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CONVERGENCE TO EQUILIBRIA IN A DIFFERENTIAL EQUATION WITH SMALL DELAY

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Abstract. Consider the delay differential equation

$$\dot{x}(t) = g(x(t), x(t-r)), \tag{1}$$

where $r > 0$ is a constant and $g\colon \mathbb{R}^2 \to \mathbb{R}$ is Lipschitzian. It is shown that if r is small, then the solutions of (1) have the same convergence properties as the solutions of the ordinary differential equation obtained from (1) by ignoring the delay.

Keywords: delay differential equation, equilibrium, convergence

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Consider the delay differential equation

$$\dot{x}(t) = g(x(t), x(t-r)), \tag{1}$$

where $r > 0$ is a constant and $g\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies the Lipschitz type condition

$$|g(x_1, y_1) - g(x_2, y_2)| \leqslant K|x_1 - x_2| + L|y_1 - y_2| \qquad \text{for } (x_i, y_i) \in \mathbb{R}^2,\ i = 1, 2, \tag{2}$$

where K, $L > 0$ are constants.

We may view K as the supremum of $|\partial g(x,y)/\partial x|$ and L as the supremum of $|\partial g(x,y)/\partial y|$, where the supremum is taken over all $(x, y) \in \mathbb{R}^2$.

Under the above hypothesis, for every continuous initial function $\varphi\colon [-r, 0] \to \mathbb{R}$, Eq. (1) has a unique solution x on $[-r, \infty)$ with the initial values

$$x(t) = \varphi(t) \qquad \text{for } -r \leqslant t \leqslant 0. \tag{3}$$

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It is of interest to compare the behavior of the solutions of (1) with the ordinary differential equation obtained from (1) by ignoring the delay. This equation, given by

$$\dot{x} = g(x, x), \tag{4}$$

has the same equilibria as (1). The set of equilibria E for (4) is given by

$$E = \{x \in \mathbb{R};\ g(x, x) = 0\}.$$

Suppose that $E \neq \emptyset$ and let

$$S = \sup E \qquad \text{and} \qquad I = \inf E.$$

Using the fact that the solutions of the ordinary differential equation (4) are monotone, it is easy to show that every solution of (4) converges, as $t \to \infty$, to an equilibrium if and only if both conditions (B^+) and (B^-) below hold.

(B^+) Either $S = \infty$ or $S < \infty$ and $g(x, x) < 0$ for $x > S$.

(B^-) Either $I = -\infty$ or $I > -\infty$ and $g(x, x) > 0$ for $x < I$.

In this paper, we shall show that under the smallness condition

$$Lr\mathrm{e}^{Kr} < \frac{1}{\mathrm{e}}, \tag{5}$$

the same conclusion is true for the solutions of the delay differential equation (1).

Theorem 1. *Suppose conditions* (2) *and* (5) *hold. Then every solution of* (1) *converges to an equilibrium if and only if conditions* (B^+) *and* (B^-) *hold.*

Theorem 1 is a consequence of a more general result concerning differential equations with distributed delays [9]. We will describe the main ideas of the proof.

The proof of Theorem 1 is based on some previous results [5], [9], formulated in Theorems 2–4 below, showing that under the smallness condition (5) the delay differential equation (1) is asymptotically equivalent to a certain scalar ordinary differential equation. (For further related results, see [1], [2], [4], [7], [10] and the references therein.)

Theorem 2. *Suppose conditions* (2) *and* (5) *hold. Then, for every* $x_0 \in \mathbb{R}$, *Eq.* (1) *has a unique solution* $\bar{x}$ *on* $(-\infty, \infty)$ *such that*

$$\bar{x}(0) = x_0 \qquad \textit{and} \qquad \sup_{t \leqslant 0} |\bar{x}(t)| \mathrm{e}^{\mu t} < \infty,$$

where $\mu = L + 1/r$.

The solution $\bar{x}$ of (1) from Theorem 2 will be called the *special solution* (passing through $(0, x_0)$) and will be denoted by $\bar{x} = \bar{x}(0, x_0)$. The following theorem states that every solution of (1) is asymptotically equivalent to some special solution as $t \to \infty$.

Theorem 3. *Under the hypotheses of Theorem* 2, *for every solution x of* (1) *on* $[-r, \infty)$ *there exists a unique special solution $\bar{x}$ of* (1) *such that*

$$\sup_{t \geqslant 0} |x(t) - \bar{x}(t)| \mathrm{e}^{\mu t} < \infty,$$

where μ has the meaning from Theorem 2. *In particular,*

$$\lim_{t \to \infty} |x(t) - \bar{x}(t)| = 0.$$

The next result states that the special solutions of (1) satisfy an ordinary differential equation.

Theorem 4. *Under the hypotheses of Theorem* 2, *the special solutions of* (1) *satisfy the ordinary differential equation*

$$\dot{x} = h(x), \tag{6}$$

where $h\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) = g(x, \bar{x}(0, x)(-r)) \qquad \text{for } x \in \mathbb{R}.$$

Moreover, h is Lipschitzian and Eqs. (1) *and* (6) *have the same equilibria.*

We are in a position to give an outline of the proof of Theorem 1.

S u f f i c i e n c y. Theorem 3 implies that in order to prove the convergence of all solutions of (1), it suffices to restrict ourselves to the special solutions. By Theorem 4, the special solutions of (1) satisfy the ordinary differential equation (6). Since the solutions of (6) are monotone, the boundedness of the solutions of (6) (which coincide with the special solutions of (1)) implies their convergence to equilibria. Consequently, it suffices to show that if conditions (B^+) and (B^-) are satisfied, then all (special) solutions of (1) are bounded. This will be shown in Lemmas 1 and 2 below.

Lemma 1. *Suppose conditions* (2), (5) *and* (B^+) *hold. Then every solution of* (1) *is bounded from above.*

Before we present the proof of Lemma 1, we summarize some facts from the oscillation theory of linear delay differential equations (see [3] and/or [9]).

Consider the linear delay differential equation

$$\dot{y}(t) = -Ky(t) - Ly(t-r). \tag{7}$$

The *fundamental solution* v of (7) is the unique solution of (7) with the initial values

$$v(t) = \begin{cases} 0 & \text{for } -r \leqslant t < 0 \\ 1 & \text{for } t = 0. \end{cases}$$

Together with v, consider the solution u of (7) with the initial values $u(t) = 1$ for $-r \leqslant t \leqslant 0$. If (5) holds, then both the solutions u and v are positive for $t \geqslant 0$. Moreover, u is a *dominant solution* of (7) in the sense that for every solution y of (7) on $[-r, \infty)$ there exists a constant $M > 0$ such that

$$|y(t)| < Mu(t) \qquad \text{for all } t \geqslant -r. \tag{8}$$

Furthermore, u and v satisfy the identity

$$u(t) + (K+L)\int_0^t v(t-s)\,\mathrm{d}s = 1 \qquad \text{for all } t \geqslant 0. \tag{9}$$

Using these facts, we can easily prove the lemma.

Let x be a solution of (1) and (3). Rewrite Eq. (1) in the form

$$\dot{x}(t) = -Kx(t) - Lx(t-r) + f(x(t), x(t-r)),$$

where f is defined by

$$f(x,y) = g(x,y) + Kx + Ly \qquad \text{for } (x,y) \in \mathbb{R}^2$$

and satisfies the monotonicity condition

$$f(x_1, y_1) \leqslant f(x_2, y_2) \qquad \text{whenever } x_1 \leqslant x_2 \text{ and } y_1 \leqslant y_2 \tag{10}$$

(see (2)). By the variation-of-constants formula [6], for $t \geqslant 0$,

$$x(t) = y(t) + \int_0^t v(t-s) f(x(s), x(s-r))\,\mathrm{d}s, \tag{11}$$

where y is the solution of (7) with the initial condition $y(t) = \varphi(t)$ for $-r \leqslant t \leqslant 0$. Let $M > 0$ be a constant satisfying (8). By assumption (B^+), there exists a constant C such that

$$C > M \qquad \text{and} \qquad g(C, C) \leqslant 0. \tag{12}$$

Clearly, for $-r \leqslant t \leqslant 0$,

$$x(t) = \varphi(t) = y(t) < Mu(t) = M < C.$$

We will show that $x(t) < C$ for all $t > 0$. Otherwise, there exists $t_1 > 0$ such that $x(t) < C$ for $-r \leqslant t < t_1$ and $x(t_1) = C$. By virtue of (8)–(12), this leads to

$$\begin{aligned} C = x(t_1) = y(t_1) &+ \int_0^{t_1} v(t_1 - s) f(x(s), x(s-r)) \,\mathrm{d}s \\ &< Mu(t_1) + \int_0^{t_1} v(t_1 - s) f(C, C) \,\mathrm{d}s \\ &< Cu(t_1) + \int_0^{t_1} v(t_1 - s)(g(C, C) + KC + LC) \,\mathrm{d}s \\ &\leqslant C\left[u(t_1) + (K + L) \int_0^{t_1} v(t_1 - s) \,\mathrm{d}s\right] = C, \end{aligned}$$

a contradiction.

By a similar argument, one can prove

Lemma 2. *Suppose conditions* (2), (5) *and* (B^-) *hold. Then every solution of* (1) *is bounded from below.*

N e c e s s i t y. Suppose by way of contradiction that every solution of (1) converges to an equilibrium and one of the conditions (B^+) and (B^-) fails. Then either

(C^+) $\quad S < \infty$ and $g(x, x) > 0$ for $x > S$,

or

(C^-) $\quad I > -\infty$ and $g(x, x) < 0$ for $x < I$.

Consider case (C^+). Choose a constant $k > S$ and consider the solution x of (1) with the initial condition $x(t) = k$ for $-r \leqslant t \leqslant 0$. By the variation-of-constants formula, for all $t \geqslant 0$, x satisfies (11), where y is the solution of the linear equation (7) with the initial values $y(t) = k$ for $-r \leqslant t \leqslant 0$. In view of the linearity of (7), $y(t) = ku(t)$ for all $t \geqslant -r$. Hence

$$x(t) = ku(t) + \int_0^t v(t - s) f(x(s), x(s-r)) \,\mathrm{d}s \qquad \text{for } t \geqslant 0.$$

Let $k_1 \in (S, k)$ be fixed. Evidently, for $-r \leqslant t \leqslant 0$, $x(t) = k > k_1$, and a similar argument as in the proof of Lemma 1 shows that $x(t) > k_1$ for all $t > 0$. Hence $\liminf_{t\to\infty} x(t) \geqslant k_1 > S$. Since S is the largest equilibrium of (1), this shows that x cannot converge to any equilibrium, which is a contradiction.

In case (C^-), we can obtain a contradiction in a similar way.

Remark. Theorem 1 is closely related to the results of Smith and Thieme [12] (see also [11], Chap. 6). In the usual phase space $C = C([-r, 0], \mathbb{R})$ for Eq. (1), they have introduced a special ordering (the so-called exponential ordering) and have shown that under the smallness condition (5) the semiflow generated by the solutions of (1) is strongly order preserving. The theory of monotone dynamical systems [11] implies that under some additional compactness assumption "most of the solutions" converge to equilibria. More precisely, there exists an open and dense set of initial functions in C corresponding to solutions of (1) which converge to an equilibrium. The conclusion of Theorem 1 is stronger, since we have obtained the convergence of all solutions.

Note that the monotonicity condition of Smith and Thieme [12] can be guaranteed under weaker conditions than (5) (see [11], Chap. 6, Remark 2.2). However, Krisztin et al. [8] have shown that in this case Eq. (1) may have a nonconstant periodic solution. In this sense the stronger condition (5) is necessary for the validity of Theorem 1.

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