NOTE ON A LOVÁSZ'S RESULT

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Abstract. In this paper, we give a generalization of a result of Lovász from [2].

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The terminology and notation used in this paper are those of [1]. So, let $\mathbf{H} = (X, \mathcal{E})$ be a hypergraph with X the set of vertices and $\mathcal{E} = \{E_i\}_{i\in I}$ the set of edges.

Theorem 1. *If $\mathbf{H} = (X, \mathcal{E})$ is a hypergraph without cycles of length greater than two then there exists a vertex belonging to a single edge, or there exist two edges E_i and E_j such that $E_i \subset E_j$.*

P r o o f. Suppose that no edge is contained in another one and that every vertex belongs to at least two edges. Let

$$(x_1, E_{i1}, x_2, E_{i2}, \ldots, x_p, E_{ip}, x_{p+1})$$

be a chain of maximum length. We may suppose that $x_1 \in E_{i1} - E_{i2}$, since otherwise x_1 could be replaced by a vertex x such that $x \in E_{i1} - E_{i2}$ (such a vertex x exists and $x \neq x_k$, $k = 2, 3$, since x_2, $x_3 \in E_{i2}$ and $x \neq x_k$, $4 \leqslant k \leqslant p+1$, since, by hypothesis, $\mathbf{H}$ does not contain cycles of length greater than or equal to three). Obviously, there exists an edge E_i such that $i \neq i_1$ and $x_1 \in E_i$. Since $x_1 \notin E_{i2}$ we have $i \neq i_2$. Moreover, if $i = i_k$, $3 \leqslant k \leqslant p$, then there exists a cycle

$$(x_1, E_{i1}, x_2, \ldots, x_k, E_{ik}, x_1)$$

of length greater than or equal to three, a contradiction. Thus, since the chain $(x_1, x_2, \ldots, x_{p+1})$ is maximal, we have $E_i \subset \{x_1, x_2, \ldots, x_{p+1}\}$ and, since $i \neq i_1$, we have $E_i - E_{i1} \neq \emptyset$. Let k be the smallest index for which $x_k \in E_i - E_{i1}$. Obviously, since $x_k \notin E_{i1}$, we have $k \neq 1, 2$. On the other hand, $k < 3$, since otherwise there exists a cycle

$$(x_1, E_{i1}, x_2, \ldots, x_k, E_i, x_1)$$

of length greater than or equal to three, a contradiction. The theorem is proved. □

Theorem 2. *If $\mathbf{H} = (X, \mathcal{E})$ is a hypergraph without cycles of length greater than two and with p connected components such that $|E_i \cap E_j| \leqslant q$ for every $E_i \neq E_j$, then*

$$\sum_{i \in I} \big(|E_i| - q\big) \leqslant |X| - pq. \tag{1}$$

Proof. We shall prove this theorem by induction. Obviously, the theorem is true for $\sum\limits_{i \in I} |E_i| = 1$. So, suppose that it is true for hypergraphs $\mathbf{H}^*$ for which $\sum\limits_{i \in I^*} |E_i^*| < \sum\limits_{i \in I} |E_i|$.

Obviously, by Theorem 1, only two situations are possible.

(a) There exists a vertex x_1 which belongs to a single edge, say E_1. By induction hypothesis, the theorem is true for the subhypergraph $\mathbf{H}^*$ induced by $X^* = X - \{x_1\}$. Thus, we have

$$\sum_{i \in I^*} \big(|E_i^*| - q\big) \leqslant |X^*| - p^* q.$$

If $E_1 \neq \{x_1\}$, then $I^* = I$, $p^* = p$, $|E_1^*| = |E_1| - 1$ and (1) is verified.

If $E_1 = \{x_1\}$, then $I^* = I - \{1\}$, $p^* = p - 1$ and (1) is also verified.

(b) There is no vertex belonging to single edge, but there exist two edges E_{i0} and E_{j0} such that $E_{j0} \subset E_{i0}$. Since, by induction hypothesis, the theorem is true for the partial hypergraph $\mathbf{H}^* = \big(X, \mathcal{E} - \{E_{j0}\}\big)$, it follows that

$$\sum_{i \in I - \{j0\}} \big(|E_i| - q\big) \leqslant |X| - pq$$

(obviously, $p^* = p$). Moreover,

$$|E_{j0}| - q = |E_{i0} \cap E_{j0}| - q \leqslant 0$$

and (1) is verified. The theorem is proved. □

Obviously, Theorem 2 for $q = 2$ yields

$$\sum_{i \in I} \big(|E_i| - 2\big) \leqslant |X| - 2p < |X| - p,$$

that is, the result of Lovász from [2].

References

[1] *C. Berge*: Graphes et Hypergraphes. Dunod, Paris, 1970.
[2] *L. Lovász*: Graphs and set-systems. Beitrage zur Graphentheorie (H. Sachs, H. S. Voss and H. Walther, eds.). Teubner, 1968, pp. 99–106.

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