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NONLINEAR ELECTROKINETICS IN NEMATIC ELECTROLYTES

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To Alex Mielke, with friendship and admiration

Abstract. In this article we study a system of nonlinear PDEs modelling the electrokinetics of a nematic electrolyte material consisting of various ions species contained in a nematic liquid crystal.

The evolution is described by a system coupling a Nernst-Planck system for the ions concentrations with a Maxwell's equation of electrostatics governing the evolution of the electrostatic potential, a Navier-Stokes equation for the velocity field, and a non-smooth Allen-Cahn type equation for the nematic director field.

We focus on the two-species case and prove apriori estimates that provide a weak sequential stability result, the main step towards proving the existence of weak solutions.

1. **Introduction.** In this paper we consider a version of the system derived in [2, (2.51)-(2.55)] describing the electrokinetics of a nematic electrolyte that consists of ions that diffuse and advect in a nematic liquid crystal environment.

The system can be written in terms of the following variables:

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- the vector n modelling the local orientation of the nematic liquid crystal molecules,
- the macroscopic velocity v of the liquid crystal molecules,
- the pressure p resulting from the incompressibility constraint on the fluid,
- the electrostatic potential Φ,
- the concentrations c_k, $k = 1, \dots, N$, with valences $z_k \in \{-1, 1\}$, of the families of charged ions present in the liquid crystal.

Actually, we consider a modified version of the system in [2], assuming certain simplifications commonly used in the mathematical literature on liquid crystals. More specifically we take equal elastic constants in the Oseen-Frank energy and use a Ginzburg-Landau configuration potential $\mathscr{F}$ of *singular* type (see below for more details) in order to avoid introducing the unit length constraint (cf. equation (2.56) of [2]) on n (and thus we can correspondingly drop the related Lagrange multiplier term λn in the system in [2]). Furthermore neglecting body forces and inertial effects acting on the director field, we can write the resulting PDE system as follows:

$$\frac{\partial c_k}{\partial t} + v \cdot \nabla c_k = \frac{1}{k_B \theta} \operatorname{div} \left(c_k \mathcal{D}_k \nabla \mu_k \right), \quad \text{for } k = 1, \dots, N, \tag{1.1}$$

$$-\operatorname{div}(\varepsilon_0 \varepsilon(n) \nabla \Phi) = \sum_{k=1}^{N} q z_k c_k, \tag{1.2}$$

$$\begin{aligned} \frac{\partial v}{\partial t} + (v \cdot \nabla) v + \nabla p = -K \operatorname{div}(\nabla n \odot \nabla n) + \operatorname{div} \sigma, \\ + \varepsilon_0 \operatorname{div} \big((\nabla \Phi \otimes \nabla \Phi) \varepsilon(n) \big), \end{aligned} \tag{1.3}$$

$$\operatorname{div} v = 0, \tag{1.4}$$

$$\gamma_1 (n_t + v \cdot \nabla n - \Omega(v) n) + \gamma_2 D(v) n = K \Delta n + \varepsilon_0 \varepsilon_a \left(\nabla \Phi \otimes \nabla \Phi \right) n - \partial \mathscr{F}, \tag{1.5}$$

where μ_k are the electrochemical potentials of the ions associated to the various ions species c_k, given by

$$\mu_k := k_B \theta (\ln(c_k) + 1) + q z_k \Phi, \tag{1.6}$$

$k_B > 0$ denotes the Boltzmann constant, $\theta > 0$ stands for the absolute temperature, and q denotes the elementary charge.

Moreover, we have indicated by

$$D(v) := \frac{1}{2} (\nabla v + \nabla v^t) \quad \text{and} \quad \Omega(v) := \frac{1}{2} (\nabla v - \nabla v^t) \tag{1.7}$$

the symmetric and antisymmetric parts of the velocity gradient. The diffusion operator in (1.2) is ruled by the matrix

$$\varepsilon(n) := \varepsilon_\perp \mathrm{Id} + \varepsilon_a n \otimes n, \tag{1.8}$$

with constants $\varepsilon_\perp > 0$ and $\varepsilon_a \geq 0$, Id denoting the identity matrix. Here $\varepsilon_a = \varepsilon_{||} - \varepsilon_\perp$, where $\varepsilon_{||}$ and $\varepsilon_\perp$ denote the electric permittivity when the electric field $\mathbf{E} = \nabla \Phi$ is parallel, respectively, perpendicular to n.

The constant $\varepsilon_0 > 0$ stands for the vacuum dielectric permeability. The matrices $\mathcal{D}_k$ are positive definite, i.e.,

$$(\mathcal{D}_k \xi) \cdot \xi > \alpha |\xi|^2 \tag{1.9}$$

for some $\alpha > 0$ and all $k = 1, \dots, N$ and $\xi \in \mathbb{R}^3$. In the above we have denoted by $\nabla n \odot \nabla n$ the 3×3 matrix whose (i, j)-component is $n_{k,i} n_{k,j}$ (here and in the sequel we assume summation over repeated indices). As customary, for $a, b \in \mathbb{R}^3$ we denote as $a \otimes b$ the 3×3 matrix with component (i, j) given by $a_i b_j$. We will further assume that the system is non-dimensionalized, so the constants are dimensionless (this can be achieved similarly as in Section 3.2 in [2]).

The Nernst-Planck type equations (1.1) correspond to the continuity equation for ions with the electric potential Φ satisfying the Maxwell's equation of electrostatics (1.2).

The Navier-Stokes equations (1.3), with the incompressibility constraint (1.4), rule the evolution of the liquid crystal flow. Note the Korteweg forces on the right-hand side being induced by the the director field n and the effects of the electric field, respectively. As in [15], we assume for the total stress tensor the following general expression:

$$\sigma = \alpha_1(D(v)n \cdot n)n \otimes n + \alpha_2 \mathring{n} \otimes n + \alpha_3 n \otimes \mathring{n} + \alpha_4 D(v) + \alpha_5 D(v)n \otimes n + \alpha_6 n \otimes D(v)n, \qquad (1.10)$$

where we have denoted $\mathring{n} := \partial_t n + v \cdot \nabla n - \Omega(v)n$ the Lie derivative of n. Here the term $\alpha_4 D(v)$ represents the classical Newtonian stress tensor, while the other terms represent the additional stress produced by the interaction of the anisotropic liquid crystal molecules, see [8, 9].

As mentioned above, we avoid to insert the unit length constraint in (1.5) and instead require $|n| \leq 1$, in the spirit of the the variable length model proposed by J. L. Ericksen in [10]. Indeed, following an approach commonly used in the context of phase-transition models, we enforce the property $|n| \leq 1$ by means of the *singular potential* $\mathscr{F}$. Namely, we assume $\mathscr{F} : \mathbb{R}^3 \to [0, +\infty]$ be a convex and lower semicontinuous function whose *effective domain* (i.e., the set where it attains finite values) is assumed to coincide with the closed unit ball $\overline{B}_1$ of $\mathbb{R}^3$, with a reference choice being given by

$$\mathscr{F}(n) = \frac{1}{2} F(|n|^2), \qquad (1.11)$$

where F is convex and has the interval $(-\infty, 1]$ as an effective domain. We will actually choose $F(r) = (1 - r)\log(1 - r)$, an expression mutuated from the Cahn-Hilliard logarithmic potential, but we point out that more general choices may be allowed.

Such an idea was introduced by J.L. Ericksen in [10] in order to enforce the physicality of a scalar order parameter and has already been applied to liquid crystal models in a number of papers (cf., e.g., [11, 12]) and has the advantage that as soon as we have proved existence of a solution, then the constraint $|n| \leq 1$ is authomatically satisfied. This helps in the estimates which actually could not be performed in this way in the case of a *classical* double-well potential.

Finally, in order to avoid complications due to the interaction with the boundary, we will settle the above system on the flat 3-dimensional torus

$$\mathcal{T}^3 = \left([-\pi, \pi]|_{\{-\pi, \pi\}} \right)^3 \qquad (1.12)$$

so assuming periodic boundary conditions. We note that more realistic choices for the boundary conditions could be likely taken. Nevertheless the above setting, beyond being the simplest one mathematically, is also consistent with the basic physical principles of conservation of charge and of momentum (indeed, we assume no external forces be present), that can be verified respectively by integrating (1.1) and (1.3) with respect to space variables.

Our main aim here is to set the ground for proving the existence of weak solutions. These are usually obtained via three steps: *'apriori estimates'*, *'approximation scheme'*, and *'compactness'*.

The apriori estimates are obtained on *presumptive* smooth solutions of the equation. Such estimates allow to control (in terms of initial data and fixed parameters of the system) certain norms, sufficiently strong, in order to allow to pass to the limit in the approximation scheme.

The approximation scheme is usually designed such that one can obtain estimates for the approximating equations that are usually very close to the apriori estimates. The construction of such a scheme can be a highly tedious and non-trivial issue in presence of complex systems as we consider (see for comparison our previous works on non-isothermal liquid crystals, with an approximation scheme [11] and without one, just with apriori estimates as in here [12]). Thus we will leave the construction of such a scheme to interested readers and focus just on the first part, namely obtaining apriori estimates that are strong enough in order to allow to pass to the limit in the approximation scheme via compactness and we will refer to this as 'weak sequential stability', the main content of Theorem 1.

In addition to that, we will focus on a simplified version of system (1.1)-(1.5), complemented with the Cauchy conditions and with periodic boundary conditions in three dimensions of space and with no restrictions on the magnitude of the initial data. The precise simplifications will be introduced in the next section, but it is worth observing that, beyond setting some physical constants equal to one, the only effective reduction we are actually going to operate concerns the number of species c_k which will be assumed to be equal to 2. Namely, we only take two species c_p and c_m, which will then denote the density of positive and negative charges, respectively. Mathematically speaking, this ansatz simplifies the nature of the system (1.1)-(1.2), and in particular permits us to prove by means of very simple maximum principle arguments the uniform boundedness of c_p and c_m, which is a key ingredient for obtaining the apriori estimates.

It is worth noting that we expect the same boundedness property to hold also in the general case of N-species, however the proof may be much more involved and require use of more technical results about invariant regions for evolutionary systems (see, e.g., [5]). We also expect that similar arguments could be applied in the more complicated systems where one uses a tensorial order parameter, that is a matrix valued function, i.e. a Q-tensor in the LC terminology, instead of the vector-valued one, n, as done for instance in [3]. The current work is related to work done in certain simpler systems that can be regarded as subsets of our equations, such as Nernst-Planck-Navier-Stokes system (see for instance [7] and the references therein) and liquid crystal equations (see for instance the review [16]).

The main ingredients of the proofs are the following: first we perform an energy estimate which is mainly based on a key Lemma (cf. Lemma 1) providing sufficient conditions on the α_i-coefficients such that the dissipation is non-negative. Then, via a maximum-principle technique, we prove pointwise bounds for c_p and c_m. The L^∞-estimate on the potential Φ follows instead by a Moser-iteration scheme proved in Lemma 2, while in Lemma 3 we state an L^p-regularity result for n. This result, based on an L^p-estimate for the potential $\partial\mathscr{F}$, is in general new in the framework of non-smooth parabolic systems, while it is quite known in case of scalar equations (cf., e.g., [6]). Finally, an additional regularity result for n (cf. Lemma 4) is shown in case the anisotropy coefficient ε is sufficiently small. In the last Section 4 the weak sequential stability property result is proved for every $\varepsilon > 0$.

The plan of the paper is as follows: in the next section 2 we introduce the simplified version of system (1.1)-(1.5) and state the precise formulation of our existence theorem. Then, the basic apriori estimates are derived in Section 3.

Finally, in Section 4 we will prove the stability result.

2. **Main results.**

We start introducing some notation. Given a space of functions defined over $\Omega = \mathcal{T}^3$, we will always use the same notation for scalar-, vector-, or tensor-valued function. For instance, we will indicate by the same letter H the spaces $L^2(\Omega)$, $L^2(\Omega)^3$ and $L^2(\Omega)^{3\times 3}$. Correspondingly, the norm in H will be simply denoted by $\|\cdot\|$. The notation actually subsumes the periodic boundary conditions. We also set $V = H^1(\Omega)$ (or $H^1(\Omega)^3$, or $H^1(\Omega)^{3\times 3}$). For two 3×3 matrices A, B, we also set $A : B := A_{ij}B_{ij}$.

In view of the discussion carried out above, we now introduce the simplified system for which we shall prove existence of weak solutions. Namely, we assume $\varepsilon_\perp = k_B\theta = K = \varepsilon_0 = q = \gamma_1 = \gamma_2 = 1$ and write ε in place of ε_a. Moreover, we only take two species c_p and c_m with $z_p = 1$ and $z_m = -1$.

Moreover we take, similarly in spirit as in [2], Section 3.1, the matrices $\mathcal{D}_p = \mathcal{D}_k = \mathrm{Id} + \varepsilon n \otimes n$.[1] Then the simplified system takes the form

$$\frac{\partial c_p}{\partial t} + v \cdot \nabla c_p = \operatorname{div}\big((\mathrm{Id} + \varepsilon n \otimes n)(\nabla c_p + c_p \nabla \Phi)\big), \tag{2.1}$$

$$\frac{\partial c_m}{\partial t} + v \cdot \nabla c_m = \operatorname{div}\big((\mathrm{Id} + \varepsilon n \otimes n)(\nabla c_m - c_m \nabla \Phi)\big), \tag{2.2}$$

$$-\operatorname{div}\big((\mathrm{Id} + \varepsilon n \otimes n)\nabla \Phi\big) = c_p - c_m, \tag{2.3}$$

$$\begin{aligned} \frac{\partial v}{\partial t} + (v \cdot \nabla) v + \nabla p = {} & \alpha_4 \operatorname{div} D(v) - \operatorname{div}(\nabla n \odot \nabla n) \\ & + \operatorname{div}\big((\nabla \Phi \otimes \nabla \Phi)(\mathrm{Id} + \varepsilon n \otimes n)\big) \\ & + \operatorname{div}\big(\alpha_1 (D(v) n \cdot n) n \otimes n + \alpha_2 \mathring{n} \otimes n + \alpha_3 n \otimes \mathring{n}\big) \\ & + \operatorname{div}\big(\alpha_5 D(v) n \otimes n + \alpha_6 n \otimes D(v) n\big), \end{aligned} \tag{2.4}$$

$$\operatorname{div} v = 0, \tag{2.5}$$

$$n_t + v \cdot \nabla n - \Omega(v) n + D(v) n = \Delta n + \varepsilon \left(\nabla \Phi \otimes \nabla \Phi\right) n - \partial \mathscr{F}(n). \tag{2.6}$$

Note that $\partial \mathscr{F}$ denotes the *subdifferential* of $\mathscr{F}$ in the sense of convex analysis. Although one can use more general assumptions on the potential here we are assuming for definiteness that

$$\mathscr{F}(n) := \begin{cases} \frac{1}{2} F(|n|^2) - F_*, & \text{if } |n| \le 1 \\ +\infty, & \text{otherwise} \end{cases} \tag{2.7}$$

where

$$F(r) := (1-r)\log(1-r) - F_*, \ \ r \in (0,1), \tag{2.8}$$

and F_* is chosen such that $\min F(r) = F(1 - 1/e) = 0$.

Moreover, in order to prove the energy estimate (cf. Lemma 1), let us suppose that there exists $\delta > 0$ such that

$$\alpha_4 > 0, \quad \alpha_4 - |\alpha_1| - |\alpha_5| - |\alpha_6| - \frac{1}{1-\delta} > 0. \tag{2.9}$$

Finally, we assume the initial data to satisfy the following conditions, where $\bar{c} > 0$ is a given constant:

$$c_{p,0}, c_{m,0} \in L^\infty(\mathcal{T}^3), \quad 0 \le c_{p,0}, c_{m,0} \le \bar{c} \ \text{ a.e. in } \ \mathcal{T}^3, \tag{2.10}$$

$$v_0 \in L^2(\mathcal{T}^3), \quad \operatorname{div} v_0 = 0, \tag{2.11}$$

$$n_0 \in H^1(\mathcal{T}^3), \quad |n_0(x)| \le 1, \forall x \in \mathcal{T}^3, \tag{2.12}$$

Let us now define the weak solutions, in a rather standard way, but emphasizing the spaces of functions used.

[1] This simplification is not necessary for obtaining the energy law in Proposition 1, but essential in deriving the maximum principle in Proposition 2

Definition 1. *[Weak solutions] Assume hypotheses (2.8), (2.10)–(2.12). Then, the functions*

$$v \in L^\infty(0,T;H) \cap L^2(0,T;V), \tag{2.13}$$

$$n \in W^{1,p_0}(0,T;L^{p_0}(\mathcal{T}^3)) \cap L^{p_0}(0,T;W^{2,p_0}(\mathcal{T}^3)) \cap L^\infty(0,T;V) \cap L^\infty((0,T)\times\mathcal{T}^3), \tag{2.14}$$

$$\mathscr{F}(n) \in L^{p_0}(0,T;L^{p_0}(\mathcal{T}^3)) \quad \textit{for some } p_0 > 1, \tag{2.15}$$

$$\Phi \in L^\infty(0,T;V) \cap L^\infty(0,T;L^\infty(\mathcal{T}^3)) \cap L^\infty(0,T;W^{1,p_M}(\mathcal{T}^3)) \quad \textit{for some } p_M > 2, \tag{2.16}$$

$$c_p,\ c_m \in W^{1,4/3}(0,T;V') \cap L^2(0,T;V) \cap L^\infty(0,T;L^\infty(\mathcal{T}^3)), \tag{2.17}$$

$$c_p,\, c_m \geq 0 \ \textit{a.e. in } \mathcal{T}^3 \times (0,T), \tag{2.18}$$

are a weak solution of (2.1)–(2.5) provided that

$$\int_0^T \left(< \frac{\partial c_p}{\partial t}, \phi_p > - \int_\Omega v c_p \cdot \nabla \phi_p \right) = \int_0^T \int_\Omega \big((\mathrm{Id} + \varepsilon n \otimes n)(\nabla c_p + c_p \nabla \Phi) \big) \nabla \phi_p, \tag{2.19}$$

$$\int_0^T \left(< \frac{\partial c_m}{\partial t}, \phi_m > - \int_\Omega v c_m \cdot \nabla \phi_m \, dx \right) = \int_0^T \int_\Omega \big((\mathrm{Id} + \varepsilon n \otimes n)(\nabla c_m - c_m \nabla \Phi) \big) \nabla \phi_m, \tag{2.20}$$

$$\int_0^T \int_\Omega \big((\mathrm{Id} + \varepsilon n \otimes n) \nabla \Phi \big) : \nabla u = \int_0^T \int_\Omega (c_p - c_m) u, \tag{2.21}$$

$$\begin{aligned}\int_0^T \int_\Omega v \frac{\partial z}{\partial t} + (v \otimes v) : \nabla z = - \int_\Omega v_0 z(0)\, dx + \int_0^T \int_\Omega \sigma : \nabla z - (\nabla n \odot \nabla n) : \nabla z \\ + \int_0^T \int_\Omega \big((\nabla \Phi \otimes \nabla \Phi)(\mathrm{Id} + \varepsilon n \otimes n) \big) : \nabla z \end{aligned} \tag{2.22}$$

$$n_t + v \cdot \nabla n - \Omega(v) n + D(v) n = \Delta n + \varepsilon \left(\nabla \Phi \otimes \nabla \Phi \right) n - \partial \mathscr{F}(n) \quad \textit{a.e. in } \mathcal{T}^3 \times (0,T), \tag{2.23}$$

with σ, $D(v)$, and $\Omega(v)$ defined as in (1.10) and (1.7), and holding true for every test functions ϕ_p, $\phi_m \in L^4(0,T;V)$, $u \in L^2(0,T;V)$, $z \in C^\infty(\mathcal{T}^3 \times [0,T])$, $\operatorname{div} z = 0$ and coupled with the initial conditions:

$$c_p(0) = c_{p,0},\ c_m(0) = c_{m,0}, \quad \textit{in } V', \qquad n(0) = n_0,\ v(0) = v_0, \quad \textit{a.e. in } \mathcal{T}^3. \tag{2.24}$$

The *weak sequential stability* theorem we aim to prove is the following:

Theorem 1. *Let us assume that there exists a family $(c_p^{(k)}, c_m^{(k)}, \Phi^{(k)}, v^{(k)}, n^{(k)})_{k\in\mathbb{N}}$ of smooth solutions of the system (2.1)–(2.5) on the flat 3-dimensional torus $\mathcal{T}^3$ subject to corresponding initial data*

$$c_p^{(k)}(0) = c_{p,0}^{(k)}, \quad c_m^{(k)}(0) = c_{m,0}^{(k)}, \quad n^{(k)}(0) = n_0^{(k)}, \tag{2.25}$$

with $(c_{p,0}^{(k)}, c_{m,0}^{(k)}, n_0^{(k)}) \in (C^\infty(\mathcal{T}^3))^3$. We furthermore assume that the conditions (2.7),(2.8), (2.10)–(2.12), (1.9) hold. Moreover we assume that there exists a constant $\tilde{C}$, independent of $k \in \mathbb{N}$, such that

$$\|c_{p,0}^{(k)}\|_{L^\infty}, \|c_{m,0}^{(k)}\|_{L^\infty}, \|n_0^{(k)}\|_{H^1(\mathcal{T}^3)}, \|v_0\|_H \leq \tilde{C} \quad \textit{and } c_{i,0}^k \to c_{i,0}, \quad n_0^k \to n_0, \tag{2.26}$$

the latter convergence relations holding, e.g., in the sense of distributions.

Then there exists a (non-relabelled) sequence of the family $(c_p^{(k)}, c_m^{(k)}, \Phi^{(k)}, v^{(k)}, n^{(k)})$ tending, in the sense explicated in relations (4.1)–(4.7) below, to a quintuple (c_p, c_m, Φ, v, n) solving system (2.1)-(2.6) in the sense specified in Definition 1.

Remark 1. *In fact one would need solutions which are not smooth but just 'sufficiently regular', but the precise minimal regularity needed is not of interest since in general the solutions obtained through approximations scheme are smooth.*

The rest of the paper is devoted to the proof of Theorem 1.

3. Apriori estimates.

We now prove a number of apriori estimates on the solutions of system (2.1)-(2.6). As noted above, we decided to perform the computations by directly working on the "limit" equations without referring to any explicit regularization or approximation scheme. Of course, in such a setting, the procedure has just a formal character because the use of some test function as well as some integration by parts is not justified (this, for instance, surely happens in connection with the Navier-Stokes system (2.4)). On the other hand, the computations we are going to develop are not trivial and involve a certain number of subtlenesses; for this reason we believe that presenting them in the simplest possible setting might help comprehension. Actually, in the last part of the paper we will provide some hints about the construction of an approximation scheme being compatible with the estimates.

The first property we prove is the basic energy estimate resulting as a consequence of the variational nature of the model. We state it in the form of a

Proposition 1 (Energy law)**.** *Let $(c_m, c_p, \Phi, v, n) : \Omega \to \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ be a sufficiently smooth solution of system (2.1)-(2.6) on $\mathcal{T}^3 \times (0,T)$ complemented with the initial conditions (2.24) and satisfying the coefficient relations (2.9)(that ensure the non-negativity of the dissipation). Then there holds the energy inequality*

$$E(t) + \int_0^t \int_{\mathcal{T}^3} \Big(\frac{1}{c_p} |\nabla c_p + c_p \nabla \Phi|^2 + \frac{1}{c_m} |\nabla c_m - c_m \nabla \Phi|^2 \Big) \tag{3.1}$$

$$+ \int_0^t \int_{\mathcal{T}^3} \big(\underbrace{\alpha_4 |D(v)|^2 + \alpha_1 (n \cdot D(v) n)^2 + 2(\mathring{n} \cdot D(v) n) + (\alpha_5 + \alpha_6) |D(v) n|^2 + |\mathring{n}|^2}_{\geq 0} \big)$$

$$\leq E(0) \tag{3.2}$$

where the energy functional is defined as

$$E(t) = \int_{\mathcal{T}^3} \Big(\frac{1}{2} |v|^2 + \frac{1}{2} |\nabla n|^2 + \mathscr{F}(n) + c_p \ln c_p + c_m \ln c_m + \frac{1}{2} (1 + \varepsilon n \otimes n) \nabla \Phi \cdot \nabla \Phi \Big). \tag{3.3}$$

Proof. We multiply the equation (2.1) by $\ln c_p + \Phi$, integrate by parts using periodic boundary conditions to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{T}^3} c_p (\ln c_p - 1) + \int_{\mathcal{T}^3} c_p' \Phi + \int_{\mathcal{T}^3} (v \cdot \nabla c_p) \Phi$$

$$+ \int_{\mathcal{T}^3} (\mathrm{Id} + \varepsilon n \otimes n)(\nabla c_p + c_p \nabla \Phi) \cdot \Big(\frac{\nabla c_p}{c_p} + \nabla \Phi \Big) = 0, \tag{3.4}$$

whence, by positive definiteness of the matrix $n \otimes n$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{T}^3} c_p (\ln c_p - 1) + \underbrace{\int_{\mathcal{T}^3} c_p' \Phi}_{:= \mathscr{A}_{11}} + \underbrace{\int_{\mathcal{T}^3} (v \cdot \nabla c_p) \Phi}_{:= \mathscr{A}_{21}} + \int_{\mathcal{T}^3} \frac{1}{c_p} |\nabla c_p + c_p \nabla \Phi|^2 \leq 0. \tag{3.5}$$

Similarly, testing (2.2) by $\ln c_m - \Phi$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{T}^3} c_m (\ln c_m - 1) - \int_{\mathcal{T}^3} c_m' \Phi - \int_{\mathcal{T}^3} (v \cdot \nabla c_p) \Phi$$

$$+ \int_{\mathcal{T}^3} (\mathrm{Id} + \varepsilon n \otimes n)(\nabla c_m - c_m \nabla \Phi) \cdot \Big(\frac{\nabla c_m}{c_m} - \nabla \Phi \Big) = 0, \tag{3.6}$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}^3} c_m(\ln c_m - 1) \underbrace{-\int_{\mathcal{T}^3} c_m'\Phi}_{:=\mathscr{A}_{12}} \underbrace{-\int_{\mathcal{T}^3}(v\cdot\nabla c_m)\Phi}_{:=\mathscr{A}_{22}} + \int_{\mathcal{T}^3}\frac{1}{c_m}|\nabla c_m - c_m\nabla\Phi|^2 \le 0. \tag{3.7}$$

We now test (2.3) by $-\partial_t\Phi$ getting, after an integration by parts,

$$-\int_{\mathcal{T}^3}(\mathrm{Id}+\varepsilon n\otimes n)\nabla\Phi\cdot\nabla\Phi_t + \int_{\mathcal{T}^3}(c_p - c_m)\Phi_t = 0, \tag{3.8}$$

which can be expanded into

$$\underbrace{-\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}^3}(n\otimes n\nabla\Phi)\cdot\nabla\Phi - \int_{\mathcal{T}^3}\nabla\Phi\cdot\nabla\Phi_t + \int_{\mathcal{T}^3}(c_p-c_m)\Phi_t}_{:=\mathscr{A}_{13}}$$
$$= \underbrace{-\frac{\varepsilon}{2}\int_{\mathcal{T}^3}\partial_t(n\otimes n)\nabla\Phi\cdot\nabla\Phi}_{:=\mathscr{A}_3}. \tag{3.9}$$

Multiplying (2.3) by $-v\cdot\nabla\Phi$ and integrating by parts we get

$$-\int_{\mathcal{T}^3}\big((\mathrm{Id}+\varepsilon n\otimes n)\nabla\Phi\big)\cdot\nabla(v\cdot\nabla\Phi) + \int_{\mathcal{T}^3}(c_p-c_m)v\cdot\nabla\Phi = 0. \tag{3.10}$$

Splitting the left-hand side and integrating by parts further, we obtain

$$\underbrace{-\int_{\mathcal{T}^3}(\nabla\Phi\otimes\nabla\Phi):\nabla v}_{:=\mathscr{A}_4} \underbrace{-\varepsilon\int_{\mathcal{T}^3}\big((n\otimes n)\nabla\Phi\big)\cdot\nabla(v\cdot\nabla\Phi)}_{:=\mathscr{A}_5} + \underbrace{\int_{\mathcal{T}^3}(c_p-c_m)v\cdot\nabla\Phi}_{:=\mathscr{B}_2} = 0. \tag{3.11}$$

Multiplying (2.4) by v and integrating by parts, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \alpha_4\|D(v)\|^2 = \underbrace{\int_{\mathcal{T}^3}(\nabla n\odot\nabla n):\nabla v}_{:=\mathscr{A}_6} \underbrace{-\int_{\mathcal{T}^3}\nabla\Phi\otimes\nabla\Phi:\nabla v}_{\mathscr{A}_4}$$
$$\underbrace{-\varepsilon\int_{\mathcal{T}^3}\big((\nabla\Phi\otimes\nabla\Phi)(n\otimes n)\big):\nabla v}_{:=\mathscr{B}_{51}}$$
$$-\int_{\mathcal{T}^3}\underbrace{\big(\alpha_1(Dn\cdot n)n\otimes n + \alpha_2\mathring{n}\otimes n + \alpha_3 n\otimes\mathring{n} + \alpha_5 Dn\otimes n + \alpha_6 n\otimes Dn\big):\nabla v}_{:=\mathscr{B}_7}. \tag{3.12}$$

Finally, multiplying (2.6) by $\dot{n} = n_t + v\cdot\nabla n$ we get

$$\int_{\mathcal{T}^3}\underbrace{\big(\mathring{n}+D(v)n\big)\cdot\dot{n}}_{:=\mathscr{A}_7} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}^3}\mathscr{F}(n) + \underbrace{\int_{\mathcal{T}^3}\nabla n\cdot\nabla(v\cdot\nabla n)}_{:=\mathscr{B}_6}$$
$$= \underbrace{\varepsilon\int_{\mathcal{T}^3}\nabla\Phi\otimes\nabla\Phi : n\otimes n_t}_{\mathscr{B}_3} + \underbrace{\varepsilon\int_{\mathcal{T}^3}\big((\nabla\Phi\otimes\nabla\Phi)n\big)\cdot(v\cdot\nabla)n}_{\mathscr{B}_{52}}. \tag{3.13}$$

We can now sum (3.5), (3.7), (3.9), (3.11), (3.12), (3.13). We combine a number of terms and may note several cancellations, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}^3}\Big(-\frac{1}{2}(\mathrm{Id}+\varepsilon n\otimes n)\nabla\Phi\cdot\nabla\Phi+(c_p-c_m)\Phi\Big)=\mathscr{A}_{11}+\mathscr{A}_{12}+\mathscr{A}_{13},$$
$$\mathscr{A}_{21}+\mathscr{A}_{22}=\mathscr{B}_2,\qquad \mathscr{A}_3=\mathscr{B}_3,\qquad \mathscr{A}_6=\mathscr{B}_6.$$

The most delicate cancellation is $\mathscr{A}_5=\mathscr{B}_{51}+\mathscr{B}_{52}$, which amounts to

$$-\int_{\mathcal{T}^3}(n\otimes n\nabla\Phi)\cdot\nabla(v\cdot\nabla\Phi)=-\int_{\mathcal{T}^3}(\nabla\Phi\otimes\nabla\Phi)n\otimes n:\nabla v+\int_{\mathcal{T}^3}(\nabla\Phi\otimes\nabla\Phi)n)\cdot(v\cdot\nabla)n,$$

which, after expanding $(n\otimes n\nabla\Phi)\cdot\nabla(v\cdot\nabla\Phi)=(n\otimes n\nabla\Phi)\cdot(\nabla v\nabla\Phi)+(n\otimes n\nabla\Phi)\cdot(v\cdot\nabla)\nabla\Phi$, simplifies to

$$-\int_{\mathcal{T}^3}n_in_j\partial_j\Phi v_k\partial_i\partial_k\Phi=\int_{\mathcal{T}^3}\partial_i\Phi\partial_j\Phi n_jv_k\partial_kn_i. \tag{3.14}$$

Then, we integrate by parts the ∂_k derivative and note that no boundary terms appear due to the choice of periodic boundary conditions. Hence, using $\partial_kv_k=0$ we obtain

$$\begin{aligned}-\int_{\mathcal{T}^3}n_in_j\partial_j\Phi v_k\partial_i\partial_k\Phi&=\int_{\mathcal{T}^3}n_{i,k}n_j\partial_j\Phi\partial_i\Phi v_k\\&+\int_{\mathcal{T}^3}n_in_{j,k}\partial_j\Phi\partial_i\Phi v_k+\int_{\mathcal{T}^3}n_in_j\partial_j\partial_k\Phi\partial_i\Phi v_k.\end{aligned} \tag{3.15}$$

We note that after permuting the indices the above turns into

$$-2\int_{\mathcal{T}^3}n_in_j\partial_j\Phi v_k\partial_i\partial_k\Phi=2\int_{\mathcal{T}^3}n_{i,k}n_j\partial_j\Phi\partial_i\Phi v_k, \tag{3.16}$$

which is exactly (3.14), thus proving the claimed cancellation $\mathscr{A}_5=\mathscr{B}_{51}+\mathscr{B}_{52}$.

Furthermore, as in [2] we have

$$\mathscr{A}_7+\mathscr{B}_7=\alpha_1(n\cdot Dn)^2+2(\mathring{n}\cdot Dn)+\alpha_4|D|^2+(\alpha_5+\alpha_6)|Dn|^2+|\mathring{n}|^2 \tag{3.17}$$

Collecting the above computations, and using also the charge conservation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}^3}(c_p+c_m)=0, \tag{3.18}$$

we finally arrive at

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}^3}\Big(\frac{1}{2}|v|^2+\frac{1}{2}|\nabla n|^2+\mathscr{F}(n)+c_p\ln c_p+c_m\ln c_m\\&-\frac{1}{2}(\mathrm{Id}+\varepsilon n\otimes n)\nabla\Phi\cdot\nabla\Phi+(c_p-c_m)\Phi\Big)\\&+\int_{\mathcal{T}^3}\Big(\frac{1}{c_p}|\nabla c_p+c_p\nabla\Phi|^2+\frac{1}{c_m}|\nabla c_m-c_m\nabla\Phi|^2+\alpha_4|D(v)|^2\Big)\\&+\int_{\mathcal{T}^3}\Big(\alpha_1(n\cdot Dn)^2+2(\mathring{n}\cdot Dn)^2+(\alpha_5+\alpha_6)|Dn|^2+|\mathring{n}|^2\Big)\le 0.\end{aligned} \tag{3.19}$$

Let us now notice that, testing (2.3) by Φ and integrating by parts, there follows

$$\int_{\mathcal{T}^3}(c_p-c_m)\Phi=\int_{\mathcal{T}^3}(\mathrm{Id}+\varepsilon n\otimes n)\nabla\Phi\cdot\nabla\Phi. \tag{3.20}$$

Replacing the above into (3.19), we obtain (3.2), which concludes the proof. ∎

The energy estimate (3.2) implies a number of apriori bounds for the solutions of system (2.1)-(2.6), provided that the dissipation term is nonnegative. In our simplified setting (where we have set $\gamma_1, \gamma_2 = 1$, this results as a restriction on the choice of the parameters α_j. Namely, we can observe the following

Lemma 1. *If (2.9) holds true, then we have, for some $\delta' > 0$,*

$$\alpha_4|D|^2 + \alpha_1(n \cdot Dn)^2 + 2(\mathring{n} \cdot Dn) + (\alpha_5 + \alpha_6)|Dn|^2 + |\mathring{n}|^2 \geq \delta' \left(|Dn|^2 + |\mathring{n}|^2\right) \tag{3.21}$$

for arbitrary $\mathring{n} \in \mathbb{R}^3, n \in \mathbb{R}^3$, $D \in \mathbb{R}^{3\times 3}$ with $|n| \leq 1$ and the matrix D symmetric and traceless.

Proof. Noting that we have (where we use that $|n| \leq 1$):

$$(n \cdot Dn)^2 \leq |n|^2|Dn|^2 \leq |D|^2, \quad |2(\mathring{n} \cdot Dn)| \leq 2|\mathring{n}||Dn| \leq (1-\delta)|\mathring{n}|^2 + \frac{1}{1-\delta}|D|^2$$

we immediately deduce that (2.9) implies the claimed (3.21). ∎

In the sequel we shall always assume (2.9). In this way, as a consequence of the energy estimate (3.2), using also the positive definiteness of the matrix $n \otimes n$ and (2.8), we can obtain a number of apriori bounds holding for any hypothetical solution of the system and independently of any eventual approximation or regularization parameter. Namely, we have

$$\|v\|_{L^\infty(0,T;H)} + \|v\|_{L^2(0,T;V)} \leq c, \tag{3.22}$$

$$\|n\|_{L^\infty(0,T;V)} \leq c, \qquad |n| \leq 1 \quad \text{a.e. in } (0,T) \times \mathcal{T}^3, \tag{3.23}$$

$$c_p, c_m \geq 0 \quad \text{a.e. in } (0,T) \times \mathcal{T}^3, \tag{3.24}$$

$$\|\nabla\Phi\|_{L^\infty(0,T;H)} \leq c. \tag{3.25}$$

where c is a constant depeding only on $E(0)$ as defined in (3.3). Note that the second bound in (3.23) directly follows from our choice of the potential F.

Proposition 2 (Maximum principle)**.** *Let $c_p^0, c_m^0 : \mathcal{T}^3 \to \mathbb{R}_+$ satisfy (2.24) and let v, n satisfy (3.22), (3.23). Then, if (c_p, c_m, Φ) solve equations (2.1), (2.2), (2.3) subject to periodic boundary conditions and initial data c_p^0, c_m^0 as above, then there follows*

$$|c_p(x,t)|, |c_m(x,t)| \leq \bar{c}, \quad \textit{a.e. in } (0,T) \times \mathcal{T}^3. \tag{3.26}$$

Proof. We multiply (2.1) by $(c_p - \bar{c})^+$ and integrate over $\mathcal{T}^3$ and by parts, to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}^3} |(c_p - \bar{c})^+|^2 + \frac{1}{2}\int_{\mathcal{T}^3} v \cdot \nabla((c_p - \bar{c})^+)^2 \\
&\quad + \int_{\mathcal{T}^3} (\mathrm{Id} + \varepsilon n \otimes n)\nabla(c_p - \bar{c})^+ \cdot \nabla(c_p - \bar{c})^+ \\
&\quad + \int_{\mathcal{T}^3} (\mathrm{Id} + \varepsilon n \otimes n)\nabla\Phi \cdot \nabla\Big(\frac{1}{2}((c_p - \bar{c})^+)^2 + \bar{c}(c_p - \bar{c})^+\Big) = 0.
\end{aligned} \tag{3.27}$$

Similarly, we get from (2.2)

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}^3} |(c_m - \bar{c})^+|^2 + \frac{1}{2}\int_{\mathcal{T}^3} v \cdot \nabla((c_m - \bar{c})^+)^2 \\
&\quad + \int_{\mathcal{T}^3} (\mathrm{Id} + \varepsilon n \otimes n)\nabla(c_m - \bar{c})^+ \cdot \nabla(c_m - \bar{c})^+ \\
&\quad - \int_{\mathcal{T}^3} (\mathrm{Id} + \varepsilon n \otimes n)\nabla\Phi \cdot \nabla\Big(\frac{1}{2}((c_m - \bar{c})^+)^2 + \bar{c}(c_m - \bar{c})^+\Big) = 0.
\end{aligned} \tag{3.28}$$

We now define

$$M(r) := \begin{cases} 0 & \text{if } r \le \bar{c}, \\ \frac{1}{2}((r-\bar{c})^+)^2 + \bar{c}(r-\bar{c})^+ & \text{if } r \ge \bar{c}, \end{cases} \tag{3.29}$$

Then, summing (3.27) and (3.28) and using incompressibility, we deduce

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}^3} \left(|(c_p-\bar{c})^+|^2 + |(c_m-\bar{c})^+|^2\right) \\ &\quad \le -\int_{\mathcal{T}^3} (\mathrm{Id} + \varepsilon n \otimes n)\nabla\Phi \cdot \nabla(M(c_p) - M(c_m)). \end{aligned} \tag{3.30}$$

The integral on the right-hand side can be computed by using (2.3). This leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}^3} \left(|(c_p-\bar{c})^+|^2 + |(c_m-\bar{c})^+|^2\right) \le -\int_{\mathcal{T}^3} (c_p - c_m)(M(c_p) - M(c_m)) \le 0, \tag{3.31}$$

the inequality following from the monotonicity of the function M. Noting that (2.24) implies that the left-hand side is null at $t = 0$, we obtain the claimed estimate. ∎

In particular, we have obtained the additional bound

$$\|c_p\|_{L^\infty(0,T;L^\infty(\mathcal{T}^3))} + \|c_m\|_{L^\infty(0,T;L^\infty(\mathcal{T}^3))} \le c. \tag{3.32}$$

where the constant c depends just on the L^∞ norm of $c_p(0)$ and $c_m(0)$. We can then test (2.1) by c_p and (2.2) by c_m. Using once more the positive definiteness of the matrix $n \otimes n$, we may note that

$$\left|\int_{\mathcal{T}^3} c_p \nabla\Phi \cdot \nabla c_p\right| \le \|c_p\|_{L^\infty(\mathcal{T}^3)} \|\nabla\Phi\|_H \|\nabla c_p\|_H \le c\|\nabla c_p\|_H \le c + \frac{1}{2}\|\nabla c_p\|_H^2, \tag{3.33}$$

with an analogous relation holding for c_m and where the constants $c > 0$ are independent of time in view of (3.25) and (3.32). Analogously we can estimate the term $-\int_{\mathcal{T}^3} \varepsilon(n \otimes n) c_p \nabla\Phi \cdot \nabla c_p$ by (3.23).

Then, it is not difficult to deduce the parabolic regularity estimate

$$\|c_p\|_{L^2(0,T;V)} + \|c_m\|_{L^2(0,T;V)} \le c. \tag{3.34}$$

In view of the fact that Φ is defined up to an additive constant, it is not restrictive to assume that

$$\Phi_\Omega = \int_{\mathcal{T}^3} \Phi(t) = 0 \quad \text{for a.e. } t \in (0,T). \tag{3.35}$$

Of course, such a normalization property, joint with (3.25), implies

$$\|\Phi\|_{L^\infty(0,T;V)} \le c. \tag{3.36}$$

We have, however, a better property which is given by the following

Lemma 2 (Uniform boundedness of Φ)**.** *We have the additional estimate*

$$\|\Phi\|_{L^\infty(0,T;L^\infty(\mathcal{T}^3))} \le c. \tag{3.37}$$

Proof. The proof follows by applying a Moser iteration argument on equation (2.3) and using the uniform boundedness of the right-hand side following from estimate (3.32). We give some highlights for the reader's convenience. As a general rule, we multiply equation (2.3) by $(\Phi)^{p-1} := |\Phi|^{p-1} \operatorname{sign} \Phi$

where the exponent p will be taken larger and larger. This gives

$$
\begin{aligned}
(p-1)\int_{\mathcal{T}^3}(\mathrm{Id}+\varepsilon n\otimes n)|\Phi|^{p-2}\nabla\Phi\cdot\nabla\Phi &= \int_{\mathcal{T}^3}(c_p-c_m)|\Phi|^{p-1}\operatorname{sign}\Phi \\
&\le c\int_{\mathcal{T}^3}|\Phi|^{p-1}\le c\int_{\mathcal{T}^3}\left(\frac{1}{p}+\frac{p-1}{p}|\Phi|^p\right) \\
&\le \frac{c}{p}+c\int_{\mathcal{T}^3}|\Phi|^p. \qquad (3.38)
\end{aligned}
$$

As a first step, we take $p=p_0=6$. Then, controlling the right-hand side by the Poincaré-Wirtinger inequality we deduce (cf. also (3.35))

$$
c\int_{\mathcal{T}^3}|\Phi|^6 = c\|\Phi-\Phi_\Omega\|_6^6\le c\|\nabla\Phi\|_2^6\le c, \qquad (3.39)
$$

the last inequality following from (3.25). Here and below, we are noting simply by $\|\cdot\|_q$ the norm in $L^q(\mathcal{T}^3)$, $1\le q\le\infty$, for notational simplicity. We also point out that all the estimates obtained in this proof are uniform with respect to the time variable, because so are (3.25) and (3.32) that serve as a starting point of the argument.

Hence, noting that

$$
(p-1)\int_{\mathcal{T}^3}(\mathrm{Id}+\varepsilon n\otimes n)|\Phi|^{p-2}\nabla\Phi\cdot\nabla\Phi\ge\frac{4(p-1)}{p^2}\int_{\mathcal{T}^3}\left|\nabla(\Phi)^{p/2}\right|^2 \qquad (3.40)
$$

at the first iteration, i.e. for $p=6$, we deduce

$$
\left\|\nabla\Phi^3\right\|_2\le c, \qquad (3.41)
$$

whence, recalling (3.25) and using Sobolev's embeddings,

$$
\|\Phi\|_{18}^3\le c(\left\|\nabla|\Phi|^3\right\|_2+\|\Phi\|_6^3)\le c. \qquad (3.42)
$$

Now, in order to take care of further iterations, we need to keep trace of the dependence on p of the various constants. Let us, then, go back to (3.38) with a generic p and combine it with (3.40) to deduce (for $p\ge 2$)

$$
\int_{\mathcal{T}^3}\left|\nabla(\Phi)^{p/2}\right|^2\le\frac{cp}{(p-1)}+\frac{cp^2}{p-1}\int_{\mathcal{T}^3}|\Phi|^p\le c+c(p+2)\int_{\mathcal{T}^3}|\Phi|^p
$$

where c is independent of p.

Adding also $\|\Phi\|_p^p$ to both hand sides and using the Sobolev embedding, we then deduce

$$
\begin{aligned}
\|\Phi\|_{3p}^p &= \left\|(\Phi)^{p/2}\right\|_6^2\le c\left\|(\Phi)^{p/2}\right\|_V^2 \\
&\le c\left\|(\Phi)^{p/2}\right\|_2^2+c\int_{\mathcal{T}^3}\left|\nabla(\Phi)^{p/2}\right|^2\le c+c(p+3)\|\Phi\|_p^p\le c+cp\|\Phi\|_p^p, \qquad (3.43)
\end{aligned}
$$

where c is still independent of p.

We define $b_p=\max(1,\|\Phi\|_p)$. Then, assuming without loss of generality that $c\ge 1$ the last inequality implies:

$$
b_{3p}^p\le cpb_p^p
$$

with $c>1$ a constant independent of p. Then, since $\ln b_{3p}\le\frac{\ln(cp)}{p}+\ln b_p$, we get

$$
\begin{aligned}
\ln b_{3^np} &\le \frac{\ln(c3^{(n-1)}p)}{3^{n-1}p}+\ln b_{3^{n-1}p} \\
&\le \frac{\ln(c3^{(n-1)}p)}{3^{n-1}p}+\frac{\ln(c3^{(n-2)}p)}{3^{n-2}p}+\cdots+\ln b_p.
\end{aligned}
$$

and hence

$$\ln b_{3^n p} \le \sum_{k=1}^{n-1} \frac{\ln(c3^k p)}{c3^k p} + \ln b_p.$$

Noting that constant c is independent of n and p and letting $n \nearrow \infty$ we obtain (3.37). ∎

It is worth observing that the bounds derived up to this point are not sufficient for passing to the limit in (a suitable approximation) of system (2.1)-(2.6), the main trouble being represented by the quadratic terms in $\nabla\Phi$ and ∇n. Indeed, at the moment such quantities are bounded only in L^2 with respect to space variables. Hence, at the limit we might expect occurrence of defect measures. Fortunately, this is not the case, because it is possible to improve a bit the regularity properties proved so far.

Lemma 3 (Additional regularity estimate)**.** *Let us assume that the initial data satisfy* (2.10)–(2.12). *Then the following additional regularity conditions hold:*

$$\|\nabla\Phi\|_{L^\infty(0,T;L^{p_M}(\mathcal{T}^3))} \le c_{p_M}, \qquad \textit{for some } p_M > 2 \tag{3.44}$$

$$\|n_t\|_{L^{p_0}(0,T;L^{p_0}(\mathcal{T}^3))} + \|\Delta n\|_{L^{p_0}(0,T;L^{p_0}(\mathcal{T}^3))} \le c, \qquad \textit{for some } p_0 > 1 \tag{3.45}$$

$$\|\partial\mathscr{F}(n)\|_{L^{p_0}(0,T;L^{p_0}(\mathcal{T}^3))} \le c \qquad \textit{for some } p_0 > 1. \tag{3.46}$$

Proof. The key point stands in the application of some refined elliptic regularity result to equation (2.3). Indeed, in view of the bound $|n| \le 1$ and of the positive definiteness of $n \otimes n$, the matrix $\mathrm{Id} + \varepsilon n \otimes n$ is strongly elliptic and has bounded coefficients. Since the right-hand side of (2.3) is uniformly bounded by (3.26), we can then apply the integrability result [17, Thm. 1, p. 198], which implies

$$\|\nabla\Phi\|_{L^\infty(0,T;L^{p_M}(\mathcal{T}^3))} \le c_{p_M} \qquad \text{for some } p_M > 2. \tag{3.47}$$

Note that, at least in three space dimensions, there is no quantitative control of p_M. Nevertheless, we know that $p_M > 2$. As a consequence of (3.44), (2.6) can be rearranged in the form

$$n_t - \Delta n + \partial\mathscr{F}(n) = \underbrace{-v\cdot\nabla n + \Omega(v)n - D(v)n + \varepsilon\left(\nabla\Phi\otimes\nabla\Phi\right)n}_{:=f}, \tag{3.48}$$

where a simple check based on the previous estimates (3.22), (3.23) shows that, at least,

$$v\cdot\nabla n + \Omega(v)n - D(v)n \in L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}}(\mathcal{T}^3))$$

which together with (3.47) implies

$$f \in L^p(0,T;L^p(\mathcal{T}^3)). \tag{3.49}$$

for all $p \le p_0$ where

$$p_0 := \min\left(\frac{3}{2}, \frac{p_M}{2}\right). \tag{3.50}$$

Recalling (2.7), we observe that, componentwise, equation (3.48) takes the form

$$\partial_t n_i - \Delta n_i + F'(|n|^2)n_i = f_i, \tag{3.51}$$

where F' is monotone because F is convex.

This property, however, has to be a bit clarified. Indeed, the function $\mathscr{F}$ may be nonsmooth, and its subdifferential $\partial\mathscr{F}$ may be (and in fact has to be, in view of assumption (2.8)) a singular operator. Hence, here and below the use of F' to represent the subdifferential $\partial\mathscr{F}$ is formal and to make the procedure fully rigorous one should rather perform some regularization of $\mathscr{F}$ and then pass to the limit. Since this kind of argument is rather standard, we omit details for brevity.

Take from now on $p =: p_0$ (for simplicity of notation). We then test (3.51) by the function $G_i(n) = |F'(|n|^2)|^{p-1} \operatorname{sign} F'(|n|^2) n_i$ to obtain

$$\frac{1}{2}\int_{\mathcal{T}^3} |F'(|n|^2)|^{p-1} \operatorname{sign} F'(|n|^2) \frac{\mathrm{d}}{\mathrm{d}t}|n_i|^2 + \int_{\mathcal{T}^3} |F'(|n|^2)|^p n_i^2 \\ + \int_{\mathcal{T}^3} |F'(|n|^2)|^{p-1} \operatorname{sign} F'(|n|^2) |\nabla n_i|^2 + \mathcal{M}_i = \int_{\mathcal{T}^3} f_i \ F'(|n|^2)|^{p-1} \operatorname{sign} F'(|n|^2) n_i, \tag{3.52}$$

where the "mixed" term $\mathcal{M}$ is given by

$$\mathcal{M}_i = (p-1)\int_{\mathcal{T}^3} |F'(|n|^2)|^{p-2} F''(|n|^2) n_i \nabla |n|^2 \cdot \nabla n_i \\ = \frac{(p-1)}{2}\int_{\mathcal{T}^3} |F'(|n|^2)|^{p-2} F''(|n|^2) \nabla |n|^2 \cdot \nabla n_i^2. \tag{3.53}$$

Let us sum (3.52) for $i = 1, 2, 3$. It is then easy to check that

$$\sum_{i=1}^{3} \mathcal{M}_i = \frac{(p-1)}{2}\int_{\mathcal{T}^3} |F'(|n|^2)|^{p-2} F''(|n|^2) \nabla |n|^2 \cdot \nabla |n|^2 \geq 0 \tag{3.54}$$

due to convexity of F. We split the term $\int_{\mathcal{T}^3} |F'(|n|^2)|^{p-1} \operatorname{sign} F'(|n|^2) |\nabla n|^2$ over two subsets of $\mathcal{T}^3$, namely

$$\mathcal{T}^3_+ := \left\{ x \in \mathcal{T}^3, |n|^2(x) \geq 1 - \frac{1}{e} \right\}, \text{ respectively } \mathcal{T}^3_- := \left\{ x \in \mathcal{T}^3, |n|^2(x) < 1 - \frac{1}{e} \right\},$$

where we neglect the dependence on t for simplicity.

Then, taking into account that $F'(r) \geq 0$ for $r \in (1 - \frac{1}{e}, 1)$, neglecting the positive term $\int_{\mathcal{T}^3_+} |F'(|n|^2)|^{p-1} \operatorname{sign} F'(|n|^2) |\nabla n|^2$ on the left-hand side, and using that $F'(|n|^2(x)) \in (-1, 0)$ for $x \in \mathcal{T}^3_-$ we deduce:

$$\frac{1}{2}\int_{\mathcal{T}^3} |F'(|n|^2)|^{p-1} \operatorname{sign} F'(|n|^2) \frac{\mathrm{d}}{\mathrm{d}t}|n|^2 + \int_{\mathcal{T}^3} |F'(|n|^2)|^p |n|^2 \\ \leq \int_{\mathcal{T}^3} F'(|n|^2)|^{p-1} \operatorname{sign} F'(|n|^2) f \cdot n + \int_{\mathcal{T}^3_-} |F'(|n|^2)|^{p-1} |\nabla n|^2 \\ \leq \big\| F'(|n|^2)|^{p-1} \operatorname{sign} F'(|n|^2) \big\|_{p/(p-1)} \|f \cdot n\|_p + \int_{\mathcal{T}^3} |\nabla n|^2 \\ \leq c \big\| F'(|n|^2)| \big\|_p^{p-1} \|f\|_p + c \\ \leq \sigma \big\| F'(|n|^2)| \big\|_p^p + c_\sigma \|f\|_p^p + c, \tag{3.55}$$

where we also used Hölder's and Young's inequalities and the apriori bounds (3.23).

Now, note that

$$\frac{1}{2}\int_{\mathcal{T}^3} |F'(|n|^2)|^{p-1} \operatorname{sign} F'(|n|^2) \frac{\mathrm{d}}{\mathrm{d}t}|n|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}^3} \Gamma_p(|n|^2), \tag{3.56}$$

where the function Γ_p is defined by the right-hand side above and it is bounded from below. Notice that $\lim_{r \to 1^-} \Gamma_p(r) < +\infty$ and that

$$\int_{\mathcal{T}^3} |F'(|n|^2)|^p |n|^2 \geq \frac{1}{2}\int_{\mathcal{T}^3} |F'(|n|^2)|^p - c \tag{3.57}$$

(to see this, split the integral into the subregions $|n|^2 \leq 1/2$, where F' is bounded and $|n| \geq 1/2$ which gives the control from below). Hence, taking $\sigma < 1/2$, we see that the first term on the right-hand side of (3.55) is controlled. On the other hand, integrating in time, we may note that the latter term in (3.55) is also controlled by (3.49). As a consequence, we obtain first

$$\|F'(|n|^2)\|_{L^p((0,T)\times\mathcal{T}^3)} \le c$$

and, as a consequence,

$$\|\partial\mathcal{F}(n)\|_{L^p((0,T)\times\mathcal{T}^3)} \le c\,.$$

Finally, comparing terms in (3.5) and applying elliptic regularity results of Agmon-Douglis-Nirenberg type, we get the bound

$$\|n_t\|_{L^p(0,T;L^p(\mathcal{T}^3))} + \|\Delta n\|_{L^p(0,T;L^p(\mathcal{T}^3))} \le c, \tag{3.58}$$

where we also used the regularity $n_0 \in W^{1,\frac{3}{2}}(\mathcal{T}^3)$ which is actually implied by our assumption (2.12). ∎

In the case when the anisotropy coefficient ε is small enough compared to the other parameters, we can prove some additional estimates. This is stated in the following

Lemma 4 (H^2-estimates)**.** *Let us assume that the initial data satisfy* (2.10)–(2.12)*. Furthermore, let* $\varepsilon > 0$ *be small enough. Then, we have*

$$\|\Phi\|_{L^2(0,T;H^2(\mathcal{T}^3))} + \|n\|_{L^2(0,T;H^2(\mathcal{T}^3))} \le c. \tag{3.59}$$

Proof. We proceed in a natural way by testing (2.6) by $-\Delta n$. Then, we can preliminarily observe that, by convexity of $\mathscr{F}$ (and consequent monotonicity of the subdifferential),

$$-\int_{\mathcal{T}^3} \partial\mathscr{F}(n)\cdot\Delta n \ge 0. \tag{3.60}$$

As already noted before, this property, due to nonsmoothness of $\partial\mathscr{F}$, may require an approximation argument to be proved rigorously.

That said, we arrive at the bound

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n\|_H^2 + \|\Delta n\|_H^2 &= \int_{\mathcal{T}^3}(v\cdot\nabla n)\cdot\Delta n - \int_{\mathcal{T}^3}(\Omega(v)n)\cdot\Delta n \\ &+ \int_{\mathcal{T}^3}(D(v)n)\cdot\Delta n - \int_{\mathcal{T}^3}\varepsilon((\nabla\Phi\otimes\nabla\Phi)n)\cdot\Delta n =: \sum_{j=1}^4 I_j.\end{aligned} \tag{3.61}$$

and we need to estimate the terms I_j on the right-hand side. A key role will be played by the inequality

$$\|\nabla z\|_{L^4(\Omega)} \le c\|z\|_{L^\infty(\Omega)}^{1/2}\|z\|_{H^2(\Omega)}^{1/2}, \tag{3.62}$$

holding for every $z \in H^2(\Omega)$, Ω being a smooth bounded domain of $\mathbb{R}^3$ (for instance $\Omega = \mathcal{T}^3$). Then, integrating by parts and using (2.5) with the periodic boundary conditions, we have

$$\begin{aligned}I_1 &= -\int_{\mathcal{T}^3}(\nabla n\odot\nabla n):\nabla v \le \|\nabla n\|_{L^4(\Omega)}^2\|\nabla v\|_H \\ &\le c\|n\|_{L^\infty(\Omega)}\big(\|n\|_H + \|\Delta n\|_H\big)\|\nabla v\|_H \\ &\le c + \frac{1}{6}\|\Delta n\|_H^2 + c\|\nabla v\|_H^2,\end{aligned} \tag{3.63}$$

where we used in an essential way the property $|n| \le 1$ almost everywhere.

Next, it is clear that

$$I_2 + I_3 \le c\|n\|_{L^\infty(\Omega)}\|\nabla v\|_H\|\Delta n\|_H \le \frac{1}{6}\|\Delta n\|_H^2 + c\|\nabla v\|_H^2, \tag{3.64}$$

and, finally,

$$\begin{aligned} I_4 &\le c\varepsilon \|n\|_{L^\infty(\Omega)} \|\nabla\Phi\|^2_{L^4(\Omega)} \|\Delta n\|_H \le c\varepsilon \|\Phi\|_{L^\infty(\Omega)} \|\Phi\|_{H^2(\Omega)} \|\Delta n\|_H \\ &\le c\varepsilon^2 \|\Phi\|^2_{H^2(\Omega)} + \frac{1}{6}\|\Delta n\|^2_H, \end{aligned} \tag{3.65}$$

where for the last inequality we implicitly used Lemma 2.

Taking (3.63)-(3.65) into account, (3.61) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n\|^2_H + \|\Delta n\|^2_H \le c + c\|\nabla v\|^2_H + c\varepsilon^2\|\Phi\|^2_H + c\varepsilon^2\|\Delta\Phi\|^2_H, \tag{3.66}$$

where we point out that the constants c, in particular the last one, may depend on the various parameters of the problem, but are independent of the coefficient ε.

In order to control the last term, we apply elliptic regularity results to (2.3) (or, in other words, we test it by $-\Delta\Phi$) to obtain

$$\begin{aligned} \|\Delta\Phi\|_H &\le c\big(\varepsilon\|\nabla n\|_{L^4(\Omega)}\|n\|_{L^\infty(\Omega)}\|\nabla\Phi\|_{L^4(\Omega)} + \varepsilon\|D^2\Phi\|_H\|n\|^2_{L^\infty(\Omega)} + \|c_p - c_m\|_H\big) \\ &\le c\big(\varepsilon\|\Delta n\|_H^{1/2}\|\Delta\Phi\|_H^{1/2}\|\Phi\|^{1/2}_{L^\infty(\Omega)} + \varepsilon\|\Delta\Phi\|_H + 1\big) \\ &\le \frac{1}{4}\|\Delta n\|_H + c\varepsilon\|\Delta\Phi\|_H + c. \end{aligned} \tag{3.67}$$

where we have repeatedly used (3.62). At this point, we may assume ε so small that $c\varepsilon \le 1/2$. Then, the second term on the right-hand side can be absorbed by the corresponding quantity on the left-hand side. Squaring the resulting relation, we then deduce

$$\|\Delta\Phi\|^2_H \le \frac{1}{4}\|\Delta n\|^2_H + c. \tag{3.68}$$

Replacing into (3.66), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n\|^2_H + \frac{3}{4}\|\Delta n\|^2_H \le c + c\|\nabla v\|^2_H + c\frac{\varepsilon^2}{2}\|\Delta n\|^2_H, \tag{3.69}$$

which, possibly assuming ε small (such that $c\varepsilon^2/2 \le 1/4$), reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n\|^2_H + \frac{1}{2}\|\Delta n\|^2_H \le c + c\|\nabla v\|^2_H. \tag{3.70}$$

Integrating in time and recalling (3.22) we obtain the estimate for n in (3.59). The estimate for Φ is then deduced by integrating in time (3.68).

∎

4. **Weak sequential stability: proof of Theorem 1.** Let us assume $(c_{p,k}, c_{m,k}, \Phi_k, v_k, n_k)$ to be a family of approximating solutions complying with the estimates derived in the previous section uniformly with respect to the parameter $k \in \mathbb{N}$. We will then prove that there exists a (non-relabelled) sequence of the above sequence tending, in a suitable way, to a quintuple (c_p, c_m, Φ, v, n) solving system (2.1)-(2.6) in the sense specified in Definition 1.

To this aim, we start deducing some convergence properties (as mentioned, we will always assume to hold up to the extraction of subsequences) arising as a consequence of the bounds (3.22)-(3.25), (3.32), (3.34), (3.36), (3.37), (3.44)–(3.46) and (3.59). Namely, we have that there exists $\lambda \in$

$L^{p_0}(0,T;L^{p_0}(\mathcal{T}^3))$ such that

$$v_k \to v \quad \text{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V), \tag{4.1}$$

$$n_k \to n \quad \text{weakly star in } L^\infty(0,T;V) \cap L^\infty(0,T;L^\infty(\mathcal{T}^3)), \tag{4.2}$$

$$\Phi_k \to \Phi \quad \text{weakly star in } L^\infty(0,T;V) \cap L^\infty(0,T;L^\infty(\mathcal{T}^3)), \tag{4.3}$$

$$c_{p,k},\ c_{m,k} \to c_p,\ c_m \quad \text{weakly star in } L^2(0,T;V) \cap L^\infty(0,T;L^\infty(\mathcal{T}^3)), \tag{4.4}$$

$$\nabla\Phi_k \to \nabla\Phi \quad \text{weakly star in } L^\infty(0,T;L^{p_M}(\mathcal{T}^3)), \tag{4.5}$$

$$\partial_t n_k,\ \Delta n_k,\ \partial\mathscr{F}(n_k) \to n_t,\ \Delta n,\ \lambda \quad \text{weakly in } L^{p_0}(0,T;L^{p_0}(\mathcal{T}^3)), \tag{4.6}$$

where in deducing (4.3) we also used the normalization $(\Phi_k)_\Omega = 0$ and p_0, p_M are the exponents introduced in Lemma 3. Of course this implies in particular $\Phi_\Omega = 0$. Let us notice that, in the limit, we preserve the boundedness conditions $0 \le c_p \le \bar{c}$, $0 \le c_m \le \bar{c}$, $|n| \le 1$ almost everywhere in $\mathcal{T}^3$. In addition to that, if ε is sufficiently small (cf. Lemma 4), we also get:

$$\Phi_k,\ n_k \to \Phi,\ n \quad \text{weakly in } L^2(0,t;H^2(\mathcal{T}^3)). \tag{4.7}$$

In the following we show how to treat the passing to the limit just for the most difficult terms. We first note that, by (3.22) and interpolation,

$$\|v_k\|_{L^4(0,T;L^3(\mathcal{T}^3))} \le c, \tag{4.8}$$

whence, using (3.34), there follows

$$\|v_k \cdot \nabla c_{p,k}\|_{L^{4/3}(0,T;L^{6/5}(\mathcal{T}^3))} + \|v_k \cdot \nabla c_{m,k}\|_{L^{4/3}(0,T;L^{6/5}(\mathcal{T}^3))} \le c. \tag{4.9}$$

Then, using uniform boundedness of $c_{p,k}$, $c_{m,k}$ as well as the bounds (3.25), (3.32) it is not difficult to deduce from (2.1), (2.2) that

$$\|\partial_t c_{p,k}\|_{L^{4/3}(0,T;V')} + \|\partial_t c_{m,k}\|_{L^{4/3}(0,T;V')} \le c. \tag{4.10}$$

Hence, taking also into account (3.45), the Aubin-Lions lemma with the uniform boundedness property gives

$$c_{p,k},\ c_{m,k},\ n_k \to c_p,\ c_m,\ n \quad \text{strongly in } L^q(0,T;L^q(\mathcal{T}^3))\ \forall\, q \in [1,\infty). \tag{4.11}$$

Then, using (4.6), (4.11), the monotonicity of $\partial\mathscr{F}$, and the result [1, Prop. 1.1, p. 42], we get $\lambda = \partial\mathscr{F}(n)$. Moreover, by (4.3) and (4.4) we get

$$\|c_{p,k}\nabla\Phi_k\|_{L^\infty(0,T;H)} + \|c_{m,k}\nabla\Phi_k\|_{L^\infty(0,T;H)} \le c,$$

whence

$$c_{p,k}\nabla\Phi_k \to c_p\nabla\Phi, \quad c_{m,k}\nabla\Phi_k \to c_m\nabla\Phi \quad \text{weakly star in } L^\infty(0,T;H),$$

where we have used also (4.11). Using the Gagliardo-Nirenberg inequality (cf. [18]) together with (4.6) and the fact that $|n_k| \le 1$, we get

$$\|\nabla n_k \odot \nabla n_k\|_{L^s(0,T;L^s(\mathcal{T}^3))} \le c \quad \text{for some exponent } s > 1,$$

Finally, using the bound on $\partial_t n_k$ in (4.6) and again the Gagliardo-Nirenberg inequality (cf. [18]) interpolating between the spaces $L^\infty(0,T;L^\infty(\mathcal{T}^3))$ and $L^{p_0}(0,T;W^{2,p_0}(\mathcal{T}^3))$ at place 1/2, we also get the convergence

$$\nabla n_k \odot \nabla n_k \to \nabla n \odot \nabla n \quad \text{weakly in } L^s(0,T;L^s(\mathcal{T}^3)), \tag{4.12}$$

which is sufficient in order to conclude the passage to the limit as $k \to \infty$ in order to obtain the claimed weak solutions.

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