|^{2p}} \, \mathrm{d}m_{2n}(z) < \infty,$$

where $w = |w|(\zeta_1, \ldots, \zeta_n)$, $0 < p, q < \infty$ and $s > -1$.

Let further $\widehat{Q}^q_{p,s,n} = \widehat{Q}^q_{p,s,n}(\mathbb{D})$ be the subspace of all $f \in H(\mathbb{D})$ such that

$$\|f\|^q_{\widehat{Q}^q_{p,s,n}} = \sup_{w\in\mathbb{D}^n} \prod_{j=1}^n (1-|w_j|)^p \int_{\mathbb{D}} \frac{|f(z)|^q (1-|z|)^{ns+2n-2}}{\prod\limits_{k=1}^n |1-\overline{w}_k z|^{2p}} \, \mathrm{d}m_2(z) < \infty,$$

where $0 < p, q < \infty$ and $s > -1$.

Finally, let $\widehat{Q}^q_{p,t,s,n}$ be the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|\hat{f}\|^q_{\widehat{Q}^q_{p,t,s,n}} = \sup_{w\in\mathbb{D}^n} \int_{\mathbb{D}} \frac{(1-|w|)^t (1-|z|)^{ns+2n-2} |f(z)|^q}{\prod\limits_{k=1}^n |1-|w|\overline{\zeta}_k z|^{2p}} \, \mathrm{d}m_2(z) < \infty,$$

where $w_k = |w|\zeta_k$, $k \in \{1, \ldots, n\}$.

An interesting question is:

Question. When does the equality

$$\operatorname{Diag} Q_{p,q,s}(\mathbb{D}^n) = \widehat{Q}^q_{p,s,n}(\mathbb{D})$$

hold, and for which p, q, s?

Case $q \leqslant 1$. In this subsection we consider the case $q \leqslant 1$. An answer to the above question for this case is given by the following theorem.

Theorem 2. *Suppose that $s > \max\{2p-2,-1\}$, $q \leqslant 1$ and $p > 0$. Then*

$$\operatorname{Diag} Q_{p,q,s}(\mathbb{D}^n) = \widehat{Q}^q_{p,s,n}(\mathbb{D}).$$

Proof. Fix $w \in \mathbb{D}^n$. By using dyadic decomposition of the polydisk for fixed $w \in \mathbb{D}^n$, the following inequality can be obtained for all $0 < p, q < \infty$ and $s > -1$, as in [11]:

$$\int_{\mathbb{D}^n} \frac{|f(z)|^q \prod_{k=1}^n (1-|z_k|)^s}{\prod_{k=1}^n |1-\overline{w}_k z_k|^{2p}} \,\mathrm{d}m_{2n}(z) \geqslant C \int_{\mathbb{D}} \frac{|f(z,\dots,z)|^q (1-|z|)^{ns+2n-2}}{\prod_{k=1}^n |1-\overline{w}_k z|^{2p}} \,\mathrm{d}m_2(z). \tag{9}$$

Multiplying (9) by $\prod_{j=1}^n (1-|w_j|)^p$, then taking the supremum over $\mathbb{D}^n$ we obtain that

$$\|\varphi(\tilde f)\|_{Q_{p,q,s}} \geqslant C \|\operatorname{Diag} f\|_{\widehat{Q}^q_{p,s,n}} \tag{10}$$

for every analytic extension $\varphi(\tilde f)$ of the function $\tilde f$ on the polydisk ($\tilde f = f(z,\dots,z) = \operatorname{Diag} f$).

The main problem is how to prove the reverse statement. As in [11] set

$$F(f)(z) = C(\alpha,n) \int_{\mathbb{D}} \frac{(1-|w|)^\alpha f(w)}{\prod_{j=1}^n (1-\overline{w} z_j)^{(\alpha+2)/n}} \,\mathrm{d}m_2(w), \quad \alpha > -1,\ n \in \mathbb{N}, \tag{11}$$

where $f \in H(\mathbb{D})$, α is sufficiently large, for example, $(\alpha+2)q/n > \max\{2p, s+2\}$, and $C(\alpha,n)$ is the well known Bergman projection constant. Obviously $\operatorname{Diag} F(f)(z) = f(z)$, by virtue of the reproducing property of the Bergman projection.

Now we prove the reverse inequality of (10), that is, $\|F(f)\|_{Q_{p,q,s}} \leqslant C\|f\|_{\widehat{Q}^q_{p,s,n}}$ (for some p,q,s).

From (11) and (2) (see also [10]) we have

$$\prod_{j=1}^n (1-|w_j|)^p \int_{\mathbb{D}^n} \frac{|F(f)(z)|^q \prod_{k=1}^n (1-|z_k|)^s}{\prod_{k=1}^n |1-\overline{w}_k z_k|^{2p}} \,\mathrm{d}m_{2n}(z)$$

$$\leqslant C \prod_{j=1}^n (1-|w_j|)^p \int_{\mathbb{D}^n} \int_{\mathbb{D}} \frac{|f(\hat z)|^q (1-|\hat z|)^{\alpha q+2q-2} \prod_{k=1}^n (1-|z_k|)^s \,\mathrm{d}m_2(\hat z)}{\prod_{k=1}^n |1-\overline{w}_k z_k|^{2p} \prod_{k=1}^n |1-z_k \overline{\hat z}|^{q(\alpha+2)/n}} \,\mathrm{d}m_{2n}(z)$$

$$\leqslant C\prod_{j=1}^{n}(1-|w_j|)^p\int_{\mathbb{D}}\frac{|f(\hat z)|^q\prod\limits_{k=1}^{n}(1-|\hat z|)^{ns+2n-2}}{\prod\limits_{k=1}^{n}|1-\overline{w}_k\hat z|^{2p}}\,\mathrm{d}m_2(\hat z),$$

where in the last inequality we have used Lemma G with $v=2p$ and $r=n^{-1}(\alpha+2)q$, that is, the inequality

$$\int_{\mathbb{D}^n}\frac{\prod\limits_{k=1}^{n}(1-|z_k|)^s\,\mathrm{d}m_{2n}(z)}{\prod\limits_{k=1}^{n}|1-\overline{w}_kz_k|^{2p}\prod\limits_{k=1}^{n}|1-z_k\bar{\hat z}|^{(\alpha+2)q/n}}\leqslant\frac{(1-|\hat z|)^{t_1}}{\prod\limits_{k=1}^{n}|1-\overline{w}_k\hat z|^{2p}}$$

where $t_1=(s+2)n-(\alpha+2)q$. Note that by the choice of α we have

$$2p-s<2<\frac{\alpha+2}{n}q-s,\quad s>-1\quad\text{and}\quad\frac{\alpha+2}{n}q+2p-s>2,$$

so that Lemma G can be applied. Using this and the fact that $t_1+\alpha q+2q-2=ns+2n-2$, the reverse inequality and consequently the theorem follow.

Note that we have proved above that

$$\begin{aligned}(12)\qquad \|f\|^q_{Q_{p,q,s}}&=\sup_{w\in\mathbb{D}^n}\prod_{j=1}^{n}(1-|w_j|)^p\int_{\mathbb{D}^n}|f(z)|^q\prod_{k=1}^{n}\frac{(1-|z_k|)^s}{|1-\overline{w}_kz_k|^{2p}}\,\mathrm{d}m_{2n}(z)\\&\geqslant\sup_{w\in\mathbb{D}^n}\int_{\mathbb{D}}\frac{(1-|w|)^{np}(1-|z|)^{ns+2n-2}|f(z,\dots,z)|^q}{\prod\limits_{k=1}^{n}|1-|w|\bar\zeta_kz|^{2p}}\,\mathrm{d}m_2(z)\\&=\|\hat f\|^q_{\widehat{Q}^q_{p,np,s,n}},\end{aligned}$$

where $w_k=|w|\zeta_k$, $k\in\{1,\dots,n\}$.

This is true since

$$\begin{aligned}\|f\|^q_{Q_{p,q,s}}&\geqslant\|f(z,\dots,z)\|^q_{\widehat{Q}^q_{p,s,n}}\\&=\sup_{w\in\mathbb{D}^n}\prod_{k=1}^{n}(1-|w_k|)^p\int_{\mathbb{D}}\frac{|f(z,\dots,z)|^q(1-|z|)^{ns+2n-2}}{\prod\limits_{l=1}^{n}|1-|w_l|\bar\zeta_lz|^{2p}}\,\mathrm{d}m_2(z)\\&\geqslant\|\hat f\|^q_{\widehat{Q}^q_{p,np,s,n}}.\end{aligned}$$

As in the proof of the previous theorem, we also have

$$(13)\qquad (1-|w|)^t \int_{\mathbb{D}^n} |F(f)(z)|^q \prod_{l=1}^{n} \frac{(1-|z_l|)^s}{|1-\overline{w}_l z_l|^{2p}}\, \mathrm{d}m_{2n}(z)$$
$$\leqslant C \int_{\mathbb{D}^n}\int_{\mathbb{D}} |f(\hat{z})|^q \frac{(1-|w|)^t(1-|\hat{z}|)^{\alpha q+2q-2}\,\mathrm{d}m_2(\hat{z}) \prod_{l=1}^{n}(1-|z_l|)^s}{\prod_{k=1}^{n} |1-\bar{\hat{z}} z_k|^{(\alpha+2)q/n}|1-\overline{w}_k z_k|^{2p}}\, \mathrm{d}m_{2n}(z),$$

$w_k = |w|\zeta_k$, $k \in \{1,\dots,n\}$.

Further, similarly to the above we have that for every $t>0$ the following inequality holds

$$(14)\qquad \sup_{w\in\mathbb{D}^n} (1-|w|)^t \int_{\mathbb{D}^n} |F(f)(z)|^q \prod_{l=1}^{n} \frac{(1-|z_l|)^s}{|1-|w|\zeta_l z_l|^{2p}}\, \mathrm{d}m_{2n}(z)$$
$$\leqslant C \sup_{w\in\mathbb{D}} (1-|w|)^t \int_{\mathbb{D}} |f(z)|^q \frac{(1-|z|)^{sn+2n-2}}{\prod_{k=1}^{n}|1-|w|\zeta_k z|^{2p}}\, \mathrm{d}m_2(z).$$

By changing the order of integration and using Lemma G, we obtain the inequality

$$(15)\qquad (1-|w|)^t \int_{\mathbb{D}^n} |F(f)(z)|^q \prod_{l=1}^{n} \frac{(1-|z_l|)^s}{|1-\overline{w}_l z_l|^{2p}}\, \mathrm{d}m_{2n}(z)$$
$$\leqslant C (1-|w|)^t \int_{\mathbb{D}} |f(z)|^q \frac{(1-|z|)^{sn+2n-2}}{\prod_{k=1}^{n}|1-|w|\zeta_k z|^{2p}}\, \mathrm{d}m_2(z).$$

Hence, we obtain the following theorem:

Theorem 3. *Assume that* $s > \max\{2p-2, -1\}$, $q \leqslant 1$, $p > 0$ *and* $t > 0$. *Then*

$$\operatorname{Diag} \widehat{Q}_{t,q,p,s}(\mathbb{D}^n) = \widehat{Q}^q_{p,t,s,n}(\mathbb{D}).$$

Indeed, one direction follows from (13) and the reverse is a consequence of (14) and (15).

From Theorem 3 we now easily obtain the following corollary:

Corollary 1. *Assume that $s > \max\{2p-2,-1\}$, $q \leqslant 1$ and $p > 0$. Then*

$$\begin{aligned}\operatorname{Diag}\widehat{Q}_{np,q,s,p}(\mathbb{D}^n) &= \sup_{w\in\mathbb{D}^n}(1-|w|)^{np}\int_{\mathbb{D}}|f(z)|^q\frac{(1-|z|)^{ns+2n-2}\,\mathrm{d}m_2(z)}{\prod\limits_{k=1}^{n}|1-|w|\zeta_k z|^{2p}}\\ &\lesssim \sup_{w\in\mathbb{D}^n}\prod_{k=1}^{n}(1-|w_k|)^p\int_{\mathbb{D}}|f(z)|^q\frac{(1-|z|)^{ns+2n-2}\,\mathrm{d}m_2(z)}{\prod\limits_{k=1}^{n}|1-\overline{w}_k\overline{z}|^{2p}}\\ &= \operatorname{Diag}Q_{p,q,s}(\mathbb{D}^n).\end{aligned}$$

Case $q \geqslant 1$. Here we consider the case $q \geqslant 1$. We use some methods from paper [8].

Theorem 4. *Suppose that $p < 1/2$, $q \geqslant 1$ and $s > -1$. Then*

$$\operatorname{Diag}Q_{p,q,s}(\mathbb{D}^n) = \widehat{Q}^q_{p,s,n}(\mathbb{D}).$$

Proof. As we have already mentioned our aim is to prove the inequality

$$\begin{aligned}\sup_{w\in\mathbb{D}^n}\prod_{k=1}^{n}(1-|w_k|)^p\int_{\mathbb{D}^n}\frac{|F(f)(z)|^q\prod\limits_{k=1}^{n}(1-|z_k|)^s\,\mathrm{d}m_{2n}(z)}{\prod\limits_{k=1}^{n}|1-\overline{w}_k z_k|^{2p}} \qquad (q\geqslant 1)\\ \lesssim \sup_{w\in\mathbb{D}^n}\prod_{k=1}^{n}(1-|w_k|)^p\int_{\mathbb{D}}\frac{|f(z)|^q(1-|z|)^{ns+2n-2}\,\mathrm{d}m_2(z)}{\prod\limits_{k=1}^{n}|1-\overline{w}_k\overline{z}|^{2p}},\end{aligned}$$

where $F(f)(z)$ is defined by (11) and α is large enough.

As in [8], when $q \geqslant 1$, then using Jensen's inequality, we have

$$M_q(\widetilde{\varphi_\alpha}(f),\tau) \lesssim \int_0^1(1-\varrho)^\alpha M_q(M_1(\varrho,\widetilde{G}),\tau)\,\mathrm{d}\varrho \tag{16}$$

where

$$G(z) = G = \frac{f(z)}{\prod\limits_{j=1}^{n}(1-\overline{z}z_j)^{(\alpha+2)/n}},\qquad \tau=(\tau_1,\ldots,\tau_n),$$

$\tau_j = |z_j|$, $\varrho = |z| \in I$,

$$(\widetilde{\varphi_\alpha}f) := \frac{F(f)(z_1,\ldots,z_n)\Big(\prod\limits_{k=1}^{n}(1-|z_k|)\Big)^{s/q}}{\prod\limits_{k=1}^{n}|1-\overline{w}_k z_k|^{2p/q}}, \tag{17}$$

and

$$\widetilde{G} := G\frac{\prod_{k=1}^{n}(1-|z_k|)^{s/q}}{\prod_{k=1}^{n}|1-\overline{w}_k z_k|^{2p/q}}.$$

Now let us write $\widetilde{G}$ as the product $\widetilde{G} = G_1G_2$, where

$$G_1 = f(z)\prod_{j=1}^{n}(1-\overline{w}_k z_k)^{-2p/q}\prod_{j=1}^{n}|1-\overline{z}z_j|^{-((\alpha+2)/n)/q}\prod_{k=1}^{n}(1-|z_k|)^{s/q}$$

and

$$G_2 = \prod_{j=1}^{n}|1-\overline{z}z_j|^{-((\alpha+2)/n)/q'}.$$

By Hölder's inequality and [8, Lemma 4.2], it follows that

$$\begin{aligned}
[M_1(\varrho,\widetilde{G})] &\lesssim [M_q(\varrho,G_1)][M_{q'}(\varrho,G_2)] \\
&\lesssim [M_q(\varrho,G_1)]\prod_{j=1}^{n}(1-|z\|z_j|)^{-((\alpha+2)/n-\delta_{j1})/q'},
\end{aligned} \tag{18}$$

where $(\delta_{j1}) = 0$, $j \neq k$, $(\delta_{j1}) = 1$, $j = k$, $1 \leqslant k \leqslant n$, $z = \varrho\xi$ and $z_k = \tau_k\varphi_k$.

We have

$$\begin{aligned}
&M_q^q(\tau, M_q(\varrho, G_1)) \\
&= \int_{T^n}\int_T \frac{|f(z)|^q\,\mathrm{d}m(\xi)}{\prod_{j=1}^{n}|1-\overline{z}z_j|^{(\alpha+2)/n}}\Big(\prod_{j=1}^{n}\frac{1}{|1-\overline{w}_j z_j|^{2p}}\Big)\,\mathrm{d}m(\varphi)\prod_{k=1}^{n}(1-|z_k|)^s \\
&\lesssim \int_T |f(z)|^q\Big(\prod_{k=1}^{n}\int_T\frac{\mathrm{d}m(\varphi_k)}{|1-\overline{z}z_k|^{(\alpha+2)/n}|1-\overline{w}_k z_k|^{2p}}\Big)\,\mathrm{d}m(\xi) \\
&\qquad\times[(1-|z_k|^s)]\prod_{k=1}^{n}(1-|z_k|)^s.
\end{aligned} \tag{19}$$

From (16) we obtain

$$\begin{aligned}
&\sup_{w\in\mathbb{D}^n}\prod_{k=1}^{n}(1-|w_k|)^p\int_{\mathbb{D}^n}\frac{|F(f)(z)|^q\prod_{k=1}^{n}(1-|z_k|)^s}{\prod_{k=1}^{n}|1-\bar{w}_k z_k|^{2p}}\,\mathrm{d}m_{2n}(z) \\
&\quad\lesssim \sup_{w\in\mathbb{D}^n}\prod_{k=1}^{n}(1-|w_k|)^p\int_{I^n}\prod_{k=1}^{n}(1-|z_k|)^s\Big(\int_0^1(1-\varrho)^\alpha M_q(\tau, M_1(\tau,\widetilde{G}))\,\mathrm{d}\varrho\Big)^q \\
&\qquad\qquad\mathrm{d}|z_1|\ldots\mathrm{d}|z_n|
\end{aligned}$$

where $I^n = [0,1)^n$.

Now we calculate the inner integral in the last expression using (18) and (19). We have

$$
\begin{aligned}
&\int_0^1 (1-\varrho)^\alpha M_q(\tau, M_1(\tau, \widetilde{G}))\,\mathrm{d}\varrho \\
&\quad\lesssim \int_0^1 (1-\varrho)^\alpha \prod_{j=1}^n (1-\varrho|z_j|)^{-((\alpha+2)/n-\delta_{j1})/q'} \\
&\quad\quad\times \bigg(\int_T |f(z)|^q \Big(\prod_{k=1}^n \int_T \frac{\mathrm{d}m(\varphi_k)}{|1-\tau_k\bar{z}\varphi_k|^{(\alpha+2)/n}|1-\overline{w}_k\tau_k\varphi_k|^{2p}}\Big)\,\mathrm{d}m(\xi)\bigg)^{1/q}\,\mathrm{d}\varrho \\
&\quad= \int_0^1 (1-\varrho)^\alpha S_1(\varrho) S_2^{1/q}(\varrho)\,\mathrm{d}\varrho,
\end{aligned}
$$

where $z=\varrho\xi$. Note that $S_2(\varrho_1)\leqslant S_2(\varrho_2)$, if $\varrho_1\leqslant\varrho_2$ and $\varrho_1,\varrho_2\in(0,1]$.

Hence, we can use the following inequality from [8]:

$$
\begin{aligned}
&\int_{I^n} \prod_{j=1}^n (1-\tau_j)^{ka_j-1}\bigg(\int_0^1 \frac{(1-\varrho)^{\delta-1}}{\prod\limits_{j=1}^n (1-\tau_j\varrho)^{b_j}} g(\varrho)\,\mathrm{d}\varrho\bigg)^k \mathrm{d}\tau_1\ldots\mathrm{d}\tau_n \\
&\quad\leqslant C(a,b,\delta,k)\int_0^1 (1-\varrho)^{k(|a|-|b|+\delta)-1} g^k(\varrho)\,\mathrm{d}\varrho
\end{aligned}
$$

when $b_j > a_j > 0$ for $j=1,\ldots,n$, $\delta>0$ and $1\leqslant k<\infty$, where $|a| = \sum\limits_{j=1}^n a_j$, $|b| = \sum\limits_{l=1}^n b_l$, $s=-1+Ka_j$, $j=1,\ldots,n$. For sufficiently large α we have

$$
\begin{aligned}
\|F(f)\|^q_{Q_{p,q,s}} &\lesssim \sup_{w\in\mathbb{D}^n} \prod_{k=1}^n (1-|w_k|)^p \int_{I^n} \Big(\prod_{j=1}^n (1-|z_j|)\Big)^s \\
&\quad\times\bigg(\int_0^1 (1-\varrho)^\alpha S_2(\varrho)^{1/q}\Big(\prod_{j=1}^n (1-\varrho|z_j|)^{-((\alpha+2)/n-\delta_{j1})/q'}\Big)\,\mathrm{d}\varrho\bigg)^q \mathrm{d}|z_1|\ldots\mathrm{d}|z_n| \\
&\lesssim \sup_{w\in\mathbb{D}^n} \prod_{k=1}^n (1-|w_k|)^p \int_0^1 S_2(\varrho)(1-\varrho)^{-1+q(\alpha+1+n(s+1)/q-(\alpha+2)/q'+1/q')}\,\mathrm{d}\varrho \\
&\lesssim \sup_{w\in\mathbb{D}^n} \prod_{k=1}^n (1-|w_k|)^p \times \int_0^1 \int_T |f(z)|^q \Big(\prod_{k=1}^n \int_T \frac{\mathrm{d}m(\varphi_k)}{|1-z\bar{\varphi}_k|^{(\alpha+2)/n}|1-\overline{w}_k\varphi_k|^{2p}}\Big) \\
&\qquad\qquad \mathrm{d}m(\xi)(1-\varrho)^{-1+q(\alpha+1+n(s+1)/q-(\alpha+2)/q'+1/q')}\,\mathrm{d}\varrho.
\end{aligned}
$$

Hence, finally we must show that under some restrictions on indices we have
(20)

$$(1-\varrho)^{-1+q(\alpha+1+n(s+1)/q-(\alpha+2)/q'+1/q')} \times \prod_{k=1}^{n} \int_{T} \frac{\mathrm{d}m(\varphi_k)}{|1-\varrho\zeta\bar{\varphi}_k|^{(\alpha+2)/n}|1-\overline{w}_k\varphi_k|^{2p}} \lesssim \frac{(1-|z|)^{ns+2n-2}}{\prod_{k=1}^{n}|1-w_k\bar{z}|^{2p}},$$

where $|z| = \varrho$ and α is large enough. This is true since it is not difficult to check that

$$\int_{T} \frac{\mathrm{d}\xi}{|1-\bar{\xi}z|^{\gamma}|1-\overline{w}\xi|^{\beta}} \lesssim \frac{(1-|z|)^{1-\widetilde{\gamma}}}{|1-\bar{z}w|^{\gamma}}; \quad z, w \in \mathbb{D}, \tag{21}$$

where $\gamma < 1$, $\beta = q - \gamma$, $\quad q \in (1+2\gamma, \infty)$, $\quad \widetilde{\gamma} = q - \gamma$.

The estimate (21) was proved in [10]. Using (21) we have

$$\prod_{k=1}^{n} \int_{T} \frac{\mathrm{d}m(\varphi_k)}{|1-z\bar{\varphi}_k|^{(\alpha+2)/n}|1-\overline{w}_k\varphi_k|^{2p}} \leqslant C\frac{(1-|z|)^{n-(\alpha+2)}}{\prod_{k=1}^{n}|1-\bar{z}w_k|^{2p}}$$

when $p < 1/2$, since α is big enough. This completes the proof of the theorem.

A closer inspection of the proofs of Theorems 2–4 shows that the following corollary holds.

Corollary 2. *Suppose that* $s > \max\{2p-2, -1\}$, $q \leqslant 1$ *and* $p > 0$, *or* $p < 1/2$, $q \geqslant 1$ *and* $s > -1$. *Then*

$$\inf_{\varphi \in M} \|\varphi(f)\|_{Q_{p,q,s}(\mathbb{D}^n)} \asymp \|f\|_{\widehat{Q}^{q}_{p,s,n}},$$

where $\varphi(f)(z_1, \ldots, z_n)$ *is an arbitrary extension of* f *from the diagonal* $(z, \ldots, z)$ *to the polydisk* $\mathbb{D}^n$.

R e m a r k. Equivalence relation (8) which provides a completely new characterization of analytic Hardy classes in the unit disk can be called Fefferman-Stein type characterization of H^p-Hardy classes, since apparently a very similar relationship for Hardy spaces in $\mathbb{R}^n$ was found for the first time in a well-known classical paper of Fefferman and Stein. The author obtained also Fefferman-Stein type characterizations of Bloch and Bergman spaces and not only in the unit disk, but also in higher dimensions: unit ball and polydisk.

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