# PRESENTATION OF THE SINGULAR PART OF THE BRAUER MONOID

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*Abstract.* We obtain a presentation for the singular part of the Brauer monoid with respect to an irreducible system of generators consisting of idempotents. As an application of this result we get a new construction of the symmetric group via connected sequences of subsets. Another application describes the lengths of elements in the singular part of the Brauer monoid with respect to the system of generators mentioned above.

*Keywords*: semigroup, presentation, symmetric group, Brauer monoid

*MSC 2000*: 20M05, 20M20

## 1. Introduction

The symmetric group $\mathcal{S}_n$ is a central object of study in many branches of mathematics. There exist several "natural" analogues (or generalizations) of $\mathcal{S}_n$ in the theory of semigroups. The most classical ones are the symmetric semigroup $\mathcal{T}_n$ and the inverse symmetric semigroup $\mathcal{IS}_n$. These arise when one tries to generalize Cayley's Theorem to the classes of all semigroups or all inverse semigroups. A less obvious semigroup generalization of $\mathcal{S}_n$ is the so-called *Brauer semigroup* $\mathfrak{B}_n$ which appears in the context of centralizer algebras in representation theory, see [2]. $\mathfrak{B}_n$ contains $\mathcal{S}_n$ as the subgroup of all invertible elements and has a nice geometric realization (see Section 2). The deformation of the corresponding semigroup algebra, the so-called *Brauer algebra*, has been intensively studied by specialists in representation theory, knot theory and theoretical physics. The semigroup properties of $\mathfrak{B}_n$ were studied in [9], [10], [8], [5], [7], [6].

Given a finitely generated semigroup, a fundamental question is to find its presentation with respect to some (irreducible) system of generators. For example, for $\mathcal{S}_n$ and $\mathfrak{B}_n$ several such presentations are known. However, for semigroups one can even make the problem more semigroup-oriented, and ask to find a presentation for the *singular part* of the semigroup which, by definition, is the set of all non-invertible elements. In the case of a finite semigroup all non-invertible elements form again a semigroup and hence the problem to find a presentation for the singular part makes sense. For example, in [4] a presentation for the singular part of $\mathcal{IS}_n$ is found (a presentation for $\mathcal{IS}_n$ itself can be found in [1]).

From [8] we know that $\mathfrak{B}_n \setminus \mathcal{S}_n$ has a natural irreducible system of generators consisting of idempotents. The main aim of the present paper is to obtain a presentation of $\mathfrak{B}_n \setminus \mathcal{S}_n$ with respect to this system of generators. Surprisingly enough, the system of the corresponding defining relations is not big and all relations have an obvious interpretation via the geometric realization of $\mathfrak{B}_n$. This result is presented in Theorem 5. As usual, a tricky part in the proof of Theorem 5 is to show that the listed system of defining relations is complete. This part of the proof is quite technical and occupies the whole Section 4. In Section 5 we present several combinatorial applications of Theorem 5. These include an interesting combinatorial realization of the symmetric group via equivalence classes of sequences of "connected" two-element subsets, and a computation of the maximal length for an element in $\mathfrak{B}_n \setminus \mathcal{S}_n$ with respect to our system of generators.

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## 2. Preliminaries about $\mathfrak{B}_n$

Let $n$ be a positive integer. Put $\mathbf{n} = \{1, \dots, n\}$ and $\mathbf{n}' = \{1', \dots, n'\}$. We consider the map $': \mathbf{n} \to \mathbf{n}'$ as a fixed bijection and denote the inverse bijection by the same symbol, that is $(x')' = x$ for all $x \in \mathbf{n}$. The elements of the Brauer semigroup $\mathfrak{B}_n$ are all possible partitions of $\mathbf{n} \cup \mathbf{n}'$ into two-element blocks. It is easy to see that $|\mathfrak{B}_n| = (2n-1)!!$.

A two-element subset $\{i, j\}$ of $\mathbf{n} \cup \mathbf{n}'$ will be called

- a *left bracket* provided that $\{i, j\} \subset \mathbf{n}$;
- a *right bracket* provided that $\{i, j\} \subset \mathbf{n}'$;
- a *line*, if $\{i, j\}$ is neither a left nor a right bracket.

Obviously, every element of $\mathfrak{B}_n$ contains the same number of left and right brackets. Let $\pi \in \mathfrak{B}_n$, and assume that $\{i_k, j_k\}$, $k \in K$, is the list of all left brackets of $\pi$;

$\{u'_k, v'_k\}$, $k \in K$, is the list of all right brackets of $\pi$; and $\{f_l, g'_l\}$, $l \in L$, is the list of all lines of $\pi$. Then we have

$$(1) \qquad \pi = \big\{\{i_k, j_k\}_{k \in K}, \{u'_k, v'_k\}_{k \in K}, \{f_l, g'_l\}_{l \in L}\big\}.$$

We say that $\pi$ has *corank* $\mathrm{corank}(\pi) = 2|K| \leqslant 2\lfloor \frac{1}{2}n \rfloor$.

It is convenient to represent the elements of $\mathfrak{B}_n$ geometrically as a kind of *microchips* as follows: we have two sets of pins (which correspond to elements in $\mathbf{n}$ and $\mathbf{n}'$ respectively), which are connected in pairs (this corresponds to the partition of $\mathbf{n} \cup \mathbf{n}'$ into two-element blocks which our element from $\mathfrak{B}_n$ represents). An example is shown in Figure 1, for convenience the same element is also written in the form (1).

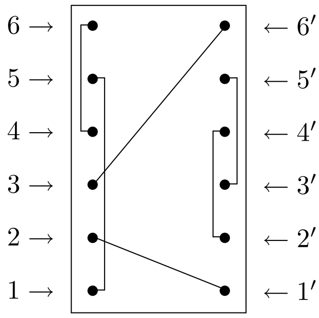



Figure 1. The element $\big\{\{1,5\},\{4,6\},\{2',4'\},\{3',5'\},\{2,1'\},\{3,6'\}\big\}$ of $\mathfrak{B}_6$.

Now we would like to define multiplication in $\mathfrak{B}_n$. To give a formal definition, for $\pi \in \mathfrak{B}_n$ and $x, y \in \mathbf{n} \cup \mathbf{n}'$ we set $x \equiv_\pi y$ provided that $x$ and $y$ are in the same block of $\pi$. The relation $\equiv_\pi$ is an equivalence relation on $\mathbf{n} \cup \mathbf{n}'$ with two-element equivalence classes. Take now $\pi, \tau \in \mathfrak{B}_n$. Define a new equivalence relation, $\equiv$, on $\mathbf{n} \cup \mathbf{n}'$ as follows:

- for $x, y \in \mathbf{n}$ we have $x \equiv y$ if and only if $x \equiv_\pi y$ or there is a sequence $c_1, \dots, c_{2s}$, $s \geqslant 1$, of elements in $\mathbf{n}$ such that $x \equiv_\pi c'_1$, $c_1 \equiv_\tau c_2$, $c'_2 \equiv_\pi c'_3$, $\dots$, $c_{2s-1} \equiv_\tau c_{2s}$ and $c'_{2s} \equiv_\pi y$;
- for $x, y \in \mathbf{n}$ we have $x' \equiv y'$ if and only if $x' \equiv_\tau y'$ or there is a sequence $c_1, \dots, c_{2s}$, $s \geqslant 1$, of elements in $\mathbf{n}$ such that $x' \equiv_\tau c_1$, $c'_1 \equiv_\pi c'_2$, $c_2 \equiv_\tau c_3$, $\dots$, $c'_{2s-1} \equiv_\pi c'_{2s}$ and $c_{2s} \equiv_\tau y'$;
- for $x, y \in \mathbf{n}$ we have $x \equiv y'$ if and only if $y' \equiv x$ if and only if there is a sequence $c_1, \dots, c_{2s-1}$, $s \geqslant 1$, of elements in $\mathbf{n}$ such that $x \equiv_\pi c'_1$, $c_1 \equiv_\tau c_2$, $c'_2 \equiv_\pi c'_3, \dots$, $c'_{2s-2} \equiv_\pi c'_{2s-1}$ and $c_{2s-1} \equiv_\tau y'$.

It is easy to see that $\equiv$ determines an equivalence relation on $\mathbf{n} \cup \mathbf{n}'$ with two-element classes and thus is an element of $\mathfrak{B}_n$. We define this element to be the product $\pi\tau$. It is straightforward that this multiplication is associative. In our geometric realization the above multiplication reduces to concatenation of chips, see an example in Figure 2.

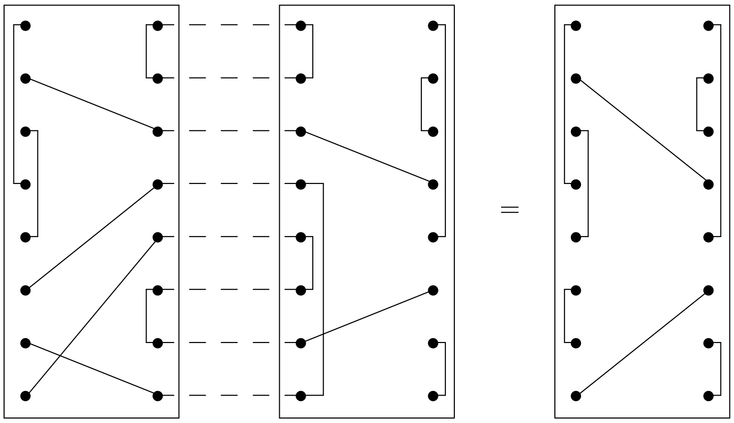


Figure 2. Elements of $\mathfrak{B}_8$ and their multiplication.

Note that the element $\{\{k, k'\}_{k\in\mathbf{n}}\}$ is the identity element in $\mathfrak{B}_n$. It is easy to see (see for example [9]) that the group of all invertible elements in $\mathfrak{B}_n$ is precisely the set of all elements of corank 0, and it is isomorphic to $\mathcal{S}_n$. We identify the elements of this subgroup of $\mathfrak{B}_n$ with $\mathcal{S}_n$ in the following way: $\pi \in \mathcal{S}_n$ corresponds to the element $\{\{k, \pi(k)\}_{k\in\mathbf{n}}\}$. Then the subsemigroup of all non-invertible elements of $\mathfrak{B}_n$ coincides with $\mathfrak{B}_n \setminus \mathcal{S}_n$.

We denote by $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ Green's relations, in particular, for a semigroup $S$ and $a \in S$, $\mathcal{H}_a$ denotes the $\mathcal{H}$-class of $S$ containing $a$ (similarly for all other relations). We will need the following description of Green's relations for $\mathfrak{B}_n$, which was obtained in [9]:

**Lemma 1.** *Let $\pi, \tau \in \mathfrak{B}_n$. Then*

(i) *$\pi\mathcal{R}\tau$ if and only if $\pi$ and $\tau$ have the same left brackets;*

(ii) *$\pi\mathcal{L}\tau$ if and only if $\pi$ and $\tau$ have the same right brackets;*

(iii) *$\pi\mathcal{H}\tau$ if and only if $\pi$ and $\tau$ have both the same left brackets and the same right brackets;*

(iv) *$\pi\mathcal{D}\tau$ if and only if $\pi\mathcal{J}\tau$ if and only if corank$(\pi)$ = corank$(\tau)$.*

## 3. An irreducible system of generators for $\mathfrak{B}_n \setminus \mathcal{S}_n$

For $i, j \in \mathbf{n}$, $i \neq j$, define $\sigma_{i,j}$ as follows:

$$\sigma_{i,j} = \big\{\{i,j\}, \{i',j'\}, \{k,k'\}_{k\neq i,j}\big\}.$$

We have $\sigma_{i,j} = \sigma_{j,i} = \sigma_{i,j}^2$ and $\operatorname{corank}(\sigma_{i,j}) = 2$. We will call these elements *atoms*. An example of an atom can be found in Figure 3.

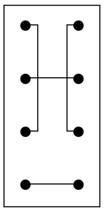


Figure 3. The atom $\sigma_{1,3}$ of $\mathfrak{B}_4$.

The following statement was proved in [8]. However, because of the poor availability of [8] we will prove it here as well.

**Proposition 2.** *The set of all atoms is an irreducible system of generators in $\mathfrak{B}_n \setminus \mathcal{S}_n$.*

To prove this statement we will need several auxiliary lemmas.

**Lemma 3.** *The semigroup $\mathfrak{B}_n \setminus \mathcal{S}_n$ is generated by the set of all elements of corank 2.*

Proof. Let $\pi \in \mathfrak{B}_n \setminus \mathcal{S}_n$ be written in the form (1) as follows:

$$\pi = \big\{\{i, \theta(i)'\}_{i\in I}, \{u_j, v_j\}_{j\in J}, \{f_j', g_j'\}_{j\in J}\big\}.$$

We have $J \neq \emptyset$, $I \subset \mathbf{n}$, and $\theta\colon I \to \mathbf{n}$ is an injection. Assume that $\operatorname{corank}(\pi) > 2$. Fix $j_0 \in J$. Construct a bijection $\vartheta\colon \mathbf{n} \setminus \{u_{j_0}, v_{j_0}\} \to \mathbf{n} \setminus \{f_{j_0}, g_{j_0}\}$ as follows:

- $\vartheta(i) = \theta(i)$ for all $i \in I$;
- $\vartheta(u_j) = f_j$ and $\vartheta(v_j) = g_j$ for all $j \in J \setminus \{j_0\}$.

Put now $\tau = \big\{\{i, \vartheta(i)'\}_{i\neq u_{j_0}, v_{j_0}}, \{u_{j_0}, v_{j_0}\}, \{f_{j_0}', g_{j_0}'\}\big\}$. We have $\operatorname{corank}(\tau) = 2$ and direct calculation shows that $\pi = \prod_{j\in J} \sigma_{u_j, v_j} \cdot \tau$. The statement follows. $\square$

**Lemma 4.** *Every element of the maximal subgroup corresponding to an atom is decomposable into a product of atoms.*

Proof. Let $\pi \in \mathfrak{B}_n$ be a group element of corank 2, $\mathcal{H}$-related to some atom. From Lemma 1 it follows that in this case $\pi = \big\{\{i, \theta(i)'\}_{i \neq u,v}, \{u, v\}, \{u', v'\}\big\}$ for some $u, v \in \mathbf{n}$, $u \neq v$, and some bijection, $\theta\colon \mathbf{n} \setminus \{u, v\} \to \mathbf{n} \setminus \{u, v\}$. We consider $\theta$ as an element of $\mathcal{S}_{\mathbf{n}\setminus\{u,v\}}$. Let

$$\theta = (i_1^{(1)}, \dots, i_{p_1}^{(1)}) \cdot \dots \cdot (i_1^{(s)}, \dots, i_{p_s}^{(s)})$$

be a cyclic decomposition of $\theta$. By direct calculation one obtains that

$$\pi = \sigma_{u,v}\sigma_{u,i_1^{(1)}} \dots \sigma_{u,i_{p_1}^{(1)}}\sigma_{u,v} \cdot \dots \cdot \sigma_{u,v}\sigma_{u,i_1^{(s)}} \dots \sigma_{u,i_{p_s}^{(s)}}\sigma_{u,v}. \tag{2}$$

The statement follows. □

Now we are ready to prove Proposition 2:

Proof of Proposition 2. First we show that atoms generate $\mathfrak{B}_n \setminus \mathcal{S}_n$. Because of Lemma 3 it is enough to show that any element $\pi \in \mathfrak{B}_n$ of corank 2 decomposes into a product of atoms. We again write $\pi$ in the form (1):

$$\pi = \big\{\{i, \theta(i)'\}_{i \in I}, \{u, v\}, \{f', g'\}\big\},$$

where $u, v \in \mathbf{n}$, $u \neq v$; $f, g \in \mathbf{n}$, $f \neq g$; and $\theta\colon \mathbf{n} \setminus \{u, v\} \to \mathbf{n} \setminus \{f, g\}$ is a bijection. Without loss of generality we may assume that $v \neq f$. Consider the element $\tau = \sigma_{v,f}\sigma_{f,g} = \big\{\{v, f\}, \{f', g'\}, \{g, v'\}, \{k, k'\}_{k \neq v,f,g}\big\}$. From Lemma 1 we have $\pi\mathcal{H}\sigma_{u,v}\tau$ and $\sigma_{u,v}\mathcal{R}\sigma_{u,v}\tau$. Hence, due to Green's Lemma, we have that the map $x \mapsto x\tau$ from $\mathcal{H}_{\sigma_{u,v}}$ to $\mathcal{H}_{\sigma_{u,v}\tau}$ is a bijection. Therefore there exists $\xi \in \mathcal{H}_{\sigma_{u,v}}$ such that $\pi = \xi\tau$. By Lemma 4, $\xi$ decomposes into a product of atoms. Hence so does $\pi$ as well.

Now we prove that no atom can be decomposed into a product of other atoms. Let $\sigma_{u,v} = \sigma_{u_1,v_1} \dots \sigma_{u_k,v_k}$. The product $\sigma_{u_1,v_1} \dots \sigma_{u_k,v_k}$ must contain the left bracket $\{u_1, v_1\}$ by the definition of multiplication in $\mathfrak{B}_n$. However, the element $\sigma_{u,v}$ contains the unique left bracket $\{u, v\}$. This implies that $\{u_1, v_1\} = \{u, v\}$ and the desired statement follows. The proof is complete. □

After Proposition 2 it is natural to ask what is the presentation of $\mathfrak{B}_n \setminus \mathcal{S}_n$ with respect to the system $\{\sigma_{u,v}\}$ of generators. We answer this question in the next section.

## 4. Main result

Denote by $T$ the semigroup generated by $\tau_{i,j}$, $i, j \in \mathbf{n}$, $i \neq j$, subject to the following relations (here $i$, $j$, $k$, $l$ are pairwise different):

$$\tau_{i,j} = \tau_{j,i}; \tag{3}$$
$$\tau_{i,j}^2 = \tau_{i,j}; \tag{4}$$
$$\tau_{i,j}\tau_{j,k}\tau_{k,l} = \tau_{i,j}\tau_{i,l}\tau_{k,l}; \tag{5}$$
$$\tau_{i,j}\tau_{i,k}\tau_{j,k} = \tau_{i,j}\tau_{j,k}; \tag{6}$$
$$\tau_{i,j}\tau_{j,k}\tau_{i,j} = \tau_{i,j}; \tag{7}$$
$$\tau_{i,j}\tau_{k,l}\tau_{i,k} = \tau_{i,j}\tau_{j,l}\tau_{i,k}; \tag{8}$$
$$\tau_{i,j}\tau_{k,l} = \tau_{k,l}\tau_{i,j}. \tag{9}$$

A straightforward calculation shows that the generators $\sigma_{i,j}$ of $\mathfrak{B}_n \setminus \mathcal{S}_n$ satisfy the relations (3)–(9) (the relations (3) and (4) are obvious, and the relations (5)–(9) are illustrated in Figures 4, 5, 6, 7 and 8). Thus there is a homomorphism $\varphi\colon T \to \mathfrak{B}_n \setminus \mathcal{S}_n$ sending $\tau_{i,j}$ to $\sigma_{i,j}$. Our main goal in the section is to prove the following theorem:

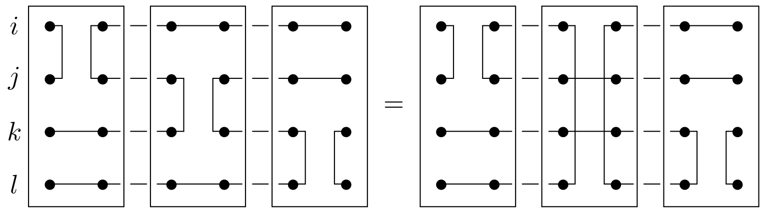



Figure 4. An example illustrating the relation (5).

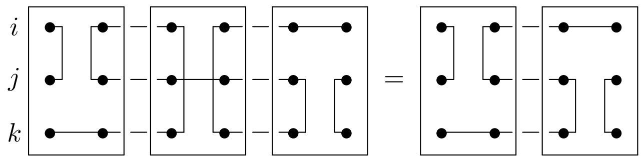



Figure 5. An example illustrating the relation (6).

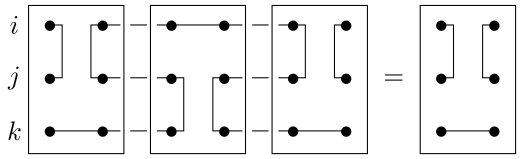



Figure 6. An example illustrating the relation (7).

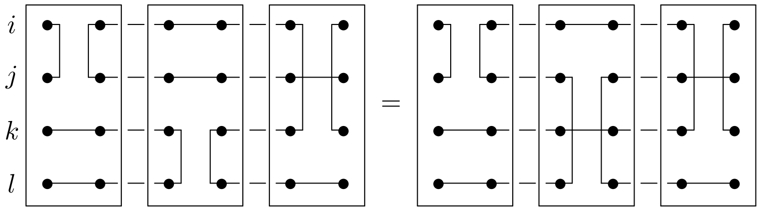



Figure 7. An example illustrating the relation (8).

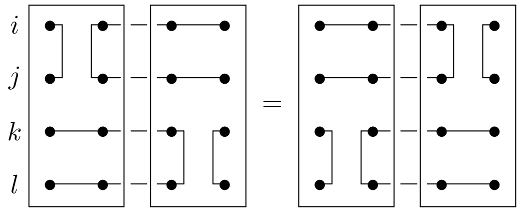



Figure 8. An example illustrating the relation (9).

**Theorem 5.** *$\varphi\colon T \to \mathfrak{B}_n \setminus \mathcal{S}_n$ is an isomorphism.*

The rest of this section is devoted to the proof of Theorem 5, which we will divide into steps formulated as lemmas and propositions. To distinguish $\tau_{i,j}$ from the atoms $\sigma_{i,j}$ we will call $\tau_{i,j}$ *quarks*. Two quarks $\tau_{i,j}$ and $\tau_{k,l}$ are said to be *connected* provided that $\{i,j\} \cap \{k,l\} \neq \emptyset$. We denote by $\mathcal{A} = \mathcal{A}_n$ the set of all quarks (the alphabet of our presentation for $T$), and by $\mathcal{A}^+$ the free semigroup over $\mathcal{A}$. In what follows we will do all our computations with words in $T$, not $\mathcal{A}^+$. In particular, $v = w$ for $v, w \in \mathcal{A}^+$ means that $v = w$ in $T$.

A word $\tau_{i_1,j_1}\tau_{i_2,j_2}\dots\tau_{i_k,j_k} \in \mathcal{A}^+$ will be called *connected* if $\tau_{i_s,j_s}$ and $\tau_{i_{s+1},j_{s+1}}$ are connected for all $1 \leqslant s \leqslant k-1$. We start with the following statement:

**Proposition 6.** *Each element of the semigroup $T$ can we written in the form $w\tau_{i_1,j_1}\tau_{i_2,j_2}\dots\tau_{i_k,j_k}$, where $w\tau_{i_1,j_1} \in \mathcal{A}^+$ is connected and all sets $\{i_s, j_s\}$, $s = 1,\dots,k$, are pairwise disjoint.*

P r o o f. We use induction on the length of element. For elements of length 1 the statement is obvious. Let $v = w\tau_{i_1,j_1}\tau_{i_2,j_2}\dots\tau_{i_k,j_k} \in T$ be such that $w\tau_{i_1,j_1} \in \mathcal{A}^+$ is connected and all sets $\{i_s, j_s\}$, $s = 1,\dots,k$, are pairwise disjoint. Let further $\tau_{i,j}$ be a generator. To complete the proof we have to show that the element $v\tau_{i,j}$ can be written in the desired form. Without loss of generality we can assume that we have one of the following cases:

C a s e 1: the set $\{i,j\}$ is disjoint with all $\{i_s, j_s\}$, $s = 1,\dots,k$. In this case the statement is trivial.

C a s e 2: the set $\{i,j\}$ is disjoint with all $\{i_s,j_s\}$, $s=2,\dots,k$, but not with $\tau_{i_1,j_1}$. In this case we can use (9) to write

$$v\tau_{i,j}=w\tau_{i_1,j_1}\tau_{i,j}\tau_{i_2,j_2}\dots\tau_{i_k,j_k}.$$

Observe that $w\tau_{i_1,j_1}\tau_{i,j}$ is connected, and the necessary statement follows again.

C a s e 3: $i=i_1$ and $j\in\bigcup\limits_{s=2}^{k}\{i_s,j_s\}$. Using (9) we can even assume $j=j_2$. Using (9) and (8) we have

$$v\tau_{i,j}=w\tau_{i,j_1}\tau_{i_2,j}\tau_{i,j}\tau_{i_3,j_3}\dots\tau_{i_k,j_k}=w\tau_{i,j_1}\tau_{i_2,j_1}\tau_{i,j}\tau_{i_3,j_3}\dots\tau_{i_k,j_k}.$$

Here $w\tau_{i,j_1}\tau_{i_2,j_1}$ is connected and the sets $\{i_2,j_1\}$, $\{i,j\}$, $\{i_s,j_s\}$, $s=3,\dots,k$, are disjoint. The claim follows.

C a s e 4: $i\in\bigcup\limits_{s=2}^{k}\{i_s,j_s\}$ and $j\notin\bigcup\limits_{s=1}^{k}\{i_s,j_s\}$. Using (9), we can even assume $i=i_2$. In this case we can use (9) to write

$$v\tau_{i,j}=w\tau_{i_1,j_1}\tau_{i,j_2}\tau_{i,j}\tau_{i_3,j_3}\dots\tau_{i_k,j_k}. \tag{10}$$

Now we have

$$\begin{aligned}\tau_{i_1,j_1}\tau_{i,j_2}\tau_{i,j}&=\tau_{i_1,j_1}\tau_{i,j_2}\tau_{i,j}\tau_{i_1,j}\tau_{i,j} \quad \text{(by (7))}\\ &=\tau_{i_1,j_1}\tau_{i,j_2}\tau_{i_1,j_2}\tau_{i_1,j}\tau_{i,j} \quad \text{(by (5))}\\ &=\tau_{i_1,j_1}\tau_{i,j_1}\tau_{i_1,j_2}\tau_{i_1,j}\tau_{i,j} \quad \text{(by (8))}\\ &=\tau_{i_1,j_1}\tau_{i_1,j_2}\tau_{i_1,j}\tau_{i,j_1}\tau_{i,j} \quad \text{(by (9))}\\ &=\tau_{i_1,j_1}\tau_{i_1,j_2}\tau_{i_1,j}\tau_{i_1,j_1}\tau_{i,j} \quad \text{(by (8))}.\end{aligned} \tag{11}$$

From (10) and (11) we have

$$v\tau_{i,j}=w\tau_{i_1,j_1}\tau_{i,j_2}\tau_{i,j}\tau_{i_3,j_3}\dots\tau_{i_k,j_k}=w\tau_{i_1,j_1}\tau_{i_1,j_2}\tau_{i_1,j}\tau_{i_1,j_1}\tau_{i,j}\tau_{i_3,j_3}\dots\tau_{i_k,j_k}.$$

Here $w\tau_{i_1,j_1}\tau_{i_1,j_2}\tau_{i_1,j}\tau_{i_1,j_1}$ is connected and the sets $\{i_1,j_1\}$, $\{i,j\}$, $\{i_s,j_s\}$, $s=3,\dots,k$, are disjoint. The claim follows.

C a s e 5: $i,j\in\bigcup\limits_{s=2}^{k}\{i_s,j_s\}$. If $\{i,j\}=\{i_s,j_s\}$ for some $s\geqslant 2$, the statement follows from (9) and (4). Otherwise, using (9) we can even assume $i=i_2$, $j=j_3$. In this case we can use (9) to write

$$v\tau_{i,j}=w\tau_{i_1,j_1}\tau_{i,j_2}\tau_{i_3,j}\tau_{i,j}\tau_{i_4,j_4}\dots\tau_{i_k,j_k}. \tag{12}$$

Now we have

$$
\begin{aligned}
(13)\qquad \tau_{i_1,j_1}\tau_{i,j_2}\tau_{i_3,j}\tau_{i,j} &= \tau_{i_1,j_1}\tau_{i,j_2}\tau_{i_3,j_2}\tau_{i,j} \quad \text{(by (8))}\\
&= \tau_{i,j_2}\tau_{i_3,j_2}\tau_{i_1,j_1}\tau_{i,j} \quad \text{(by (9))}\\
&= \tau_{i,j_2}\tau_{i_3,j_2}\tau_{i_1,j_1}\tau_{j_2,j_1}\tau_{i_1,j_1}\tau_{i,j} \quad \text{(by (7))}\\
&= \tau_{i,j_2}\tau_{i_1,j_1}\tau_{j_2,i_3}\tau_{j_2,j_1}\tau_{i_1,j_1}\tau_{i,j} \quad \text{(by (9))}\\
&= \tau_{i,j_2}\tau_{i_1,j_1}\tau_{i_1,i_3}\tau_{j_2,j_1}\tau_{i_1,j_1}\tau_{i,j} \quad \text{(by (8))}\\
&= \tau_{i_1,j_1}\tau_{i_1,i_3}\tau_{i,j_2}\tau_{j_2,j_1}\tau_{i_1,j_1}\tau_{i,j} \quad \text{(by (9))}\\
&= \tau_{i_1,j_1}\tau_{i_1,i_3}\tau_{i,j_2}\tau_{i,i_1}\tau_{i_1,j_1}\tau_{i,j} \quad \text{(by (5))}\\
&= \tau_{i_1,j_1}\tau_{i_1,i_3}\tau_{i_3,j_2}\tau_{i,i_1}\tau_{i_1,j_1}\tau_{i,j} \quad \text{(by (8))}\\
&= \tau_{i_1,j_1}\tau_{i_1,i_3}\tau_{i,i_1}\tau_{i_1,j_1}\tau_{i_3,j_2}\tau_{i,j} \quad \text{(by (9))}.
\end{aligned}
$$

From (12) and (13) we have

$$v\tau_{i,j} = w\tau_{i_1,j_1}\tau_{i,j_2}\tau_{i_3,j}\tau_{i,j}\tau_{i_4,j_4}\dots\tau_{i_k,j_k} = w\tau_{i_1,j_1}\tau_{i_1,i_3}\tau_{i,i_1}\tau_{i_1,j_1}\tau_{i_3,j_2}\tau_{i,j}\tau_{i_4,j_4}\dots\tau_{i_k,j_k}.$$

Here $w\tau_{i_1,j_1}\tau_{i_1,i_3}\tau_{i,i_1}\tau_{i_1,j_1}$ is connected and the sets $\{i_1,j_1\}$, $\{i_3,j_2\}$, $\{i,j\}$, $\{i_s,j_s\}$, $s=4,\dots,k$, are disjoint. The claim follows.

Now the proof is completed by induction. □

**Lemma 7.** *There is a unique anti-involution* $*\colon T\to T$ *satisfying* $\tau_{i,j}^* = \tau_{i,j}$ *for all* $i,j\in\{1,2,\dots,n\}$, $i\neq j$.

P r o o f. Existence follows from the fact that the relations (3)–(9) are stable with respect to $*$. Uniqueness follows from the fact that $T$ is generated by $\tau_{i,j}$, $i\neq j\in\{1,2,\dots,n\}$. □

**Lemma 8.** *Let* $\tau_{i,j}w\in\mathcal{A}^+$ *be connected. Then* $(\tau_{i,j}w)(\tau_{i,j}w)^* = \tau_{i,j}$.

P r o o f. Let $w=\tau_{i_1,j_1}\dots\tau_{i_k,j_k}$. Since $\tau_{i,j}w$ is connected, applying (7), (4) and the definition of $*$ we compute

$$
\begin{aligned}
(\tau_{i,j}w)(\tau_{i,j}w)^* &= \tau_{i,j}\tau_{i_1,j_1}\dots\tau_{i_k,j_k}\tau_{i_k,j_k}\dots\tau_{i_1,j_1}\tau_{i,j}\\
&= \tau_{i,j}\tau_{i_1,j_1}\dots\tau_{i_{k-1},j_{k-1}}\tau_{i_k,j_k}\tau_{i_{k-1},j_{k-1}}\dots\tau_{i_1,j_1}\tau_{i,j}\\
&= \tau_{i,j}\tau_{i_1,j_1}\dots\tau_{i_{k-2},j_{k-2}}\tau_{i_{k-1},j_{k-1}}\tau_{i_{k-2},j_{k-2}}\dots\tau_{i_1,j_1}\tau_{i,j}\\
&\dots\\
&= \tau_{i,j}\tau_{i_1,j_1}\tau_{i,j}\\
&= \tau_{i,j}.
\end{aligned}
$$

□

If $\{i_s, j_s\}$, $s = 1, \dots, k$, are pairwise disjoint, the element $\tau_{i_1,j_1} \dots \tau_{i_k,j_k}$ will be called a *standard idempotent*. That such element is indeed an idempotent, follows immediately from (9) and (4).

**Corollary 9.**
(i) *Every element of $T$ is $\mathcal{L}$-equivalent to a standard idempotent.*
(ii) *$T$ is regular.*
(iii) *The map $\varphi$ induces a bijection between the sets of $\mathcal{L}$-classes for the semigroups $T$ and $\mathfrak{B}_n \setminus \mathcal{S}_n$. Similarly for the $\mathcal{R}$-, $\mathcal{H}$-, and $\mathcal{D}$-classes.*

Proof. Let $v \in \mathcal{A}^+$. By Proposition 6 we can write $v = w\tau_{i_1,j_1}\tau_{i_2,j_2} \dots \tau_{i_k,j_k}$, where $w\tau_{i_1,j_1} \in \mathcal{A}^+$ is connected and all sets $\{i_s, j_s\}$, $s = 1, \dots, k$, are pairwise disjoint. By definition, the element $\varepsilon = \tau_{i_1,j_1}\tau_{i_2,j_2} \dots \tau_{i_k,j_k}$ is standard. We obviously have $v = w\tau_{i_1,j_1}\varepsilon$. By Lemma 8 we have

$$\tau_{i_1,j_1} w^* v = \tau_{i_1,j_1} w^* w \tau_{i_1,j_1}\tau_{i_2,j_2} \dots \tau_{i_k,j_k} = \tau_{i_1,j_1}\tau_{i_2,j_2} \dots \tau_{i_k,j_k} = \varepsilon.$$

Hence $v\mathcal{L}\varepsilon$, which proves (i). (i) implies that every $\mathcal{L}$-class of $T$ contains an idempotent, and hence (ii) follows.

By Lemma 1, the images of standard idempotents under $\varphi$ belong to different $\mathcal{L}$-classes of $\mathfrak{B}_n \setminus \mathcal{S}_n$. Hence different standard idempotents of $T$ belong to different $\mathcal{L}$-classes of $T$. In particular, there is a bijection between $\mathcal{L}$-classes of $T$ and standard idempotents. Since $\varphi$ is surjective, there is also a bijection between $\mathcal{L}$-classes of $\mathfrak{B}_n \setminus \mathcal{S}_n$ and standard idempotents. This implies (iii) for $\mathcal{L}$-classes. For $\mathcal{R}$-classes the statement now follows by applying $*$. For $\mathcal{H}$- and $\mathcal{D}$-classes the statement follows from the definition and the corresponding statements for $\mathcal{L}$- and $\mathcal{R}$-classes. This completes the proof. □

For $k = 1, \dots, \lfloor \frac{1}{2}n \rfloor$ set $\varepsilon_k = \tau_{1,2}\tau_{3,4} \dots \tau_{2k-1,2k}$ and let $\mathcal{H}_k$ denote the $\mathcal{H}$-class of $T$ containing the element $\varepsilon_k$. For $i, j \in \{3, \dots, n\}$, $i \neq j$, set $\gamma_{i,j} = \tau_{1,2}\tau_{1,i}\tau_{1,j}\tau_{1,2}$. Note that, using (5) and (6), we have

$$(14) \qquad \tau_{1,2}\tau_{1,i}\tau_{1,j}\tau_{1,2} = \tau_{1,2}\tau_{1,i}\tau_{i,j}\tau_{1,j}\tau_{1,2} = \tau_{1,2}\tau_{2,j}\tau_{i,j}\tau_{2,i}\tau_{1,2} = \tau_{1,2}\tau_{2,j}\tau_{2,i}\tau_{1,2}.$$

**Lemma 10.** *The elements $\gamma_{i,j}$, $i, j \in \{3, \dots, n\}$, $i \neq j$, generate $\mathcal{H}_1$ as a monoid.*

Proof. Let $w \in \mathcal{A}^+$ be such that $w \in \mathcal{H}_1$. Since $\tau_{1,2}$ is the unit element in the group $\mathcal{H}_1$, we have $w = \tau_{1,2}w\tau_{1,2}$ and hence we can assume that $w$ has the form $\tau_{1,2}w'\tau_{1,2}$ for some $w' \in \mathcal{A}^+$. We claim that $w$ is connected. Indeed, assume that $w$ is not connected. Then direct calculation shows that $\varphi(w) \in \mathfrak{B}_n$ has corank at least 4. At the same time the corank of $\varphi(\varepsilon_1)$ is 2. This contradicts Lemma 1.

We prove our lemma by induction on the length of $w' = \tau_{i_1,j_1} \dots \tau_{i_k,j_k}$ (note that $w'$ is connected since $w$ is). Because of (4) we can always assume that $\tau_{i_s,j_s} \neq \tau_{i_{s+1},j_{s+1}}$ for all $s = 1, \dots, k-1$, $\tau_{i_1,j_1} \neq \tau_{1,2}$ and $\tau_{i_k,j_k} \neq \tau_{1,2}$. The basis of our induction will be the cases $k = 0, 1, 2$. If $k = 0, 1$, then from (4) and (7) it follows that $w = \tau_{1,2}$, and the statement is obvious.

Let $k = 2$. If either 1 or 2 occurs in both $\{i_1, j_1\}$ and $\{i_2, j_2\}$, we are done by (14). If not, without loss of generality and up to the application of $*$ we can assume that $i_1 = 1$ and $i_2 = 2$. Then $j_1 = j_2$ since $w$ is connected. Hence, using (6) we get

$$\tau_{1,2}\tau_{1,j_1}\tau_{2,j_1}\tau_{1,2} = \tau_{1,2}\tau_{1,j_1}\tau_{1,2},$$

reducing everything to the case $k = 1$.

Now we proceed by induction and prove the step $k-1 \Rightarrow k$, where $k > 2$. If $\{i_2, j_2\} \cap \{1, 2\} \neq \emptyset$, using (7) we can write

$$\tau_{1,2}\tau_{i_1,j_1}\tau_{i_2,j_2}\tau_{i_3,j_3} \dots \tau_{i_k,j_k}\tau_{1,2} = \tau_{1,2}\tau_{i_1,j_1}\tau_{i_2,j_2}\tau_{1,2}\tau_{i_2,j_2}\tau_{i_3,j_3} \dots \tau_{i_k,j_k}\tau_{1,2}$$

and the statement follows from the induction hypothesis. If $\{i_2, j_2\} \cap \{1, 2\} = \emptyset$ then, using (5) if necessary, we may assume $i_1 = 1$ and $j_1 = j_2$. Assume first that $j_1 \in \{i_3, j_3\}$, say $j_3 = j_1$. Then by (5) we have

$$\tau_{1,2}\tau_{1,j_1}\tau_{i_2,j_1}\tau_{i_3,j_1} = \tau_{1,2}\tau_{2,i_2}\tau_{i_2,j_1}\tau_{i_3,j_1}.$$

If $i_3 = 2$, then (6) gives $\tau_{2,i_2}\tau_{i_2,j_1}\tau_{2,j_1} = \tau_{2,i_2}\tau_{2,j_1}$ and reduces our expression to the case $k-1$. If $i_3 \neq 2$, using (5) we have $\tau_{2,i_2}\tau_{i_2,j_1}\tau_{i_3,j_1} = \tau_{2,i_2}\tau_{2,i_3}\tau_{i_3,j_1}$, which reduces our expression to the case $\{i_2, j_2\} \cap \{1, 2\} \neq \emptyset$ considered above.

Finally, assume that $j_1 \notin \{i_3, j_3\}$. Then without loss of generality we can assume $i_3 = i_2$. If $j_3 = 1$, then by (6) we have $\tau_{1,j_1}\tau_{i_2,j_1}\tau_{i_2,1} = \tau_{1,j_1}\tau_{i_2,1}$, which reduces our expression to the case $k-1$. If $j_3 \neq 1$, using (5) we have $\tau_{1,j_1}\tau_{i_2,j_1}\tau_{i_2,j_3} = \tau_{1,j_1}\tau_{1,j_3}\tau_{i_2,j_3}$, which reduces our expression to the case $\{i_2, j_2\} \cap \{1, 2\} \neq \emptyset$ considered above. Now the proof is completed by induction. $\square$

For $3 \leqslant i \leqslant n-1$ set $\gamma_i = \gamma_{i,i+1}$.

**Lemma 11.** *Let $i, j \in \{3, \dots, n\}$, $i \neq j$.*

(i) $\gamma_{i,j} = \gamma_{j,i}$.

(ii) $\gamma_{i,j}$ *decomposes into a product of* $\gamma_k$*'s.*

Proof. We have

$$\begin{aligned}
\gamma_{i,j} &= \tau_{1,2}\tau_{1,i}\tau_{1,j}\tau_{1,2} \\
&= \tau_{1,2}\tau_{2,i}\tau_{1,i}\tau_{1,j}\tau_{1,2} \quad (\text{by } (6)) \\
&= \tau_{1,2}\tau_{2,i}\tau_{2,j}\tau_{1,j}\tau_{1,2} \quad (\text{by } (5)) \\
&= \tau_{1,2}\tau_{2,i}\tau_{2,j}\tau_{1,2} \quad (\text{by } (6)) \\
&= \tau_{1,2}\tau_{1,j}\tau_{1,i}\tau_{1,2} \quad (\text{by } (14)),
\end{aligned}$$

which proves (i).

Because of (i) we can assume $j > i$. If $j = i+1$ then (ii) is obvious. We proceed by induction on $j - i$ and assume that some $\gamma_{i,j}$ decomposes into a product of $\gamma_k$'s. We have

$$\begin{aligned}
\gamma_{i,j}\gamma_j\gamma_{i,j} &= \tau_{1,2}\tau_{1,i}\tau_{1,j}\tau_{1,2}\tau_{1,j}\tau_{1,j+1}\tau_{1,2}\tau_{1,i}\tau_{1,j}\tau_{1,2} \quad (\text{by } (4)) \\
&= \tau_{1,2}\tau_{1,i}\tau_{1,j}\tau_{1,j+1}\tau_{1,2}\tau_{1,i}\tau_{1,j}\tau_{1,2} \quad (\text{by } (7)) \\
&= \tau_{1,2}\tau_{1,i}\tau_{1,j}\tau_{1,j+1}\tau_{1,2}\tau_{2,j}\tau_{2,i}\tau_{1,2} \quad (\text{by } (14)) \\
&= \tau_{1,2}\tau_{1,i}\tau_{1,j}\tau_{1,j+1}\tau_{j,j+1}\tau_{2,j}\tau_{2,i}\tau_{1,2} \quad (\text{by } (5)) \\
&= \tau_{1,2}\tau_{1,i}\tau_{1,j}\tau_{j,j+1}\tau_{2,j}\tau_{2,i}\tau_{1,2} \quad (\text{by } (6)) \\
&= \tau_{1,2}\tau_{1,i}\tau_{1,j}\tau_{j,j+1}\tau_{i,j+1}\tau_{2,i}\tau_{1,2} \quad (\text{by } (5)) \\
&= \tau_{1,2}\tau_{1,i}\tau_{1,j}\tau_{1,i}\tau_{i,j+1}\tau_{2,i}\tau_{1,2} \quad (\text{by } (5)) \\
&= \tau_{1,2}\tau_{1,i}\tau_{i,j+1}\tau_{2,i}\tau_{1,2} \quad (\text{by } (7)) \\
&= \tau_{1,2}\tau_{1,i}\tau_{i,j+1}\tau_{1,j+1}\tau_{1,2} \quad (\text{by } (5)) \\
&= \tau_{1,2}\tau_{1,i}\tau_{1,j+1}\tau_{1,2} \quad (\text{by } (6)) \\
&= \gamma_{i,j+1}.
\end{aligned} \tag{15}$$

The statement (ii) now follows by induction. □

**Lemma 12.** *The elements $\gamma_i$, $i = 3, \ldots, n-1$, satisfy the following relations:*
(a) $\gamma_i^2 = \tau_{1,2}$;
(b) $\gamma_i\gamma_j = \gamma_j\gamma_i$, $|i-j| > 1$;
(c) $\gamma_i\gamma_j\gamma_i = \gamma_j\gamma_i\gamma_j$, $|i-j| = 1$.

Proof. We have

$$\begin{aligned}
\gamma_i^2 &= \tau_{1,2}\tau_{1,i}\tau_{1,i+1}\tau_{1,2}\tau_{1,i}\tau_{1,i+1}\tau_{1,2} \quad (\text{by } (4)) \\
&= \tau_{1,2}\tau_{2,i+1}\tau_{2,i}\tau_{1,2}\tau_{1,i}\tau_{1,i+1}\tau_{1,2} \quad (\text{by } (14)) \\
&= \tau_{1,2}\tau_{2,i+1}\tau_{2,i}\tau_{1,i}\tau_{1,i+1}\tau_{1,2} \quad (\text{by } (6)) \\
&= \tau_{1,2}\tau_{2,i+1}\tau_{1,i+1}\tau_{1,i}\tau_{1,i+1}\tau_{1,2} \quad (\text{by } (5)) \\
&= \tau_{1,2}\tau_{2,i+1}\tau_{1,i+1}\tau_{1,2} \quad (\text{by } (7))
\end{aligned}$$

$$= \tau_{1,2}\tau_{1,i+1}\tau_{1,2} \quad \text{(by (6))}$$
$$= \tau_{1,2} \quad \text{(by (7))},$$

which implies (a).

To prove (b) we may assume $j \geqslant i+2$. We have

$$\begin{aligned}
\gamma_i\gamma_j &= \tau_{1,2}\tau_{1,i}\tau_{1,i+1}\tau_{1,2}\tau_{1,j}\tau_{1,j+1}\tau_{1,2} \quad \text{(by (4))}\\
&= \tau_{1,2}\tau_{1,i}\tau_{1,i+1}\tau_{1,2}\tau_{2,j}\tau_{1,j}\tau_{1,j+1}\tau_{1,2} \quad \text{(by (6))}\\
&= \tau_{1,2}\tau_{1,i}\tau_{1,i+1}\tau_{i+1,j}\tau_{2,j}\tau_{1,j}\tau_{1,j+1}\tau_{1,2} \quad \text{(by (5))}\\
&= \tau_{1,2}\tau_{1,i}\tau_{i,j}\tau_{i+1,j}\tau_{2,j}\tau_{2,j+1}\tau_{1,j+1}\tau_{1,2} \quad \text{(by (5))}\\
&= \tau_{1,2}\tau_{1,i}\tau_{i,j}\tau_{i+1,j}\tau_{2,j}\tau_{2,j+1}\tau_{1,2} \quad \text{(by (6))}\\
&= \tau_{1,2}\tau_{2,j}\tau_{i,j}\tau_{i+1,j}\tau_{i+1,j+1}\tau_{2,j+1}\tau_{1,2} \quad \text{(by (5))}\\
&= \tau_{1,2}\tau_{2,j}\tau_{i,j}\tau_{i,j+1}\tau_{i+1,j+1}\tau_{2,j+1}\tau_{1,2} \quad \text{(by (5))}\\
&= \tau_{1,2}\tau_{2,j}\tau_{2,j+1}\tau_{i,j+1}\tau_{i+1,j+1}\tau_{1,i+1}\tau_{1,2} \quad \text{(by (5))}\\
&= \tau_{1,2}\tau_{2,j}\tau_{2,j+1}\tau_{i,j+1}\tau_{1,i}\tau_{1,i+1}\tau_{1,2} \quad \text{(by (5))}\\
&= \tau_{1,2}\tau_{2,j}\tau_{2,j+1}\tau_{1,2}\tau_{1,i}\tau_{1,i+1}\tau_{1,2} \quad \text{(by (5))}\\
&= \tau_{1,2}\tau_{1,j+1}\tau_{1,j}\tau_{1,2}\tau_{1,i}\tau_{1,i+1}\tau_{1,2} \quad \text{(by (14))}\\
&= \tau_{1,2}\tau_{1,j}\tau_{1,j+1}\tau_{1,2}\tau_{1,i}\tau_{1,i+1}\tau_{1,2} \quad \text{(by Lemma 11 (i))}\\
&= \gamma_j\gamma_i. \quad \text{(by (4))}
\end{aligned}$$

This gives (b).

Finally, to prove (c) we may assume $j = i+1$. We have

$$\begin{aligned}
\gamma_{i+1}\gamma_i\gamma_{i+1} &= \tau_{1,2}\tau_{1,i+1}\tau_{1,i+2}\tau_{1,2}\tau_{1,i}\tau_{1,i+1}\tau_{1,2}\tau_{1,i+1}\tau_{1,i+2}\tau_{1,2} \quad \text{(by (4))}\\
&= \tau_{1,2}\tau_{1,i+1}\tau_{1,i+2}\tau_{1,2}\tau_{1,i}\tau_{1,i+1}\tau_{1,i+2}\tau_{1,2} \quad \text{(by (7))}\\
&= \tau_{1,2}\tau_{2,i+2}\tau_{2,i+1}\tau_{1,2}\tau_{1,i}\tau_{1,i+1}\tau_{1,i+2}\tau_{1,2} \quad \text{(by (14))}\\
&= \tau_{1,2}\tau_{2,i+2}\tau_{2,i+1}\tau_{i,i+1}\tau_{1,i}\tau_{1,i+1}\tau_{1,i+2}\tau_{1,2} \quad \text{(by (5))}\\
&= \tau_{1,2}\tau_{2,i+2}\tau_{2,i+1}\tau_{i,i+1}\tau_{1,i+1}\tau_{1,i+2}\tau_{1,2} \quad \text{(by (6))}\\
&= \tau_{1,2}\tau_{2,i+2}\tau_{2,i+1}\tau_{i,i+1}\tau_{i,i+2}\tau_{1,i+2}\tau_{1,2} \quad \text{(by (5))}\\
&= \tau_{1,2}\tau_{2,i+2}\tau_{2,i+1}\tau_{2,i+2}\tau_{i,i+2}\tau_{1,i+2}\tau_{1,2} \quad \text{(by (5))}\\
&= \tau_{1,2}\tau_{2,i+2}\tau_{i,i+2}\tau_{1,i+2}\tau_{1,2} \quad \text{(by (7))}\\
&= \tau_{1,2}\tau_{1,i}\tau_{i,i+2}\tau_{1,i+2}\tau_{1,2} \quad \text{(by (5))}\\
&= \tau_{1,2}\tau_{1,i}\tau_{1,i+2}\tau_{1,2} \quad \text{(by (6))}\\
&= \gamma_{i,i+2}.
\end{aligned}$$

Now (c) follows from (15). This completes the proof. □

**Corollary 13.**

(i) $\mathcal{H}_1 \cong \mathcal{S}_{n-2}$.

(ii) *Let $\pi \in T$ be such that $\pi \mathcal{D} \varepsilon_1$. Then the restriction of $\varphi$ to $\mathcal{H}_\pi$ is injective.*

Proof. $\mathcal{H}_1$ contains $\varepsilon_1$ and hence is a group. By Lemmas 11 and 10, $\mathcal{H}_1$ is generated by $\gamma_i$, $i = 3, \ldots, n-1$. By Lemma 12, $\gamma_i$'s satisfy Coxeter relations of type $A_{n-3}$. Hence $\mathcal{H}_1$ is a quotient of $\mathcal{S}_{n-2}$. However, $\varphi(\mathcal{H}_1)$ is a maximal subgroup of $\mathfrak{B}_n$, which is isomorphic to $\mathcal{S}_{n-2}$ by [9, Theorem 1]. The statement (i) follows.

(i) implies that the restriction of $\varphi$ to $\mathcal{H}_1$ is injective. Then for arbitrary $\pi \in T$ such that $\pi \mathcal{D} \varepsilon_1$ the statement (ii) follows from Green's Lemma. □

To prove Theorem 5 we have to generalize the statement of Corollary 13 (ii) to all other $\mathcal{H}$-classes. For this we will use the following statement:

**Proposition 14.** $|\mathcal{H}_k| = (n-2k)!$ *for all $k$, $1 \leqslant k \leqslant \lfloor \frac{1}{2} n \rfloor$.*

Proof. We proceed by induction on $k$. The case $k = 1$ follows from Corollary 13 (i). Let us prove the induction step $k-1 \Rightarrow k$. From the induction hypothesis and Green's Lemma it follows that every $\mathcal{H}$-class, which is $\mathcal{D}$-equivalent to $\mathcal{H}_{k-1}$, has cardinality $(n-2(k-1))!$.

For $i = 1, 2, \ldots, k$ set

$$\theta_i = \tau_{1,2}\tau_{3,4}\ldots\tau_{2i-3,2i-2}\tau_{2i+1,2i+2}\ldots\tau_{2k-1,2k}.$$

If $\pi \in \mathcal{L}_{\theta_i}$ then, using Proposition 6, one shows that $\pi\tau_{2i-1,2i} \in \mathcal{L}_{\varepsilon_k}$. Let $f_i\colon \mathcal{L}_{\theta_i} \to \mathcal{L}_{\varepsilon_k}$ denote the map $f_i(\pi) = \pi\tau_{2i-1,2i}$. This induces the map

$$f\colon \coprod_i \mathcal{L}_{\theta_i} \to \mathcal{L}_{\varepsilon_k}$$

such that the restriction of $f$ to $\mathcal{L}_{\theta_i}$ coincides with $f_i$. By Proposition 6 we have that $f$ is surjective. Our aim is to prove that even $f_1$ is surjective.

Let $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$. Define $\alpha\colon \mathcal{L}_{\theta_i} \to \mathcal{L}_{\theta_j}$ via $\alpha(\pi) = \pi\tau_{2i,2j}\tau_{2i-1,2i}$ and $\beta\colon \mathcal{L}_{\theta_j} \to \mathcal{L}_{\theta_i}$ via $\beta(\pi) = \pi\tau_{2i,2j}\tau_{2j-1,2j}$. Consider the diagram

$$\begin{tikzcd} \mathcal{L}_{\theta_i} \arrow[rr, bend right, "\alpha"'] \arrow[dr, "f_i"'] & & \mathcal{L}_{\theta_j} \arrow[ll, bend right, "\beta"'] \arrow[dl, "f_j"] \\ & \mathcal{L}_{\varepsilon_k} & \end{tikzcd} \quad . \tag{16}$$

Every element $\pi \in \mathcal{L}_{\theta_i}$ satisfies $\pi\tau_{2j-1,2j} = \pi$ by the definition of $\theta_i$. Every element $\pi \in \mathcal{L}_{\theta_j}$ satisfies $\pi\tau_{2i-1,2i} = \pi$ by the definition of $\theta_j$. Further,

$$\begin{aligned}
\tau_{2j-1,2j}\tau_{2i,2j}\tau_{2i-1,2i}\tau_{2j-1,2j} &= \tau_{2j-1,2j}\tau_{2i-1,2j-1}\tau_{2i-1,2i}\tau_{2j-1,2j} \quad \text{(by (5))} \\
&= \tau_{2j-1,2j}\tau_{2i-1,2j-1}\tau_{2j-1,2j}\tau_{2i-1,2i} \quad \text{(by (9))} \\
&= \tau_{2j-1,2j}\tau_{2i-1,2i} \quad \text{(by (7))}.
\end{aligned}$$

This implies for all $\pi \in \mathcal{L}_{\theta_i}$ the following equalities

$$\begin{aligned}
(f_j\alpha)(\pi) &= (f_j\alpha)(\pi\tau_{2j-1,2j}) \\
&= \pi\tau_{2j-1,2j}\tau_{2i,2j}\tau_{2i-1,2i}\tau_{2j-1,2j} \\
&= \pi\tau_{2j-1,2j}\tau_{2i-1,2i} \\
&= \pi\tau_{2i-1,2i} \\
&= f_i(\pi).
\end{aligned}$$

Hence $f_j\alpha = f_i$. Analogously one shows that $f_i\beta = f_j$. Thus the diagram (16) is commutative, which implies that the map $f_1$ is surjective.

**Lemma 15.** *For any $\pi \in \mathcal{L}_{\theta_1}$ there exists $\omega \in T$ such that $\omega\mathcal{H}\pi$, $\omega \neq \pi$, and $f_1(\pi) = f_1(\omega)$.*

Proof. Set $\omega = \pi\tau_{3,4}\tau_{1,3}\tau_{2,3}\tau_{3,4}$. Direct calculation shows that $\varphi(\pi)\mathcal{L}\varphi(\omega)$. Hence $\omega \in \mathcal{L}_{\theta_1}$ by Corollary 9 (iii). Further, we have $(\tau_{3,4}\tau_{1,3}\tau_{2,3}\tau_{3,4})^2 = \tau_{3,4}$ by the statement analogous to that of Lemma 12 (a), which implies $\omega\mathcal{R}\pi$, that is $\omega\mathcal{H}\pi$. Direct calculation shows that $\varphi(\pi) \neq \varphi(\pi)\varphi(\tau_{3,4}\tau_{1,3}\tau_{2,3}\tau_{3,4})$ and hence $\pi \neq \omega$. On the other hand,

$$\begin{aligned}
f_1(\omega) &= \pi\tau_{3,4}\tau_{1,3}\tau_{2,3}\tau_{3,4}\tau_{1,2} \\
&= \pi\tau_{3,4}\tau_{1,3}\tau_{2,3}\tau_{1,2}\tau_{3,4} \quad \text{(by (9))} \\
&= \pi\tau_{3,4}\tau_{1,3}\tau_{1,2}\tau_{3,4} \quad \text{(by (6))} \\
&= \pi\tau_{3,4}\tau_{1,3}\tau_{3,4}\tau_{1,2} \quad \text{(by (9))} \\
&= \pi\tau_{3,4}\tau_{1,2} \quad \text{(by (7))} \\
&= \pi\tau_{1,2} \quad (\text{since } \pi \in \mathcal{L}_{\theta_1}) \\
&= f_1(\pi).
\end{aligned}$$

$\square$

**Lemma 16.** *Assume that* $\pi, \tau \in \mathcal{L}_{\theta_1}$ *are such that* $f_1(\pi)\mathcal{H}f_1(\tau)$. *Then there exists* $\eta \in \mathcal{H}_\pi$ *such that* $f_1(\eta) = f_1(\tau)$.

Proof. If $\tau \in \mathcal{H}_\pi$, we have nothing to prove, hence we assume that $\tau \notin \mathcal{H}_\pi$. We have $\pi\tau_{1,2}\mathcal{H}\tau\tau_{1,2}$. In particular, $\pi\tau_{1,2}\mathcal{R}\tau\tau_{1,2}$. Moreover, we also have that $\operatorname{corank}(\varphi(\tau\tau_{1,2})) = 2k$. Then, applying $*$ to the statement of Proposition 6, we obtain that there exist $w, w' \in \mathcal{A}^+$, pairwise distinct $i_1, j_1, \dots, i_k, j_k$, and $a, b \in \{1, 2, \dots, k\}$ such that $\pi\tau_{1,2} = \tau_{i_1,j_1} \dots \tau_{i_k,j_k} w$, $\tau\tau_{1,2} = \tau_{i_1,j_1} \dots \tau_{i_k,j_k} w'$, the word $\tau_{i_a,j_a} w$ is connected, and the word $\tau_{i_b,j_b} w'$ is connected. Since both $\operatorname{corank}(\varphi(\pi)) = \operatorname{corank}(\varphi(\tau)) = 2k-2$ and $\tau \notin \mathcal{R}_\pi$, without loss of generality we may assume $\tau_{i_l,j_l}\pi = \pi$ for all $l = 1, \dots, k-1$ and $\tau_{i_l,j_l}\tau = \tau$ for all $l = 2, \dots, k$. Then, applying Proposition 6, we get some $v \in \mathcal{A}^+$ and $c \in \{2, 3, \dots, k\}$ such that $\tau = \tau_{i_2,j_2} \dots \tau_{i_k,j_k} v$ and the word $\tau_{i_c,j_c} v$ is connected. Put $\eta = \tau_{i_1,j_1}\tau_{i_1,i_k}\tau$. Since $\tau_{i_k,j_k}\tau = \tau$ by the above and

$$\tau_{i_k,j_k}\tau_{i_1,i_k}\tau_{i_1,j_1}\tau_{i_1,i_k}\tau_{i_k,j_k} = \tau_{i_k,j_k}$$

(by two applications of (7)), we have $\eta\mathcal{L}\tau$.

Further, since $\tau_{i_c,j_c} v$ is connected, we have $(\tau_{i_c,j_c} v)(\tau_{i_c,j_c} v)^* = \tau_{i_c,j_c}$ by Lemma 8. Using (9), this implies $\eta\mathcal{R}\tau_{i_1,j_1}\tau_{i_1,i_k}\tau_{i_2,j_2} \dots \tau_{i_k,j_k}$. Using (9), we further have

$$\tau_{i_1,j_1}\tau_{i_1,i_k}\tau_{i_2,j_2} \dots \tau_{i_k,j_k} = \tau_{i_2,j_2} \dots \tau_{i_{k-1},j_{k-1}}\tau_{i_1,j_1}\tau_{i_1,i_k}\tau_{i_k,j_k}.$$

Since $\tau_{i_1,j_1}\tau_{i_1,i_k}\tau_{i_k,j_k}$ is connected, by the same argument as above we have $\tau_{i_1,j_1}\tau_{i_1,i_k}$ $\tau_{i_2,j_2} \dots \tau_{i_k,j_k}\mathcal{R}\tau_{i_1,j_1}\tau_{i_2,j_2} \dots \tau_{i_{k-1},j_{k-1}}$. Hence $\eta\mathcal{R}\tau_{i_1,j_1} \dots \tau_{i_{k-1},j_{k-1}}$. It follows that $\eta\mathcal{R}\pi$ and hence $\eta\mathcal{H}\pi$.

The statement now follows from the following computation (using (7)):

$$\begin{aligned} f_1(\eta) = \eta\tau_{1,2} = \tau_{i_1,j_1}\tau_{i_1,i_k}\tau\tau_{1,2} &= \tau_{i_1,j_1}\tau_{i_1,i_k}\tau_{i_1,j_1} \dots \tau_{i_k,j_k} w' \\ &= \tau_{i_1,j_1} \dots \tau_{i_k,j_k} w' = \tau\tau_{1,2} = f_1(\tau). \end{aligned}$$

□

Since $f_1 \colon \mathcal{L}_{\theta_1} \to \mathcal{L}_{\varepsilon_k}$ is surjective, Lemma 16 implies that the restriction of $f_1$ to $\mathcal{H}_{\theta_1}$ is a surjection on a union of $\mathcal{H}$-classes in $\mathcal{L}_{\varepsilon_k}$. By Corollary 9 (iii), the number of $\mathcal{H}$-classes in the latter union can be computed in the semigroup $\mathfrak{B}_n$ via $\varphi$, and it is easy to see that it equals $\binom{n-(2k-2)}{2}$.

We know by induction that $|\mathcal{H}_{\theta_1}| = (n - 2(k-1))!$. Hence, taking into account Lemma 15 and Green's Lemma, we compute

$$|\mathcal{H}_{f_1(\theta_1)}| \leqslant \frac{1}{\binom{n-(2k-2)}{2}} \cdot \frac{(n-2(k-1))!}{2} = (n-2k)!. \tag{17}$$

Since $|\mathcal{H}_{\varphi(f_1(\theta_1))}| = (n-2k)!$ by [9, Theorem 1], (17) and Corollary 9 (iii) imply $|\mathcal{H}_{f_1(\theta_1)}| = (n-2k)!$. This forces $|\mathcal{H}_k| = (n-2k)!$ by Green's Lemma and the statement follows by induction. $\square$

Proof of Theorem 5. Let $1 \leqslant k \leqslant \lfloor \frac{1}{2}n \rfloor$. By Proposition 14 we have $|\mathcal{H}_k| = (n-2k)!$. By [9, Theorem 1] we have $|\varphi(\mathcal{H}_k)| = (n-2k)!$ as well. Hence the restriction of $\varphi$ to $\mathcal{H}_k$ is injective. From Green's Lemma it follows that the restriction of $\varphi$ to $\mathcal{H}_\pi$ is injective for every $\pi \in T$ such that $\pi \mathcal{D} \varepsilon_k$. From Corollary 9 (iii) it therefore follows that $\varphi$ is injective, and hence bijective. This completes the proof. $\square$

## 5. Combinatorial applications

**5.1. Connected sequences.** Two elements $\{i,j\}$ and $\{k,l\}$ of $\binom{\mathbf{n}}{2}$ are said to be *connected* provided that $\{i,j\} \cap \{k,l\} \neq \emptyset$. A *connected sequence* is a non-empty sequence $\{i_1,j_1\}$, $\{i_2,j_2\}$,..., $\{i_m,j_m\}$ of elements from $\binom{\mathbf{n}}{2}$ such that $\{i_l,j_l\}$ and $\{i_{l+1},j_{l+1}\}$ are connected for all $l = 1,\dots,m-1$. Two connected sequences will be called *equivalent* provided that one of them can be obtained from the other by a finite number of the following operations:

(I) replacing the fragment $\{i,j\},\{i,j\}$ by $\{i,j\}$ and vice versa;

(II) replacing the fragment $\{i,j\},\{j,k\},\{k,l\}$ by $\{i,j\},\{i,l\},\{k,l\}$ and vice versa, where $i \neq l$;

(III) replacing the fragment $\{i,j\},\{j,k\},\{k,i\}$ by $\{i,j\},\{k,i\}$ and vice versa;

(IV) replacing the fragment $\{i,j\},\{j,k\},\{i,j\}$ by $\{i,j\}$ and vice versa.

It is obvious that each of the operations (I)–(IV), applied to a connected sequence, produces a new connected sequence. As an immediate corollary of Theorem 5 we have the following result:

**Proposition 17.** *Let* $n \in \{2,3,\dots\}$.

(i) *There exist only finitely many, namely* $\frac{1}{4}n(n-1)n!$, *equivalence classes of connected sequences.*

(ii) *For all* $\{i,j\},\{k,l\} \in \binom{\mathbf{n}}{2}$ *the number of connected sequence whose first element is* $\{i,j\}$ *and whose last element is* $\{k,l\}$, *equals* $(n-2)!$.

Proof. Let $S$ denote the set of all equivalence classes of connected sequences. Define a semigroup structure on $S \cup \{0\}$ as follows: 0 is the zero element of $S \cup \{0\}$, and for $f,g \in S$

$$f \cdot g = \begin{cases} fg & \text{if } fg \text{ is connected,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\overline{\mathfrak{B}}$ denote the Rees quotient of $\mathfrak{B}_n \backslash \mathcal{S}_n$ modulo the ideal containing all elements of corank at least 4. By Theorem 5, mapping $\sigma_{i,j}$ to the connected sequence $\{i, j\}$ defines an epimorphism $\psi$ from $\overline{\mathfrak{B}}$ to $S \cup \{0\}$. On the other hand, from the definition of the equivalence relation on the connected sequences we have that mapping $\{i, j\}$ to $\sigma_{i,j}$ defines an epimorphism $\psi'\colon\ S \cup \{0\} \to \overline{\mathfrak{B}}$. Thus $\psi$ and $\psi'$ induce a pair of mutually inverse bijections between the set of all elements in $\mathfrak{B}_n$ of corank 2 and the set of equivalence classes of connected sequences. The claim now follows by direct computation in $\mathfrak{B}_n$. $\square$

It might be interesting to find a purely combinatorial proof for the statement of Proposition 17.

**5.2. Paths in the graph $\Gamma_n$.** There is another interesting combinatorial interpretation of the elements of $\mathfrak{B}_n$ of corank 2. Consider a non-oriented graph $\Gamma_n$ whose vertex set is $\binom{\mathbf{n}}{2}$, and such that two vertices $\{i, j\}$ and $\{k, l\}$ are connected by an edge if and only if $\{i, j\} \cap \{k, l\} \neq \emptyset$. The graph $\Gamma_4$ is shown in Figure 9.

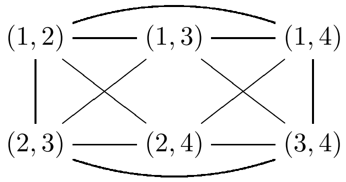
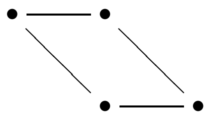



Figure 9: The graph $\Gamma_4$.

Obviously, the paths in $\Gamma_n$ can be interpreted as connected sequences as defined in the previous subsection. Then the equivalence relation on the connected sequences, defined by the operations (I)–(IV), has the following interpretation in terms of the graph $\Gamma_n$:

(I) the trivial path in each vertex is an idempotent;

(II) if the full subgraph of $\Gamma_n$ corresponding to a quadruple of vertices has the form



then the paths of length 2 in this subgraph with the same initial and the same terminal points are equivalent;

(III) for any triple $\{i, j\}$, $\{j, l\}$, $\{i, l\}$ of vertices the paths in the full subgraph $\Gamma_n$, corresponding to these vertices, with the same initial and the same terminal points are equivalent;

(IV) the path consisting of going along the same edge in two different directions coincides with the trivial path in the starting point.

These relations generate an equivalence relation on the set of all paths in $\Gamma_n$. From Proposition 17 it thus follows that the number of non-equivalent paths in $\Gamma_n$ equals $\frac{1}{4}n(n-1)n!$, and the number of non-equivalent loops at each point equals $(n-2)!$.

**5.3. The maximal length of an element from $\mathfrak{B}_n \setminus \mathcal{S}_n$.** For $w \in \mathfrak{B}_n \setminus \mathcal{S}_n$ we define the *length* $\mathrm{ls}(w)$ of $w$ as the length of the shortest possible presentation of $w$ as a product of the generators $\sigma_{i,j}$'s. For $w \in \mathcal{A}_n^+$ we define the *length* $\mathfrak{l}(w)$ of $w$ as the length of presentation of $w$ as a product of the generators $\tau_{i,j}$'s. The aim of this subsection is to prove the following statement about the maximal value $f(n)$ of $\mathrm{ls}(w)$ on $w \in \mathfrak{B}_n \setminus \mathcal{S}_n$.

**Theorem 18.** *Let $n \geqslant 2$. Then $f(n) = \lfloor \frac{3}{2}n \rfloor - 2$.*

For the proof of Theorem 18 we will need several auxiliary statements. Set $g(n) = \lfloor \frac{3}{2}n \rfloor - 2$. We will show that $f(n) \leqslant g(n)$. For $n \neq 3$ we will then find an element $w \in \mathcal{H}_{\sigma_{1,2}}$ such that $\mathrm{ls}(w) = g(n)$. For $n = 3$ we have $\mathrm{ls}(\sigma_{1,2}\sigma_{2,3}) = 2 = g(3)$. By Theorem 5 we have $\mathfrak{B}_n \setminus \mathcal{S}_n \simeq T$ and hence in the sequel we can work with the semigroup $T$ and the generators $\tau_{i,j}$'s. The function ls on $T$ is defined in the obvious way, and we consider all elements of $\mathcal{A}^+$ as elements of $T$ via the natural projection.

**Lemma 19.** *Let $u_i \in \{3, \dots, n\}$, $1 \leqslant i \leqslant k$. Then*

$$\tau_{1,2}\tau_{1,u_1}\dots\tau_{1,u_k}\tau_{1,2} = \tau_{1,2}\tau_{2,u_1}\dots\tau_{2,u_k}\tau_{1,2}.$$

P r o o f. Because of (4) we may assume that $u_i \neq u_{i+1}$ for all $i = 1, \dots, k-1$. We have

$$\begin{aligned}
\tau_{1,2}\tau_{1,u_1}\dots\tau_{1,u_k}\tau_{1,2} &= \tau_{1,2}\tau_{2,u_1}\tau_{1,u_1}\tau_{1,u_2}\dots\tau_{1,u_k}\tau_{1,2} \quad \text{(by (6))}\\
&= \tau_{1,2}\tau_{2,u_1}\tau_{2,u_2}\tau_{1,u_2}\dots\tau_{1,u_k}\tau_{1,2} \quad \text{(by (5))}\\
&\dots\\
&= \tau_{1,2}\tau_{2,u_1}\tau_{2,u_2}\dots\tau_{2,u_k}\tau_{1,u_k}\tau_{1,2} \quad \text{(by (5))}\\
&= \tau_{1,2}\tau_{2,u_1}\tau_{2,u_2}\dots\tau_{2,u_k}\tau_{1,2} \quad \text{(by (6))}.
\end{aligned}$$

□

**Corollary 20.** *Let $u_i \in \{2,\dots,n\}$, $1 \leqslant i \leqslant k$. Then there exist elements $v_i \in \{1,\dots,n\} \setminus \{2\}$, $1 \leqslant i \leqslant k$, such that*

$$w = \tau_{1,2}\tau_{1,u_1}\dots\tau_{1,u_k}\tau_{1,2} = \tau_{1,2}\tau_{2,v_1}\dots\tau_{2,v_k}\tau_{1,2}.$$

P r o o f. If $2 \notin \{u_1,\dots,u_k\}$ then the statement follows from Lemma 19. Otherwise let $u_{i_1} = u_{i_2} = \dots = u_{i_p} = 2$, $i_1 < i_2 < \dots < i_p$, be all occurrences of 2 among $u_1,\dots,u_k$. Set $u_0 = u_{k+1} = 2$, $i_0 = 0$, $i_{p+1} = k+1$. The statement now follows by applying Lemma 19 to each element $\tau_{1,u_{i_j}}\tau_{1,u_{i_j+1}}\dots\tau_{1,u_{i_{j+1}}}$, $j = 0,\dots,p$. □

**Lemma 21.** *Let $w \in \mathcal{A}^+$ be such that $w \in \mathcal{H}_1$. Assume that $\mathfrak{l}(w) = \mathrm{ls}(w) \geqslant 4$ and set $m = \mathfrak{l}(w) - 2$. Then there exist $u_i \in \{2,\dots,n\}$, $i = 1,\dots,m$, such that $w = \tau_{1,2}\tau_{1,u_1}\dots\tau_{1,u_m}\tau_{1,2}$.*

P r o o f. We use induction on $m = \mathfrak{l}(w)-2$. If $m = 2$, we have $w = \tau_{1,2}\tau_{a,b}\tau_{c,d}\tau_{1,2}$. Without loss of generality we may assume $a, d \in \{1,2\}$. If $a = d = 1$, we have nothing to prove. If $a = d = 2$, the statement follows from Lemma 19. If $a \neq d$, using (6) we see that $\mathrm{ls}(w) < 4$, a contradiction.

Now we prove the induction step $m-1 \Rightarrow m$. Let $w = \tau_{1,2}\tau_{i_1,j_1}\dots\tau_{i_m,j_m}\tau_{1,2}$. Without loss of generality we may assume $i_1 \in \{1,2\}$.

C a s e 1: $i_1 = 1$. If $1 \in \{i_l, j_l\}$ for all $l \leqslant m$, we have nothing to prove. Otherwise let $p < m$ be such that $1 \in \{i_l, j_l\}$ for all $l \leqslant p$ and $1 \notin \{i_{p+1}, j_{p+1}\}$. Thus, without loss of generality we may assume $i_l = 1$ for all $l \leqslant p$. If $j_p = 2$, the statement follows from the induction hypothesis.

Assume that $j_p \neq 2$. Without loss of generality we may write

$$w = \tau_{1,2}\tau_{1,j_1}\dots\tau_{1,j_p}\tau_{j_p,j_{p+1}}\dots\tau_{j_p,j_{p+q}}\tau_{j_{p+q},j_{p+q+1}}\dots\tau_{i_m,j_m}\tau_{1,2}.$$

Observe that $j_{p+q+1} \neq j_p$ since $\mathfrak{l}(w) = \mathrm{ls}(w)$.

Now we apply successively the relation (5) starting from $\tau_{j_p,j_{p+q}}$ and moving to the left until we reach the element $\tau_{j_p,j_{p+2}}$. We get

$$w = \tau_{1,2}\tau_{1,j_1}\dots\tau_{1,j_p}\tau_{j_p,j_{p+1}}\tau_{j_{p+1},j_{p+q+1}}\dots\tau_{j_{p+q-1},j_{p+q+1}}\tau_{j_{p+q},j_{p+q+1}}\dots\tau_{i_m,j_m}\tau_{1,2}.$$

If $j_{p+q+1} = 1$, then we can reduce the length of $w$ by (6), a contradiction. Otherwise, using (5) we can change $\tau_{j_p,j_{p+1}}$ to $\tau_{1,j_{p+q+1}}$. Since we have not changed the length of $w$, the proof in Case 1 is now completed by induction on $p$.

C a s e 2: $i_1 = 2$. Analogously to Case 1, we get existence of $u_i \in \{1,3,\dots,n\}$, $i = 1,\dots,m$, such that $w = \tau_{1,2}\tau_{2,u_1}\dots\tau_{2,u_m}\tau_{1,2}$. Now the claim follows from Corollary 20. □

Note that for every $w \in \mathcal{H}_1$ there exists a unique permutation $\pi \in \mathcal{S}_{n-2}$ such that

$$w = \big\{\{1,2\}, \{1',2'\}, \{k, \pi(k-2)'+2\}_{k\neq 1,2}\big\}. \tag{18}$$

Let $({i'}_1^{(1)}, \dots, {i'}_{p_1}^{(1)}), \dots, ({i'}_1^{(s)}, \dots, {i'}_{p_s}^{(s)})$ be a complete list of cycles of $\pi$ which have length at least 2. Set $i_a^{(b)} = {i'}_a^{(b)} + 2$ for all possible $a, b$. Then, by (2), we have the following decomposition of $w$:

$$w = \tau_{1,2}\tau_{1,i_1^{(1)}} \cdots \tau_{1,i_{p_1}^{(1)}}\tau_{1,2} \cdots \tau_{1,2}\tau_{1,i_1^{(s)}} \cdots \tau_{1,i_{p_s}^{(s)}}\tau_{1,2}.$$

We will call this decomposition a *cyclic* decomposition of $w$. We will also say that $\tau_{1,2}$ is the cyclic decomposition of $\tau_{1,2}$. In the obvious way we now define *cycles* in $\mathcal{H}_1$.

**Lemma 22.** *Let $w \in \mathcal{H}_1$ be a non-trivial cycle. Then $\mathrm{ls}(w)$ equals the length of the cyclic decomposition of $w$.*

Proof. By Lemma 21 there exist $u_1, \dots, u_{\mathrm{ls}(w)-2} \in \{2, \dots, n\}$ such that

$$w = \tau_{1,2}\tau_{1,u_1} \cdots \tau_{1,u_{\mathrm{ls}(w)-2}}\tau_{1,2}.$$

But then all elements from $\{3, \dots, n\}$ moved by the cycle $w$ should obviously occur among $u_1, \dots, u_{\mathrm{ls}(w)-2}$. The claim now follows from the formula (2). $\square$

Finally, for Theorem 5 we obtain for pairwise distinct $1, u, u_1, \dots, u_k$ and for any $l \in \mathbf{k}$ that

$$\tau_{1,u}\tau_{1,u_1} \cdots \tau_{1,u_k}\tau_{1,u} = \tau_{1,u}\tau_{1,u_l} \cdots \tau_{1,u_k}\tau_{1,u_1} \cdots \tau_{1,u_{l-1}}\tau_{1,u}. \tag{19}$$

**Lemma 23.** *Let $1, a, u, u_1, \dots, u_k$ be pairwise distinct. Then*

$$\tau_{1,a}\tau_{1,u}\tau_{1,u_1} \cdots \tau_{1,u_k}\tau_{1,u} = \tau_{1,a}\tau_{1,u_1} \cdots \tau_{1,u_k}\tau_{1,a}\tau_{1,u}.$$

Proof.

$$\begin{aligned}
\tau_{1,a}\tau_{1,u}\tau_{1,u_1} \cdots \tau_{1,u_k}\tau_{1,u} &= (\text{by } (7)) \\
\tau_{1,a}\tau_{1,u}\tau_{1,u_1} \cdots \tau_{1,u_k}\tau_{1,a}\tau_{1,u_k}\tau_{1,u} &= (\text{by } (19)) \\
\tau_{1,a}\tau_{1,u_1} \cdots \tau_{1,u_k}\tau_{1,u}\tau_{1,a}\tau_{1,u_k}\tau_{1,u} &= (\text{by } (19)) \\
\tau_{1,a}\tau_{1,u_1} \cdots \tau_{1,u_k}\tau_{1,u}\tau_{1,u_k}\tau_{1,a}\tau_{1,u} &= (\text{by } (7)) \\
\tau_{1,a}\tau_{1,u_1} \cdots \tau_{1,u_k}\tau_{1,a}\tau_{1,u}.&
\end{aligned}$$

$\square$

**Lemma 24.** *Let $w \in \mathcal{H}_1$. Then the cyclic decomposition of $w$ is of length* $\operatorname{ls}(w)$.

Proof. We use induction on $\operatorname{ls}(w)$. If $\operatorname{ls}(w) \leqslant 3$ then the statement is trivial since, by (4) and (7), the only possibility is $w = \tau_{1,2}$. Let us now prove the induction step $m+1 \Rightarrow m+2$.

Let $w \in \mathcal{H}_1$ be such that $\operatorname{ls}(w) = m+2$. By Lemma 21 we may write $w = \tau_{1,2}\tau_{1,u_1}\dots\tau_{1,u_m}\tau_{1,2}$ for some $u_i \in \{2,\dots,n\}$, $i = 1,\dots,m$. We set $u_0 = u_{m+1} = 2$. If all of $u_i$'s are pairwise distinct, the word $w$ is a cycle and the statement follows. Suppose now that there are some repetitions among $u_0, u_1, \dots, u_m$. Take the leftmost element which repeats in this series, say $u_i = u$. Let $j > i$ be the minimal possible such that $u_j = u$. Consider the element

$$w' = \tau_{1,u_i}\dots\tau_{1,u_j} = \tau_{1,u}\tau_{1,u_{i+1}}\dots\tau_{1,u_{j-1}}\tau_{1,u} \in \mathcal{H}_{\tau_{1,u}}.$$

Since $\operatorname{ls}(w) = m+2$, $\operatorname{ls}(w') = j-i+1 < m+2$. Hence, using the induction hypothesis, the cyclic decomposition of $w'$ has length $j-i+1$. Without loss of generality we hence may assume that the subword $w'$ of $w$ already coincides with the corresponding cyclic decomposition, that is, it is a cycle.

Now we claim that $u_{i-1}, u_i, \dots, u_{j-1}$ are pairwise distinct. Indeed, if not, then $u_{i-1}$ coincides with one of $u_{i+1}, \dots, u_{j-1}$. Then applying (19) we can obtain the fragment $\tau_{1,u_{i-1}}\tau_{1,u_i}\tau_{1,u_{i-1}}$ which can be shortened by (7), a contradiction. Hence Lemma 23 gives

$$\begin{aligned}\tau_{1,u_{i-1}}\tau_{1,u_i}\tau_{1,u_{i+1}}\dots\tau_{1,u_{j-2}}\tau_{1,u_{j-1}}\tau_{1,u_i}\tau_{1,u_{j+1}}\\ = \tau_{1,u_{i-1}}\tau_{1,u_{i+1}}\tau_{1,u_{i+2}}\dots\tau_{1,u_{j-1}}\tau_{1,u_{i-1}}\tau_{1,u_i}\tau_{1,u_{j+1}}.\end{aligned}$$

This operation makes the index of the first letter with repetition smaller. Hence, applying this procedure as many times as necessary, we may assume that $i = 0$. This means that $w$ is a product of a cycle with some element $v$ from $\mathcal{H}_1$ of strictly smaller length. By induction hypothesis, we may assume that $v$ is written in its cyclic decomposition. We are left to prove that none of the elements $u_1, \dots, u_{j-1}$ occurs among cycles in $v$. Assume that some of these elements does occur. Then, using (19), we may assume that this is $u_{j-1}$. At the same time, the cycles of any cyclic decomposition commute and hence using this and (19) we may assume that $u_{j-1} = u_{j+1}$. In this case we can make $w$ shorter by applying (7), a contradiction. This completes the proof. □

**Proposition 25.** *Let $w \in \mathcal{H}_1$. Then $\mathrm{ls}(w) \leqslant g(n)$. Moreover, if $n \geqslant 4$ then there exists $v \in \mathcal{H}_1$ such that $\mathrm{ls}(v) = g(n)$.*

Proof. If $w = \tau_{1,2}$ or $n \leqslant 3$ then the statement is obvious. Suppose now that $w \neq \tau_{1,2}$ and $n \geqslant 4$. Let $\pi \in \mathcal{S}_{n-2}$ be the permutation which corresponds to $w$ by (18). Let $c$ and $s$ be the number of non-trivial and trivial cycles in $\pi$ respectively. From Lemma 24 it follows that $\mathrm{ls}(w) = (n-2) - s + c + 1$.

Case 1: $n = 2k$, $k \in \mathbb{N}$. Then $\mathrm{ls}(w) = (2k-1) + c - s \leqslant (2k-1) + \frac{1}{2}(2k-2) = 3k-2 = g(n)$ and the equality holds if and only if $\pi$ contains $k-1$ transpositions.

Case 2: $n = 2k+1$, $k \in \mathbb{N}$. If $s = 0$ then there should exist a cycle in $\pi$ of length at least 3. Then $\mathrm{ls}(w) = 2k + c \leqslant 2k + \frac{1}{2}(n-2-3) + 1 = 3k-1 = g(n)$ and the equality holds if and only if $\pi$ contains one cycle of length 3 and $k-2$ transpositions. If $s \geqslant 1$ then $\mathrm{ls}(w) = 2k + c - s \leqslant 2k - 1 + c \leqslant 2k - 1 + \frac{1}{2}(n-2-1) = 3k-2 < g(n)$. The proof is complete. $\square$

**Lemma 26.** *Let $i_1, j_1, \ldots, i_k, j_k$ be pairwise distinct elements from $\mathbf{n}$. Then there exists a word $\mu \in \mathcal{A}^+$ such that $\mathfrak{l}(\mu) \leqslant 2k$,*

$$\tau_{1,2} \ldots \tau_{2k-1,2k}\mu \in \mathcal{L}_{\tau_{i_1,j_1} \ldots \tau_{i_k,j_k}},$$

*and $\{m, m'\} \in \varphi(\mu)$, $m \in \mathbf{n} \setminus \{i_1, j_1, \ldots, i_k, j_k, 1, 2, \ldots, 2k-1, 2k\}$.*

Proof. For $a, b, c, d \in \mathbf{n}$, $a \neq b$, $c \neq d$, set

$$\mu_{a,b,c,d} = \begin{cases} \tau_{a,c}\tau_{c,d} & \text{if } \{a,b\} \cap \{c,d\} = \emptyset, \\ \tau_{c,d} & \text{otherwise.} \end{cases}$$

Note that $\{m, m'\} \in \varphi(\mu_{a,b,c,d})$ provided that $m \neq a, b, c, d$.

Now direct calculation implies that $\tau_{a,b}\mu_{a,b,c,d}\mu_{c,d,a,b} = \tau_{a,b}$ for all $a, b, c, d$ such that $a \neq b$ and $c \neq d$. In particular, it follows that for any $w \in T$ such that $w\tau_{a,b} = w$ we have that the coranks of the elements $\varphi(w)$ and $\varphi(w\mu_{a,b,c,d})$ coincide.

In particular, the element $\varphi(\mu_1)$, where $\mu_1 = \tau_{1,2} \ldots \tau_{2k-1,2k}\mu_{1,2,i_1,j_1}$, has corank $2k$. Further, $\mu_1$ satisfies $\mu_1\tau_{i_1,j_1} = \mu_1$ by the definition of $\mu_{1,2,i_1,j_1}$. Hence there exist pairwise distinct $a_1, b_1, \ldots, a_{k-1}, b_{k-1}$ from $\mathbf{n} \setminus \{i_1, j_1\}$ such that

$$\mu_1 \in \mathcal{L}_{\tau_{i_1,j_1}\tau_{a_1,b_1} \ldots \tau_{a_{k-1},b_{k-1}}}.$$

Analogously, the element $\varphi(\mu_2)$, where $\mu_2 = \mu_1\mu_{a_1,b_1,i_2,j_2}$, also has corank $2k$. The element $\mu_2$ satisfies $\mu_2\tau_{i_1,j_1} = \mu_2$ and $\mu_2\tau_{i_2,j_2} = \mu_2$ by the definition of $\mu_{a_1,b_1,i_2,j_2}$. Hence there exist pairwise distinct $c_1, d_1, \ldots, c_{k-2}, d_{k-1}$ from $\mathbf{n} \setminus \{i_1, j_1, i_2, j_2\}$ such that

$$\mu_2 \in \mathcal{L}_{\tau_{i_1,j_1}\tau_{i_2,j_2}\tau_{c_1,d_1} \ldots \tau_{c_{k-2},d_{k-2}}}.$$

Continuing this process for $k-2$ more steps we will construct the element $\mu_k$ with the desired properties. $\square$

P r o o f o f T h e o r e m 1 8. Let $n \geqslant 4$. Then, by Proposition 25, there is $w \in T$ such that $\mathrm{ls}(w) = g(n)$. For $n = 2, 3$ an example of $w \in T$ such that $\mathrm{ls}(w) = g(n)$ was constructed immediately after the formulation of Theorem 18. Hence we are left to show that for any $w \in T$ we have $\mathrm{ls}(w) \leqslant g(n)$. Without loss of generality it is even enough to consider those $w$ for which $\tau_{1,2}w = w$.

Let now $w \in T$ be such that $\tau_{1,2}w = w$. Assume first that $\mathrm{corank}(\varphi(w)) = 2$. Then there exists a unique $\{i, j\}$, $i \neq j$, such that $w\tau_{i,j} = w$. Without loss of generality we have one of the following cases:

C a s e 1: $\{i, j\} = \{1, 2\}$. Then the statement follows from Proposition 25.

C a s e 2: $i = 1$ and $j \neq 2$. Then, applying (4) and (7), we obtain $w = w\tau_{1,j}\tau_{1,2}\tau_{1,j}$. Setting $w' = w\tau_{1,j}\tau_{1,2} = w\tau_{1,2}$ we have $w = w'\tau_{1,j}$ by (7). It follows that $w'\mathcal{H}\tau_{1,2}$. Consider the cyclic decomposition of $w'$. Assume that the cycles occurring in this decomposition do not move the element $j$. Then, by Lemma 24 and Proposition 25, we have that the length of this decomposition is at most $g(n-1)$. Since $w = w'\tau_{1,j}$, we have $\mathrm{ls}(w) \leqslant 1 + g(n-1) \leqslant g(n)$. Assume now that there is a cycle in $w'$ which moves $j$. Using (19) and the fact that the cycles in the cyclic decomposition commute, we may write $w' = w''\tau_{1,j}\tau_{1,2}$ for some $w''$ such that $\mathrm{ls}(w'') = \mathrm{ls}(w') - 2$. Since $w = w'\tau_{1,j}$, we have $w = w''\tau_{1,j}$ by (7), and thus $\mathrm{ls}(w) < \mathrm{ls}(w') \leqslant g(n)$.

C a s e 3: $\{i, j\} \cap \{1, 2\} = \emptyset$. Then we can write $w = w'\tau_{1,i}\tau_{i,j}$, $w' \in \mathcal{H}_{\tau_{1,2}}$. Consider again the cyclic decomposition of $w'$. If neither $i$ nor $j$ are moved by all the cycles, we have $\mathrm{ls}(w) \leqslant 2 + g(n-2) \leqslant g(n)$. If $i$ is moved, then, as in Case 2, we can write $w' = w''\tau_{1,i}\tau_{1,2}$ for some $w''$ such that $\mathrm{ls}(w'') = \mathrm{ls}(w') - 2$ and, using (7), we obtain $\mathrm{ls}(w) \leqslant \mathrm{ls}(w'') + 2 = \mathrm{ls}(w') \leqslant g(n)$. Finally, let us assume that $i$ is not moved but $j$ is. Assume that $x_1, \dots, x_p, j$, $p > 0$, is a cycle in the cyclic decomposition of $w'$. Then, using (19) and the fact that the cycles in the cyclic decomposition commute, we may assume that this cyclic decomposition has the form

$$\tau_{1,2} \dots \tau_{1,2}\tau_{1,x_1} \dots \tau_{1,x_p}\tau_{1,j}\tau_{1,2}.$$

Now we can compute the following expression containing the last cycle of this decomposition:

$$\begin{aligned}
\tau_{1,2}\tau_{1,x_1} \dots \tau_{1,x_p}\tau_{1,j}\tau_{1,2}\tau_{1,i}\tau_{i,j} &= (\text{by } (5))\\
\tau_{1,2}\tau_{1,x_1} \dots \tau_{1,x_p}\tau_{1,j}\tau_{1,2}\tau_{2,j}\tau_{i,j} &= (\text{by } (6))\\
\tau_{1,2}\tau_{1,x_1} \dots \tau_{1,x_p}\tau_{1,j}\tau_{2,j}\tau_{i,j} &= (\text{by } (5))\\
\tau_{1,2}\tau_{1,x_1} \dots \tau_{1,x_p}\tau_{2,x_p}\tau_{2,j}\tau_{i,j} &= (\text{by } (5))\\
&\dots
\end{aligned}$$

$$\tau_{1,2}\tau_{1,x_1}\tau_{2,x_1}\dots\tau_{2,x_{p-1}}\tau_{2,x_p}\tau_{2,j}\tau_{i,j} = (\text{by } (6))$$
$$\tau_{1,2}\tau_{2,x_1}\dots\tau_{2,x_{p-1}}\tau_{2,x_p}\tau_{2,j}\tau_{i,j}.$$

It follows that $\mathrm{ls}(w) \leqslant \mathrm{ls}(w') \leqslant g(n)$.

Assume now that $\mathrm{corank}(\varphi(w)) = 2k$, $k > 1$. We may further assume that $\tau_{1,2}\dots\tau_{2k-1,2k}w = w$ and $w\tau_{i_1,j_1}\dots\tau_{i_k,j_k} = w$ for some pairwise distinct elements $i_1, j_1, \dots, i_k, j_k$. Without loss of generality we may assume that we have one of the following cases:

Case 1: $\{i_1, j_1, \dots, i_k, j_k\} = \{1, \dots, 2k\}$. Since the map

$$\begin{aligned}\mathcal{H}_{\tau_{1,2}\dots\tau_{2k-1,2k}} &\to \mathcal{H}_{\tau_{i_1,j_1}\dots\tau_{i_k,j_k}}\\ x &\mapsto x\tau_{i_1,j_1}\dots\tau_{i_k,j_k}\end{aligned}$$

is obviously a bijection, there exists $w' \in \mathcal{H}_{\tau_{1,2}\dots\tau_{2k-1,2k}}$ for which we have $w = w'\tau_{i_1,j_1}\dots\tau_{i_k,j_k}$. From the definition of $g$ we have $g(n+2p) = g(n) + 3p$ for all positive integers $n$ and $p$. Hence

$$\mathrm{ls}(w) \leqslant k + \mathrm{ls}(w') \leqslant k + (k-1) + g(n - 2(k-1)) = g(n) + 2 - k \leqslant g(n).$$

Case 2: $|\{i_k, j_k\} \cap \{1, \dots, 2k\}| \leqslant 1$. Without loss of generality we may also further assume $\{i_k, j_k\} \cap \{1, \dots, 2k-2\} = \emptyset$. Then, using Lemma 26, we see that there exists $\mu$ such that $\mathfrak{l}(\mu) \leqslant 2(k-1)$ and

$$\tau_{1,2}\dots\tau_{2k-3,2k-2}\tau_{i_k,j_k}\mu\mathcal{L}\tau_{i_1,j_1}\dots\tau_{i_k,j_k}.$$

This implies that the map

$$\begin{aligned}\mathcal{L}_{\tau_{1,2}\dots\tau_{2k-3,2k-2}\tau_{i_k,j_k}} &\to \mathcal{L}_{\tau_{i_1,j_1}\dots\tau_{i_k,j_k}}\\ x &\mapsto x\mu\end{aligned}$$

is a bijection. In particular, there exists $v \in \mathcal{L}_{\tau_{1,2}\dots\tau_{2k-3,2k-2}\tau_{i_k,j_k}}$ such that $v\mu = w$. Since $\tau_{1,2}\dots\tau_{2k-1,2k}w = w$, it follows that $\tau_{1,2}\dots\tau_{2k-1,2k}v = v$ (because of $\mathrm{corank}(\varphi(v)) = \mathrm{corank}(\varphi(w))$). Hence $v \in \mathcal{R}_{\tau_{1,2}\dots\tau_{2k-1,2k}}$.

From the above we derive $\tau_{2a-1,2a}v = v\tau_{2a-1,2a} = v$ for all $a = 1, \dots, k-1$. Hence we can write $v = \tau_{1,2}\dots\tau_{2k-3,2k-2}v'$, where $v'$ is such that $\mathrm{corank}(\varphi(v')) = 2$ and $\{m, m'\} \in \varphi(v')$ for all $m \leqslant 2k-2$. Using induction on $n$ and the case of corank 2 considered above, we obtain $\mathrm{ls}(v') \leqslant g(n - 2(k-1))$. Hence $\mathrm{ls}(v) \leqslant (k-1) + g(n - 2(k-1))$ and from $w = v\mu$ we get

$$\mathrm{ls}(w) \leqslant \mathrm{ls}(v) + \mathrm{ls}(\mu) \leqslant (k-1) + g(n - 2(k-1)) + 2(k-1) = g(n).$$

This completes the proof. $\square$

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