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TO PROVE THE EVIDENT: ON THE INFERENTIAL ROLE OF EUCLIDEAN DIAGRAMS

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Abstract

Diagrams have been rightly acknowledged to license inferences in Euclid's geometric practice. However, if on one hand purely visual proofs are to be found nowhere in the Elements, on the other, fully fledged proofs of diagrammatically evident statements are offered, as in El. I. 20: "In any triangle the sum of two sides is greater than the third." In this paper I will explain, taking as a starting point Kenneth Manders' analysis of Euclidean diagram, how exact and co-exact claims enter proposition I. 20. Then, I will ultimately argue that this proposition serves broader explanatory purposes, enhancing control on diagram appearance.

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Introduction

There is a broad consensus on the fact that visual inspection of the diagram is a primary form of ancient mathematical thought and reasoning.[1]

Euclid's plane geometry,[2] for instance, offers several examples of geometric reasoning contexts in which diagrams have the standing to license inferences, substituting to the written text.

Moreover the use of diagrams in Euclid's geometrical reasoning is rightly claimed to be necessary for the deployment of the demonstration, because without them not only our knowledge of spatial relations would be extremely poor, but many proofs would turn into *non sequitur.*[3]

Things are not so straightforward, though. Even if perceptual cues have a primary role in Euclid's plane geometry, and texts of mathematical works are almost always accompanied by figures (not only in the geomet-

[1] Historians agree on the role of "seeing in the diagram" for the development of pre-Euclidean mathematics. As Thomas Heath remarks: "Many propositions were doubtless first discovered by drawing all sorts of figures and lines in them, and observing apparent relations of equality [...] between parts". Thomas HEATH, *A History of Greek mathematics*. Vol. 1. Oxford: Clarendon Press 1921, p. 136. On the other side, indirect sources indicate that the first mathematical discoveries attributable to 6–5th century B.C. mathematicians dealt with properties easily recognizable in diagrams, by reasoning, for instance, on symmetries. See Árpád SZABO, *The beginning of Greek mathematics*. Translated by A. M. Ungar. Dordrecht – Boston – London: Reidel 1978).

[2] I agree with Marco Panza on his consideration of the notion of *Euclid's plane geometry*: "With 'Euclid's plane geometry' I mean plane geometry as it is expounded by Euclid, especially in the Elements (but the Data are also relevant for understanding some crucial feature or this geometry), and was largely practiced up to early modern age. This should be confounded neither with plane Euclidean geometry in general, nor with elementary synthetic plane geometry. [...] The text of the Elements I refer to is that established by Heiberg (Euclid HM). This leaves open the possibility of confirming or refuting some of my statements based on philological evidences that Heiberg's edition does not reflect." Marco PANZA, "The Twofold Role of Diagrams in Euclid's Plane Geometry." Forthcoming, 2007. In the following text I will keep using that notion with the same sense.

[3] See for some interesting examples the discussion of Reviel Netz in Karine CHEMLA, (ed.), *History of Science, History of the Text*. Dordtrecht: Springer 2005.

ric and arithmetic books of the *Elements*), exclusively visual proofs are to be found nowhere in the *Elements*.[4] On the contrary, geometric arguments are always composed by a verbal part, which we may call discursive text, which interacts with the diagram via cross reference (letters or groups of letters to indicate points, lines, ...), even when the claims to be proved are considered diagrammatically evident.

It is the case of the Euclid's proposition I. 20: "In any triangle the sum of two sides is greater than the third", which seems to have aroused, in antiquity, dispute around the necessity of the fully fledged proof that we meet in the *Elements*.[5]

Indeed, it is hard to see why Euclid troubled himself with such a proof if in other loci of the *Elements* he would also allow himself "to draw conclusion from diagram in an intuitive and unconstrained way".[6]

[4] In this, I guess that pre-Euclidean tradition must have been rather different. The discussion between Socrates and the slave reported in Plato's *Meno* stands as evidence of a geometric practice in which "verbal argument is only an accompaniment to diagrammatic manipulation and the diagram is both the source of conviction and the court of last resort in deciding the truth or falsity of a geometric assertion," (Ian MUELLER, "Euclids Elements and the Axiomatic Method." *The British Journal for the Philosophy of Science*, vol. 20, 1969, p. 292) However, according to Mueller, Meno's discussion is not really in conceptual contrast to Euclidean reasoning style (as argued, for instance, by Szabò): "Euclid's formalism is much more like formalism in literature, which focuses on stylistic niceties, than like formalism in mathematics which is motivated by a philosophical conception of mathematics," (*ibid*., p. 161).

[5] So, for instance, Proclus claims that "[T]he Epicureans are wont to ridicule this theorem, saying it is evident even to an ass and needs no proof [...] that the present theorem is known to an ass they make out from the observation that, if straw is placed at one extremity of the sides, an ass in quest of provender will make his way along the one side, but not by way of the two others." PROCLUS, *A Commentary on the First Book of Euclid's Elements*. Translated with Introduction and Notes by G. R. Morrow. Princeton: Princeton University Press 1970, § 322, p. 251.

[6] See John MUMMA, *Proofs, Pictures, and Euclid* [online]. 2007. Available at: http://www.andrew.cmu.edu/user/jmumma/Euclid-pictures.pdf [quoted 24/09/2009]. On the same vein, O. Veblen recognized that evident results were nonetheless proved in the *Elements*, and Kenneth Manders agrees that "the somewhat maligned requirement that this diagrammatically evident fact be proved

I will propose in my paper a twofold explanation. At first, in Euclidean plane geometry diagram-based inferences are subject to a set of constraints that relieve from the strictly controlled use of diagrams in the geometric practice. Because of these constraints, visual inspection of the diagram, despite its persuasiveness, cannot alone stand as a proof of the claim in I 20.

My second point is that proofs enter Euclidean practice to serve broader explanatory purposes than justificatory ones. Proof of I. 20, for instance, besides justifying the attribution of certain properties to given geometric entities, plays a decisive role within a strategy for enhancing control on diagram appearance.

Exact and co-exact diagrammatic conditions

I will now give a brief description of the set of constraints in action within Euclid's plane geometry.

As recent works have shown[7] in Euclid's plane geometry claims can be read directly in the diagram when they are based on those diagrammatic conditions insensitive to the effects of a range of continuous variations in diagram entries: the so-called *co-exact diagrammatic conditions*. Co-exact claims regard, for instance, part-whole relations of regions, segments bounding regions, lower dimensional counterparts, and intersection of curves.

On the contrary, claims are always assumed or established explicitly in the text when they are based on those diagrammatic conditions which

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no doubt reflects a relatively advanced theoretical foundational development in Greek geometry, but it is that tradition so developed that we are to characterize" (Kenneth MANDERS, "The Euclidean Diagram." In: MANCOSU, P. (ed.), *The Philosophy of Mathematical Practice*, New York: Oxford University Press 2008, p. 91).

7 MANDERS, "The Euclidean Diagram". See also: Kenneth MANDERS, "Diagram-based Geometric Practice." In: MANCOSU, P. (ed.), *The Philosophy of Mathematical Practice*, pp. 65–79; MUMMA, *Proofs, Pictures, and Euclid*; PANZA, "The Twofold Role of Diagrams in Euclid's Plane Geometry."

fail upon the slightest variation in the appearance of a diagram. These last conditions are called *exact diagrammatic conditions*, and exact claims include equality and inequalities between lines and angles (except coincidence), parallelism, rightness of angles, and proportionalities[8].

I basically share Kenneth Manders' position according to which these strictures on inferential standards are directly related to the physical and cognitive capacities that participants deploy in attaining uniformity for controlling production and reading of diagrams in a shared way.

A little practice with diagramming can show how agreement on their appearance is easily attained in responding to several diagrammatic conditions: I suppose that any Euclidean geometer, as agent endowed with normal cognitive capacities, has enough cognitive skills to agree on many co-exact conditions indicated in a diagram, and have enough physical skills to produce diagrams that meet many co-exact conditions.

On the contrary, the appearance of a diagram is extremely sensitive to exact conditions: it is humanly impossible to trace segments equal one another within the least degree of accuracy, perfectly straight or parallel lines, circles or right angles, so that disagreement on judging those features in a diagram is common and expected.

Since written language can be employed successfully to express exact properties, Euclidean geometers can invoke textual resources to supply weakness in appearance control: in the *Elements* the construction of *straight* lines and circles is postulated, lines *are said* to be parallel or perpendicular in a given geometric context, equality and inequality between distant elements in the diagram are usually obtained via prior entries in the text.[9]

[8] See MANDERS, "Diagram-based Geometric Practice", p. 67.
[9] See MANDERS, "The Euclidean Diagram", p. 125.

Further problems with appearance control

However, in Euclid's plane geometry control of diagram appearance does not limit itself to charge the text with responsibility for exact claims.

Diagrams are compositional objects,[10] and it is common that spurious sub-diagrams just "pop up" at any step of a geometric construction. It is only to trace a line inside a triangle and two new triangles appear without their being recorded in the text, or to cut two circles and their intersection point "happens" in the diagram.

It has been rightly remarked[11] that auxiliary constructions, being an "essential" part of geometric proof, always introduce new individuals with respect to the beginning and the end of a geometric argument.

However, to admit that something "just happens" when we draw a straight line or cross two circles is to admit a break in control "along a broad front".[12] In fact, as constructions pile up heuristic uncertainty about their outcome will probably grow, leading to the possibility of multiple and dissimilar results under the development of a given proof.[13]

We look then for alternative measures that might have been available within Euclid's plane geometry in order to overcome what might have been felt as an "impotence" or failure in control.

[10] See PANZA, "The Twofold Role of Diagrams in Euclid's Plane Geometry", p. 21.
[11] See Jan VON PLATO – Petri MAENPAA, "The Logic of Euclidean Construction Procedures." In: HAAPARTNA, L. – KUSCH, M. – NINILUOTO, I., "Language Knowledge and Intentionality. Perspectives on the Philosophy of Jaakko Hintikka." *Acta Philosophica Fennica*, vol. 49, 1990, pp. 274–293.
[12] MANDERS, "The Euclidean Diagram", p. 130.
[13] Which must have been a felt as threat in ancient geometrical practice, if I interpret well Plato's words in Cratylus: "Geometric diagrams [...] often have a slight and invisible flaw in the first part of the process, and are consistently mistaken in the long deductions that follow." Plato, 436D, Jowett tr., quoted in MANDERS, "The Euclidean Diagram", p. 88.

To provide critical scrutiny for variant diagrams, or, as it may be said, to "probe"[14] them seems to be one of the most promising means for challenging control within ancient geometric practice.

In the following I will try to elaborate, according with the guidelines given by K. Manders, a local strategy for probing diagram alternatives in order to show how control on spurious sub-diagrams[15] can be attained.

I called the strategy *local* because it applies only to the emergence of spurious triangles under a geometric construction, but it is not of small importance: a little practice with diagram-drawing shows that they emerge "easily" from given configurations of lines,[16] so I guess that challenging this situation would represent, alone, a great enhancement in control.

An example of control on diagram appearance based on testing diagram alternatives (probing)

It is a fact of intuition that if we take a triangle and if we stretch its base preserving the length of the other sides, at some point the triangle disappears or "splits up". This may suggest that the appearance of a triangle in a configuration of lines is tied with metric conditions concerning the length of its sides, but still we have to clarify how these metric conditions govern the production of such a bounded region.

I may start by moving two segments around the endpoints of a given one,[17] to record circumstances in which a triangle appears, and subsequently try to get them to fail.

[14] MANDERS, "The Euclidean Diagram", p. 131.
[15] I distinguish "construction" from "production", taking the second term to refer to "non regulated" construction of a figure, as in the cases exemplified above.
[16] See EUCLID, *The Thirteen Books of the Elements*. Vol I. Translated and edited by Sir Thomas Heath. New York: Dover Publications 1956. Book I., Propositions I.: 1, 5, 6, 7, 9, 10, 16,18,19, 20, 21, 33, 34, 35, 40, 41, 42, 43, 44, 45, 47.
[17] See MANDERS, "The Euclidean Diagram", p. 131.

We call the given segment AB and the moving ones AM and BN. We notice that if we decide to circle the moving segments, we will soon arrive at three situations distinct from a co-exact point of view: in one case the two circling segments intersect the given initial segment in one recognizable point, otherwise they intersect it in two points M and N, such that either point M is between A and N, or point N is between A and M.

If we consider the traced circles instead of the circling segments, we notice that three cases can occur: circle with ray AM either touches externally circle with ray BN, or circle with ray AM is secant to circle with ray BN, or finally, circle AM does not intersect circle with ray BN.[18]

Now, when the circles cross each other a triangle (or, more exactly, a couple of them) can be constructed by joining the intersection points with the extremities A and B. In the other two cases, no possibility to connect the endpoints is given. We have thus come up with the following result: *if circles with rays AM and BN have an intersection point (so that point N is between A and M), and their centers A and B are joined to it by "straight" lines, a triangle is produced.*

The clear fact that this geometric context does not exhaust all the imaginable contexts in which a triangle can "pop up" shows that it is a sufficient, but not a necessary condition for producing a triangle.

Secondly, we notice that the first condition, concerning the pointwise circle-circle intersection, is indeed a co-exact diagrammatic property, and that the second one is a diagram entry not controlled propositionally, that is the reason of my brackets on the word "straight" (in other terms, lines need not be drawn perfectly straight or equal to one another to produce a clearly readable diagram).

We have enough elements, now, to try and ask the question about the metric relationships that hold between the three sides of the triangle thus produced.

[18] I omit here all the problems concerning the existence of the intersection point, to which neither ancient readers were not insensitive.

Inspection of the diagram shows that when circles do cross each other in exactly one point not on AB (taking just the region of the plane upside AB), then they cross the given segment AB on two points, M and N, such that N is between A and M.

In Euclidean plane geometry, judgments of equality and inequality among segments can be made directly on the diagram when a segment is a proper part of another or two segments are coincident (co-exact conditions). Both co-exact situations are obtained in our diagram, so that we can conclude that AN, NM and MB taken together are equal to AB. From this, we can infer via a simple reasoning that AM and BN taken together exceed AB.[19] Thus, we can get the following co-exact result: *(a) if circles with rays AM and BN do cross each other in exactly one point (and thus a triangle can be produced), AM and BN taken together are greater than the segment AB*.

On the contrary, the following statement, which can be considered the converse of the preceding one: *(b) given a segment AB and two circles with rays AM and BN, if the sum of AM and BN exceeds AB, then the circles with rays AM and BN do intersect in exactly one point* – is no more based on co-exact diagrammatic conditions. In fact we can get it to fail imagining that circle BN has been drawn so as to cross circle AM in more than one point. That circle BN is in itself co-exactly equivalent to another circle B'N' which respects the pointwise circle-circle condition, but in the overall diagrammatic context it gives rise to an utterly different geometric situation, given the same disposition of the points on AB. In order to trace circles that always intersect in one point only, it is required that rays AM and BN hold the same length when circling around, and this last condition is an exact one.

[19] This reasoning does not presuppose proper geometric knowledge. It seems to me that it has rather the same status of that kind of the "common knowledge" listed by Euclid under the banner of "common notions".

According to Euclidean standards, we need a verbal argument in order to prove this claim. This is done in the first book of the *Elements*, where with slight variations in the terminology, proposition I. 20 subsumes the preceding claim *(b)*. The latter is indeed explicitly proved in the Euclidean text, as a necessary condition for the regulated construction of a triangle given its three sides.[20]

We could expect that if probing had ever been an historical practice, Euclid's proof of claim *(b)* should in principle recover our preceding steps. However, the peculiar procedure adopted by Euclid is utterly different.[21]

But this fact may be motivated by meta-theoretical reasons. Reasons of deductive orderings, for instance, could have forced the author to place our proposition at a particular point in the structure of the *Elements*, limiting the available resources to those claims already proved in the text.

What's more, as an historical practice probing can only conjecturally be ascribed to Euclid's plane geometry, where we can find, at most, responses to it. So, for instance, the diorism of I. 22 can also be seen (with Proclus) as having an objection-refuting role,[22] thus excluding purportedly all the variant diagrams examined before.

[20] EUCLID, *The Thirteen Books of the Elements*, I. 22.

[21] "For let ABC be a triangle; I say that in the triangle ABC two sides taken together in any manner are greater than the remaining one, namely

BA, AC greater than BC,
AB, BC greater than AC,
BC, CA greater than AB.

For let BA be drawn through the point D, let DA be made equal to CA, and let DC be joined. Then, since DA is equal to AC, the angle ADC is also equal to the angle ACD; [I. 5] therefore the angle BCD is greater than the angle ADC. [C.N. 5] And, since DCB is a triangle having the angle BCD greater than the angle BDC, and the greater angle is subtended by the greater side, [I. 19] therefore DB is greater than BC. Similarly we can prove that AB, BC are also greater than CA, and BC, CA than AB. Therefore etc. Q. E. D."

[22] See PROCLUS, *A Commentary on the First Book of Euclid's Elements*, § 330.

Conclusions

I will return, briefly, to my initial question: how can we explain that Euclid gives proofs even when we see clearly what's "going on" in the diagram?

Our preceding discussion shows that, even if we admitted that the non propositional acquaintance with the diagram generates a reliable belief state about the assertion stated in I. 20, it does not automatically license a proof of the claim. As it was shown, if we want to prove the claim in I. 20, we must invoke a verbal argument, because the claim rests on exact diagrammatic property. And these constraints on diagrammatic properties are, as far as we know, in force not only within proposition I. 20, but they represent a standard of practice within all geometric books of the *Elements*.

To return to the second part of the question, once our claim is demonstrated, any participant to Euclid's plane geometry by accepting a number of discursive results contained in propositions I. 1–19 is endowed with adequate resources to predict whether a given configuration of lines will realize the necessary (exact) conditions in order to produce a triangle.

Since the practice of testing diagram alternatives gives the sufficient (co-exact) conditions too, it is reasonable to suppose that in the impossibility of stating all construction postulates needed for Euclidean plane geometry,[23] a significant enhancement on controlling the emergence of a certain class of spurious sub-diagrams has been nevertheless acquired.

[23] This is, according to von Plato and Maenpaa, the very "problem of synthetic geometry": "Euclid [...] takes for granted that the two circles have an intersection point there is no easy remedy to this situation. In fact, Euclid's failure to state all the construction postulates needed for Euclidean geometry was great serendipity. For no one has been able to state them, in the more than two thousand years after Euclid." PLATO J. von, MAENPAA P., "The Logic of Euclidean Construction Procedures", p. 288.

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