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EXTENSION OF MEASURES: A CATEGORICAL APPROACH

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Dedicated to the memory of my advisor Professor Josef Novák (1905–1999)

Abstract. We present a categorical approach to the extension of probabilities, i.e. normed σ-additive measures. J. Novák showed that each bounded σ-additive measure on a ring of sets $\mathbb{A}$ is sequentially continuous and pointed out the topological aspects of the extension of such measures on $\mathbb{A}$ over the generated σ-ring $\sigma(\mathbb{A})$: it is of a similar nature as the extension of bounded continuous functions on a completely regular topological space X over its Čech-Stone compactification βX (or as the extension of continuous functions on X over its Hewitt realcompactification υX). He developed a theory of sequential envelopes and (exploiting the Measure Extension Theorem) he proved that $\sigma(\mathbb{A})$ is the sequential envelope of $\mathbb{A}$ with respect to the probabilities. However, the sequential continuity does not capture other properties (e.g. additivity) of probability measures. We show that in the category ID of D-posets of fuzzy sets (such D-posets generalize both fields of sets and bold algebras) probabilities are morphisms and the extension of probabilities on $\mathbb{A}$ over $\sigma(\mathbb{A})$ is a completely categorical construction (an epireflection). We mention applications to the foundations of probability and formulate some open problems.

Keywords: extension of measure, categorical methods, sequential continuity, sequential envelope, field of subsets, D-poset of fuzzy sets, effect algebra, epireflection

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1. Introduction

Having in mind categorical aspects of the extension of probability measures, in Section 1 we discuss the need to enlarge the category of classical fields of sets to a suitable category of fuzzy sets. In Section 2 we analyze Novák's construction and describe our goal in categorical terms. Section 3 is devoted to the extension of

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measures in the category ID. In Section 4 we mention applications to the foundations of probability and formulate some open problems.

The basic notions of probability are events, random variables (dually observables), and probabilities. Classical events can be modelled by fields of sets and generalized events by various algebraic structures: logics, MV-algebras, effect algebras, D-posets, etc. (cf. [27], [4], [28], [12], [17], [2], [3]). An observable (as the preimage map induced by a random variable) is a map of one field of events into another one and it preserves the operations on events. From the categorical viewpoint, fields of events can be considered as objects and observables as morphisms. The problem is with probability measures!

O b s e r v a t i o n 1.1. Let X be a set, let $\mathbb{A}$ be a field of subsets of X, and let p be a probability on $\mathbb{A}$. The domain of p carries the structure of a Boolean algebra and its range is the unit interval $I = [0, 1]$. We would like to treat p as a morphism and hence we have to *enlarge* the category of fields of sets so that I (carrying a suitable structure) becomes an object. Further, if p is not a $\{0, 1\}$-valued measure, then "p preserves the algebraic operations only partially", e.g., $p(A \cup B) = p(A) + p(B)$ is guaranteed only when $A \cap B = \emptyset$.

O b s e r v a t i o n 1.2. The sequential envelope of Novák, likewise the Čech-Stone compactification or the Hewitt realcompactification can be constructed via categorical products (powers of I or $\mathbb{R}$) and then the continuous extension of functions under question follows from the properties of products and the projections onto factors. Hence we need to equip the factors with a suitable *continuity structure* (sequential convergence in case of Novák).

This leads to an "evaluation" category (see Section 2) in which all spaces and maps involved in the construction of the extension become objects and morphisms, respectively. In our case it is the category ID of D-posets of fuzzy sets co-generated by the unit interval I carrying the usual difference (subtraction) and the usual convergence of sequences (cf. [9], [25]).

Recall (cf. [20], [4]) that a D-*poset* is a quintuple $(E, \leqslant, \ominus, 0_E, 1_E)$ where E is a set, $\leqslant$ is a partial order, 0_E is the least element, 1_E is the greatest element, $\ominus$ is partial operation on E such that $a \ominus b$ is defined iff $b \leqslant a$, and the following axioms are assumed:

(D1) $a \ominus 0_E = a$ for each $a \in E$;

(D2) If $c \leqslant b \leqslant a$, then $a \ominus b \leqslant a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

If no confusion can arise, then the quintuple $(E, \leqslant, \ominus, 0_E, 1_E)$ is condensed to E. A map h of a D-poset E into a D-poset F which preserves the D-structure is said to be a D-*homomorphism*.

It is known that D-posets are equivalent to effect algebras introduced in [5]. Interesting results about effect algebras, D-posets, and other quantum structures can be found in [4].

Unless stated otherwise, I will denote the closed unit interval carrying the usual linear order and the usual D-structure: $a \ominus b$ is defined whenever $b \leqslant a$ and then $a \ominus b = a - b$. Analogously, if X is a set and I^X is the set of all functions on X into I, then we consider I^X as a D-poset in which the partial order and the partial operation $\ominus$ are defined pointwise: $b \leqslant a$ iff $b(x) \leqslant a(x)$ for all $x \in X$ and $a \ominus b$ is defined by $(a \ominus b)(x) = a(x) - b(x)$, $x \in X$. A subset $\mathcal{X} \subseteq I^X$ containing the constant functions 0_X, 1_X and closed with respect to the inherited partial operation "$\ominus$" is a typical D-poset we are interested in; we shall call it a D-*poset of fuzzy sets*.

Clearly, if we identify $A \subseteq X$ and the corresponding characteristic function $\chi_A \in I^X$, then each field $\mathbb{A}$ of subsets of X can be considered as a D-poset $\mathbb{A} \subseteq I^X$ of fuzzy sets: $\mathbb{A}$ is partially ordered ($\chi_B \leqslant \chi_A$ iff $B \subseteq A$) and then $\chi_A \ominus \chi_B$ is defined as $\chi_{A \setminus B}$ provided $B \subseteq A$.

Further, assume that I carries the usual sequential convergence and that I^X and other D-posets of fuzzy sets carry the pointwise sequential convergence. In what follows, we identify I and $I^{\{x\}}$, where $\{x\}$ is a singleton. Let $\mathbb{A}$ be a field of subsets of X considered as a D-poset of fuzzy sets and let p be a probability measure on $\mathbb{A}$. Lemma 2 in [22] states that p as a map of $\mathbb{A} \subseteq I^X$ into I is *sequentially continuous*. For more information concerning the σ-additivity and the sequential continuity of measures see [10].

The category ID consists of the reduced D-posets of fuzzy sets carrying the pointwise convergence as objects and the sequentially continuous D-homomorphisms as morphisms. Note that the assumption that all objects of ID are reduced (each two points a, b of the underlying set X are separated by some fuzzy set $u \in \mathcal{X} \subseteq I^X$, i.e. $u(a) \neq u(b)$) plays the same role as the Hausdorff separation axiom T_2: limits are unique and the continuous extensions from dense subobjects are uniquely determined.

Additional information about category theory, generalized measure, sequential envelopes and their generalizations can be found, e.g., in [1], [8], [11], [18], [6], [15], [16], [19], [21], [13], [14], respectively.

2. The evaluation category

Let us start with the Novák's construction of the sequential envelope of a field of sets with respect to all probability measures (cf. [24]).

Construction 2.1. Let X be a set, let $\mathbb{A}$ be a field of subsets of X, and let $\sigma(\mathbb{A})$ be the generated σ-field. If we consider $\mathbb{A}$ as a subset of I^X, then $\sigma(\mathbb{A})$ can be considered as the smallest subset of I^X containing $\mathbb{A}$ and sequentially closed with respect to the pointwise convergence of sequences (cf. [23], [24]). Let P be the set of all probabilities on $\mathbb{A}$. Since each $x \in X$ can be considered as the degenerated one-point probability $p_x \in P$, we shall consider X as a subset of P. The evaluation of $\mathbb{A}$ is a map ev_P of $\mathbb{A}$ into I^P defined as follows: for $A \in \mathbb{A}$ put $\mathrm{ev}_P(A)(p) = p(A)$, $p \in P$. Denote $\mathrm{ev}_P(\mathbb{A}) = \{\mathrm{ev}_P(A);\ A \in \mathbb{A}\} \subset I^P$ and denote $\mathcal{X}$ the smallest of all subsets $\mathcal{Y}$ of I^P such that: (1) $\mathrm{ev}_P(\mathbb{A}) \subseteq \mathcal{Y}$ and (2) $\mathcal{Y}$ is closed with respect to the pointwise convergence in I^P. Clearly, $\mathcal{X}$ is the intersection of all such $\mathcal{Y}$.

The set $\mathcal{X}$ carrying the inherited pointwise convergence is a sequential envelope of $\mathbb{A}$ with respect to P: roughly, $\mathcal{X}$ represents a maximal larger "object" over which each probability measure on $\mathbb{A}$ can be uniquely extended to a sequentially continuous map to I in a reasonable way. First, $\mathbb{A} \subseteq I^X$ and $\mathrm{ev}_P(\mathbb{A}) \subseteq I^P$ are "isomorphic" as objects of a generalized probability theory (see the last Section). Second, $\mathcal{X}$ is a "categorical" extension of $\mathrm{ev}_P(A)$ (probabilities are simultaneously extended via "powers and projections" and each probability has a unique extension). Third, $\mathcal{X}$ has some absolute properties and, "surprisingly", $\mathcal{X}$ and $\sigma(\mathbb{A})$ are "isomorphic". Consequently, $\sigma(\mathbb{A})$ has the same absolute properties as $\mathcal{X}$ does have.

In the category of sequential convergence spaces and sequentially continuous maps the transition from $\mathbb{A}$ to $\mathcal{X} \subseteq I^P$ via ev_P has the same nature as the transition from a completely regular space S to its Čech-Stone compactification $\beta S \subseteq I^{C(S,I)}$ via embedding S into the Tikhonov cube $I^{C(S,I)}$. In the former case we work with the pointwise sequential convergence (it is the categorical product convergence in I^P) and we extend functions in P. Note that P is "just a set" of morphisms of $\mathbb{A}$ into I and sequential convergence spaces like $\mathcal{X}$ form "just a class" of objects. In the latter case we work with Tikhonov topologies and we extend continuous functions $C(S,I)$ on S into I, i.e. all morphisms of S into I, and hence βS is an epireflection (completely regular spaces are reflected into the subcategory of compact spaces).

As proved in [24], the sequential envelope of $\mathbb{A}$ with respect to P exists and (up to a sequential homeomorphism pointwise fixed on $\mathbb{A}$) it is uniquely determined. Observe that $\mathcal{X} \subseteq I^P$ does not carry any natural Boolean structure. We claim that $\sigma(\mathbb{A})$ is the sequential envelope of $\mathbb{A}$, hence $\sigma(\mathbb{A}) \subseteq I^X$ and $\mathcal{X} \subseteq I^P$ have to be "equivalent". To prove the equivalence we have to ignore the algebraic structure of $\sigma(\mathbb{A})$ and to make use of the METHM (Measure Extension Theorem). Indeed, METHM implies that each probability on $\mathbb{A}$ (as a bounded sequentially continuous function on $\mathbb{A}$) can be extended to a probability on $\sigma(\mathbb{A})$. To show that $\sigma(\mathbb{A})$ is the maximal extension of the domain for all probabilities it suffices to verify that in $\sigma(\mathbb{A})$ there is no totally divergent P-Cauchy sequence (having a potential limit outside $\sigma(\mathbb{A})$; we say that $\{A_n\}$ is P-*Cauchy* if for each $p \in P$ the sequence $\{p(A_n)\}$

is a Cauchy sequence of real numbers). But this is trivial. Let $\{A_n\}$ be a P-Cauchy sequence in $\sigma(\mathbb{A})$. Then each sequence $\{p_x(A_n)\}$, $x \in X$, converges and hence $\{A_n\}$ converges (pointwise) in $\sigma(\mathbb{A})$. The uniqueness of the sequential envelopes means that $\sigma(\mathbb{A})$ and $\mathcal{X}$ are equivalent from the viewpoint of the extension of probabilities as sequentially continuous functions. Hence $\sigma(\mathbb{A})$ is the sequential envelope of $\mathbb{A}$ with respect to P.

Observe that probabilities on $\mathbb{A}$ and their extensions on $\sigma(\mathbb{A})$ can be distinguished from other sequentially continuous functions only via the algebraic (Boolean) structure of fields of sets—the additivity is not defined in terms of continuity.

Our strategy is to find a category (as simple as possible) in which $\mathbb{A}$ and $\mathrm{ev}_P(\mathbb{A})$ (resp. $\sigma(\mathbb{A})$ and $\mathcal{X}$) "live" as equivalent objects, probabilities are exactly the morphisms the extension of which we are interested in, and the construction of an epireflection via powers and projections can be carried out. We shall call it the *evaluation category* over fields of sets and probability measures.

Observation 2.2. Let $\mathbb{A}$ be a (separated) field of subsets of X. Let $P(\mathbb{A})$ be the set of all probability measures on $\mathbb{A}$; if no confusion can arise, then $P(\mathbb{A})$ will be condensed to P. Barred the trivial case, $\mathrm{ev}_P(\mathbb{A})$ fails to be a field of subsets of P. However, $\mathrm{ev}_P(\mathbb{A}) \subset I^P$ can be considered as an object of the category ID (observe that if $p \in P$, then for $A, B \in \mathbb{A}$, $B \subset A$, we have $p(A \setminus B) = p(A) - p(B)$ and hence $\mathrm{ev}_P(A \setminus B) = \mathrm{ev}_P(A) \ominus \mathrm{ev}_P(B)$) and each probability measure p on $\mathbb{A}$ can be considered as a morphism of $\mathrm{ev}_P(\mathbb{A})$ into I (first, the sequential continuity of p as a map of $\mathbb{A}$ into I follows from Lemma 2 in [22]; second, put $p(\mathrm{ev}_P(A)) = (\mathrm{ev}_P(A))(p) = p(A)$; third, it is easy to verify that p as a map of $\mathrm{ev}_P(\mathbb{A})$ into I preserves the D-poset structure; fourth, a sequence $\{A_n\}$ converges to A in $\mathbb{A}$ iff the corresponding sequence $\{\mathrm{ev}_P(A_n)\}$ converges to $\mathrm{ev}_P(A)$ in $\mathrm{ev}_P(\mathbb{A})$). Let $\mathbb{B}$ be a field of subsets of Y and let f be an $(\mathbb{A}, \mathbb{B})$-measurable map of Y into X. Then f induces the preimage map $f^{\leftarrow}$ sending $A \in \mathbb{A}$ into $f^{\leftarrow}(A) = \{y \in Y;\ f(y) \in A\}$. It is known (cf. [7]) that $f^{\leftarrow}$ is a sequentially continuous Boolean homomorphism, hence a D-homomorphism, of $\mathbb{A}$ into $\mathbb{B}$. Moreover, f induces a map f^* of the set $P(\mathbb{B})$ (of all probability measures on $\mathbb{B}$) into $P(\mathbb{A})$ defined by $(f^*(p))(A)) = p(f^{\leftarrow}(A))$, $A \in \mathbb{A}$, $p \in P(\mathbb{B})$. If we consider points of X and Y as point probability measures, then f is the restriction of f^* to $Y \subset P(\mathbb{B})$. In fact, f induces a sequentially continuous D-homomorphism $f^{\triangleleft}$ of $\mathrm{ev}_P(\mathbb{A})$ into $\mathrm{ev}_P(\mathbb{B})$ sending $\mathrm{ev}_P(A) \in \mathrm{ev}_P(\mathbb{A}) \subset I^{P(\mathbb{A})}$ into $\mathrm{ev}_P(f^{\leftarrow}(A)) \in \mathrm{ev}_P(\mathbb{B}) \subset I^{P(\mathbb{B})}$. Natural questions arise:

1. Can each sequentially continuous D-homomorphism (i.e. a morphism of ID) of $\mathrm{ev}_P(\mathbb{A})$ into I be considered as a probability measure on $\mathbb{A}$?

2. Is each sequentially continuous D-homomorphism (i.e. a morphism of ID) h of $\mathrm{ev}_P(\mathbb{A})$ into $\mathrm{ev}_P(\mathbb{B})$ of the form $f^{\triangleleft}$ for some measurable map f of Y into X?

The answers are positive (cf. Lemma 2.2 and Lemma 3.3 in [25]). This means that the category ID is a good candidate. Still, we have to prove that $\sigma(\mathbb{A})$ is the desired epireflection in ID.

3. Extending measure

Let X be a set and let $\mathbb{A}$ be a field of subsets of X. Denote $P(\mathbb{A})$ the set of all probability measures on $\mathbb{A}$. Let $\sigma(\mathbb{A})$ be the generated σ-field. In [24] J. Novák proved that $\sigma(\mathbb{A})$ is the sequential envelope of $\mathbb{A}$ with respect to $P(\mathbb{A})$. In this section we describe how this result is related to the Measure Extension Theorem (METHM).

Theorem 3.1 (METHM-classical)**.** *Let $\mathbb{A}$ be a field of sets, let $\sigma(\mathbb{A})$ be the generated σ-field, and let p be a probability measure on $\mathbb{A}$. Then there exists a unique probability measure $\bar{p}$ on $\sigma(\mathbb{A})$ such that $\bar{p}(A) = p(A)$ for all $A \in \mathbb{A}$.*

The proof (usually based on the outer measure) can be found in any treatise on measure. However, additional properties of $\sigma(\mathbb{A})$ are usually not mentioned there. J. Novák pointed out that from the "topological viewpoint" $\sigma(\mathbb{A})$ can be viewed as a maximal object over which all probability measures on $\mathbb{A}$ can be extended.

Let $\mathbb{A}$, $\mathbb{B}$ be fields of subsets of X and let $\mathbb{A} \subseteq \mathbb{B}$. Recall that a sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\mathbb{A}$ is said to be P-*Cauchy* if for each probability measure p on $\mathbb{A}$ the sequence $\{p(A_n)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers. If for each probability measure p on $\mathbb{A}$ there exists a probability measure $\bar{p}$ on $\mathbb{B}$ such that $\bar{p}(A) = p(A)$ for all $A \in \mathbb{A}$, then $\mathbb{A}$ is said to be P-*embedded* in $\mathbb{B}$.

Theorem 3.2. *The following are equivalent*

(i) $\mathbb{A} = \sigma(\mathbb{A})$;

(ii) *Each* P-*Cauchy sequence converges in* $\mathbb{A}$;

(iii) $\mathbb{A}$ *is sequentially closed in each field of subsets* $\mathbb{B}$ *in which* $\mathbb{A}$ *is* P-*embedded.*

Proof. (i) implies (ii). Assume (i) and let $\{A_n\}_{n=1}^{\infty}$ be a P-Cauchy sequence in $\mathbb{A}$. Since each $x \in X$ represents a point-probability, the sequence $\{A_n\}_{n=1}^{\infty}$ (pointwise) converges in $\{0,1\}^X$. From $\mathbb{A} = \sigma(\mathbb{A})$ it follows that $\mathbb{A}$ is sequentially closed and hence $\{A_n\}_{n=1}^{\infty}$ converges in $\mathbb{A}$.

(ii) implies (iii). Let $\mathbb{A}$ be P-embedded in $\mathbb{B}$ and let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{A}$ which converges in $\mathbb{B}$. Since each $\bar{p} \in P(\mathbb{B})$ is sequentially continuous, $\{A_n\}_{n=1}^{\infty}$ is P-Cauchy and hence converges in $\mathbb{A}$.

(iii) implies (i). From the classical METHM it follows that $\mathbb{A}$ is P-embedded in $\sigma(\mathbb{A})$. Thus (iii) implies that $\mathbb{A}$ is sequentially closed in $\sigma(\mathbb{A})$ and hence $\mathbb{A} = \sigma(\mathbb{A})$. This completes the proof. $\square$

Theorem 3.3 (METHM-Novák). *Let $\mathbb{A}$ be a field of sets and let $\sigma(\mathbb{A})$ be the generated σ-field. Then $\sigma(\mathbb{A})$ is a maximal field of subsets in which $\mathbb{A}$ is* P*-embedded and sequentially dense.*

Proof. The assertion follows from the preceding theorem. Let $\mathbb{A}$ be a field of subsets of X. Assume that $\mathbb{A}$ is P-embedded and sequentially dense in a field $\mathbb{B}$. Clearly, $\mathbb{A}$ is P-embedded and sequentially dense in $\sigma(\mathbb{B})$. Since the generated σ-field of a field of subsets of X is the smallest sequentially closed system in $\{0,1\}^X$ containing the field in question, necessarily $\sigma(\mathbb{B}) = \sigma(\mathbb{A})$. Thus $\sigma(\mathbb{A})$ is maximal. This completes the proof. □

Observe that σ-fields form a special class of fields of sets. Indeed, $\mathbb{A} = \sigma(\mathbb{A})$ means that $\mathbb{A}$ has the following absolute property with respect to the extension of probability measures (cf. [10]): $\mathbb{A}$ is sequentially closed in each field of subsets in which it is P-embedded (in this respect this absolute property is similar to the compactness).

Finally, we show that in the realm of the category ID the embedding of a field of sets into the generated σ-field and the extension of probability measures can be characterized in categorical terms. The characterization is based on three facts:

1. Fields of sets form a special subcategory FS of ID;
2. The epireflection for sober objects of ID (into the subcategory CSID of closed sober objects) constructed by M. Papčo in [25] can be extended to a subcategory containing FS (fields of sets are not sober);
3. The epireflection applied to a field of sets $\mathbb{A}$ yields the generated σ-field $\sigma(\mathbb{A})$.

Denote FS the full subcategory of ID consisting of objects the underlying sets of which are reduced fields of sets.

Lemma 3.4. *The category* FS *and the category of reduced fields of sets and continuous Boolean homomorphisms are isomorphic.*

Proof. Let $\mathbb{A}$ be a field of subsets of X. Identifying $A \in \mathbb{A}$ and the corresponding characteristic function χ_A, we can reorganize $\mathbb{A}$ into an ID-poset in a natural way: order and convergence are defined pointwise, χ_X is the top element, $\chi_\emptyset$ is the bottom element, for $A, B \in \mathbb{A}$, $B \subseteq A$, put $\chi_A \ominus \chi_B = \chi_{A \setminus B}$; further, if $\mathbb{A}$ and $\mathbb{B}$ are fields of sets and h is a Boolean homomorphism of $\mathbb{A}$ into $\mathbb{B}$, then h preserves the D-poset structure and it can be considered as a D-homomorphism. Conversely, if $\mathbb{A}$ and $\mathbb{B}$ are ID-posets of subsets closed under the usual (finite) field operations (union, intersection, $\ldots$), then both $\mathbb{A}$ and $\mathbb{B}$ can be reorganized into fields of subsets in a natural way and each D-homomorphism of $\mathbb{A}$ into $\mathbb{B}$ can be considered as a Boolean homomorphism. □

In the sequel, the category of fields of sets and sequentially continuous Boolean homomorphisms and the full subcategory FS of ID will be treated as identical.

In [25] two important subcategories of ID have been studied: SID consisting of sober objects and CSID consisting of closed sober objects. Recall that $\mathcal{X} \subseteq I^X$ is said to be *sober* if each morphism of $\mathcal{X}$ into I is fixed, i.e. for each $h \in \hom(\mathcal{X}, I)$ there exists a unique $x \in X$ such that h is the evaluation of $\mathcal{X}$ at x. Each $\mathrm{ev}_P(\mathbb{A})$ is sober and sobriety plays a key role in categorical constructions. Further, $\mathcal{X} \subseteq I^X$ is said to be *closed*, if $\mathcal{X}$ is sequentially closed in I^X with respect to the pointwise sequential convergence. Again, each $\sigma(\mathbb{A})$ is closed and closed objects in ID generalize σ-fields of sets.

Corollary 2.17 in [25] states that CSID is epireflective in SID. We define a larger subcategory of ID, containing both SID and FS, to which the epireflection can be generalized.

Let $\mathcal{X} \subseteq I^X$ be an object of ID. According to [25], there exists the minimal of all objects $\mathcal{Y}$ of ID such that $\mathcal{X} \subseteq \mathcal{Y} \subseteq I^X$ and $\mathcal{Y}$ is sequentially closed. Denote it $\sigma(\mathcal{X})$.

Definition 3.5. Let $\mathcal{X}$ be an object of ID. If for each morphism h of $\mathcal{X}$ into I there exists a morphism $\bar{h}$ of $\sigma(\mathcal{X})$ into I such that $\bar{h}(u) = h(u)$ for all $u \in \mathcal{X}$, then $\mathcal{X}$ is said to be *sufficient*.

Recall (cf. Lemma 2.7 in [25]), that if h and h' are two morphisms of $\sigma(\mathcal{X})$ into an object $\mathcal{Y}$ of ID such that $h(u) = h'(u)$ for all $u \in \mathcal{X}$, then $h = h'$. Consequently, $\bar{h}$ in the definition above is determined uniquely.

Denote STID the full subcategory of ID consisting of sufficient objects. Clearly, each closed object of ID is sufficient. We prove that both SID and FS are subcategories of STID.

Lemma 3.6. (i) *Each sober object of* ID *is sufficient.*
(ii) *Each object of* FS *is sufficient.*

P r o o f. (i) Let $\mathcal{X}$ be sober. According to Corollary 2.17 in [25], CSID is epireflective in SID and $\sigma(\mathcal{X})$ is the epireflection of $\mathcal{X}$. Hence each morphism h of $\mathcal{X}$ into I can be (uniquely) extended over $\sigma(\mathcal{X})$ and the assertion follows.

(ii) follows from the fact that sequentially continuous D-homomorphisms of a field of sets into I are exactly probability measures and hence can be (uniquely) extended over the generated σ-field. □

Theorem 3.7. CID *is an epireflective subcategory of* STID.

P r o o f. The proof is based on the categorical properties of a product. Let $\mathcal{X} \subseteq I^X$ be an object of STID. We claim that the embedding of $\mathcal{X}$ into $\sigma(\mathcal{X})$ is the desired epireflection (remember, $\sigma(\mathcal{X})$ is closed), i.e. each morphism of $\mathcal{X}$ into a closed object of ID can be extended to a unique morphism over $\sigma(\mathcal{X})$.

(i) Let h be a morphism of $\mathcal{X}$ into I. Since $\mathcal{X}$ is sufficient, it follows that h can be uniquely extended to a morphism $\overline{h}$ of $\sigma(\mathcal{X})$ into I.

(ii) Let $\mathcal{Y} \subseteq I^Y$ be an object of CID. Let h be a morphism of $\mathcal{X}$ into $\mathcal{Y}$. Since I^Y is the categorical product in ID, the composition of h (considered as a morphism of $\mathcal{X}$ into I^Y) and each projection of I^Y into a factor I is a morphism of $\mathcal{X}$ into I and, according to (i), it can be uniquely extended to a morphism of $\sigma(\mathcal{X})$ to I. From the definition of a product and from Lemma 2.7 in [25] it follows that there exists a unique morphism $\overline{h}$ of $\sigma(\mathcal{X})$ into $\sigma(\mathcal{Y}) = \mathcal{Y}$ such that $\overline{h}(u) = h(u)$ for all $u \in \mathcal{X}$. This completes the proof. □

Theorem 3.8 (METHM-categorical)**.** *Let $\mathbb{A}$ be a field of sets and let $\sigma(\mathbb{A})$ be the generated σ-field. Then $\sigma(\mathbb{A})$ is the epireflection of $\mathbb{A}$ as an object of* STID *into* CID.

4. Concluding remarks

Details about fuzzy probability theory can be found, e.g., in [17], [2], [3], [12], [14], [26].

Let $(\Omega, \mathbb{A}, p)$ be a probability space in the classical Kolmogorov sense. A measurable map f of Ω into the real line $\mathbb{R}$, called *random variable*, sends p into a probability measure p_f, called the *distribution* of f, on the real Borel sets $\mathbb{B}$ via $p_f(B) = p(f^{\leftarrow}(B))$, $B \in \mathbb{B}$. In fact, f induces a map sending probability measures on $\mathbb{A}$ into probability measures on $\mathbb{B}$ (each point $\omega \in \Omega$, or $r \in \mathbb{R}$ is considered as a degenerated point probability measure). The preimage map $f^{\leftarrow}$, called *observable*, maps $\mathbb{B}$ into $\mathbb{A}$ and it is a sequentially continuous Boolean homomorphism. A *fuzzy random variable* (or operational r.v.) is a "measurable" map sending probability measures on one probability space into probability measures on another probability space, but it can happen that a point $\omega \in \Omega$ is mapped to a nondegenerated probability measure. The corresponding observable is still sequentially continuous, but sends fuzzy subsets into fuzzy subsets (the image of a crisp set need not be crisp) and preserves some operations on fuzzy sets. The category ID (as an evaluation category) is suitable for modelling fundamental notions of fuzzy probability theory (cf. [14]).

We conclude with some problems concernig ID.

Problem 1.1. Is each object I^X of ID sufficient?

Problem 1.2. Is STID epireflective in ID?

Note that SID is a monocoreflective subcategory of ID. If $\mathcal{X}$ is an object of ID then the monocoreflection $\mathcal{X}^*$ of $\mathcal{X}$ is called the sobrification of $\mathcal{X}$ (cf. [25]).

Problem 2.1. Does there exist an object $\mathcal{X}$ of ID such that $\sigma(\mathcal{X})$ and $\sigma(\mathcal{X}^*)$ are not isomorphic?

Problem 2.2. Does there exist an object $\mathcal{X} \subseteq \{0,1\}^X$ of ID such that $\sigma(\mathcal{X})$ and $\sigma(\mathcal{X}^*)$ are not isomorphic?

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