A destabilizing effect of stable stratification

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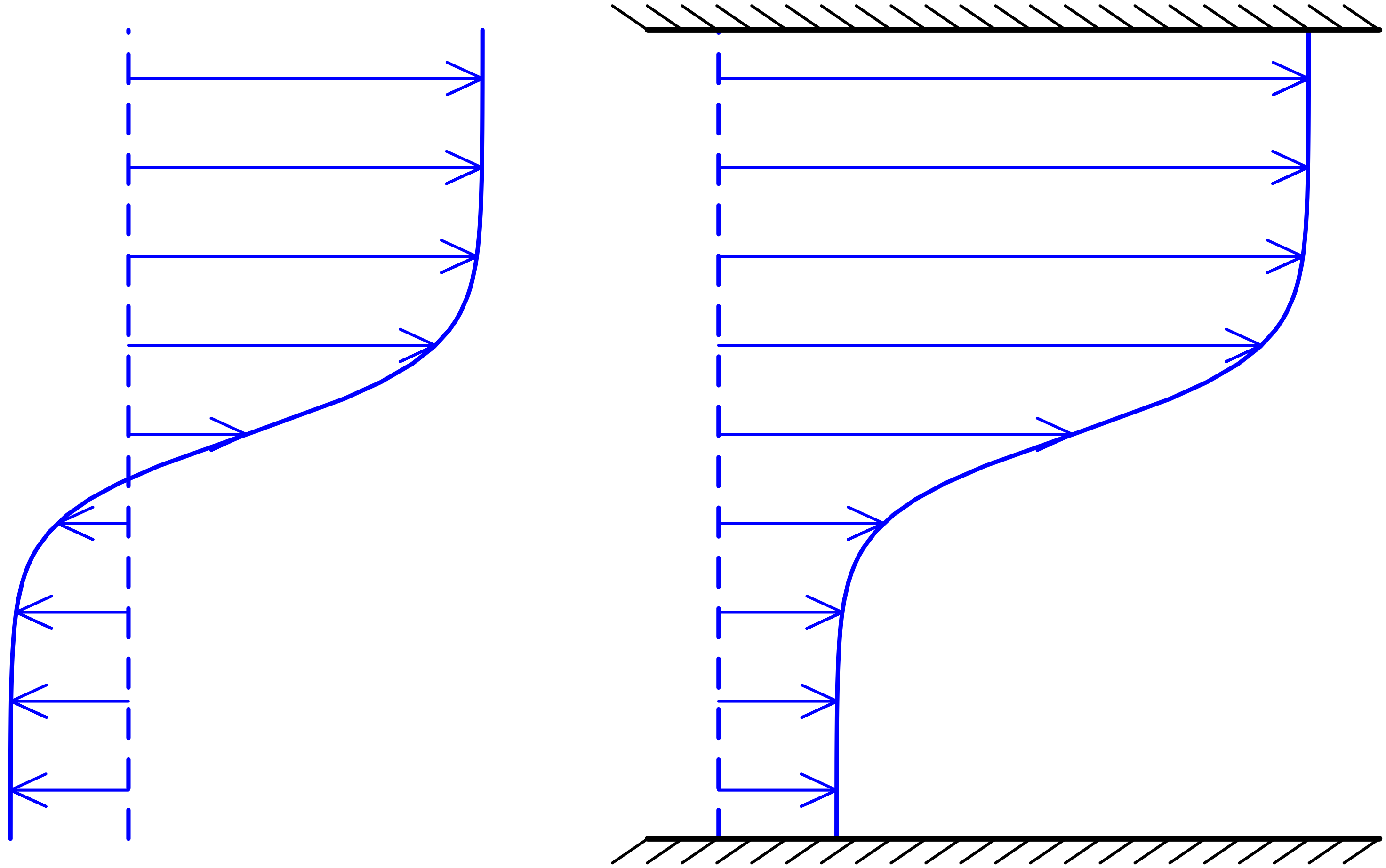
Motivation

- The instabilities of a 3-layered flow are investigated:
  - change in velocity across one interface;
  - change in density across the other interface.
- This study of stratified flow originated from an unexpected effect of confinement on homogeneous mixing layers.
- (First presented here last year, now published in *J. Fluid Mech.*, 2009).
- Likely applications in atmospheric and oceanic flows.

Homogeneous mixing layers



Unconfined counter-flow mixing layer

Confined co-flow mixing layer

Linearized waves

- Add a small disturbance to a parallel shear layer:

$$\begin{aligned}\tilde{u} &= U(y) + \epsilon u(y) \exp \mathrm{i}(\alpha x - \omega t) \\ \tilde{v} &= \epsilon v(y) \exp \mathrm{i}(\alpha x - \omega t) \\ \tilde{p} &= \epsilon p(y) \exp \mathrm{i}(\alpha x - \omega t)\end{aligned}$$

  where $\epsilon \ll 1$.
- Substitute into the Navier–Stokes equations.
- Neglect $O(\epsilon^2)$ terms (linearize), and viscosity.
- Eliminate u and p to give the Rayleigh equation:

$$(U - c)(v'' - \alpha^2 v) - U'' v = 0$$

  where $c = \omega/\alpha$ and $v = 0$ on boundaries.

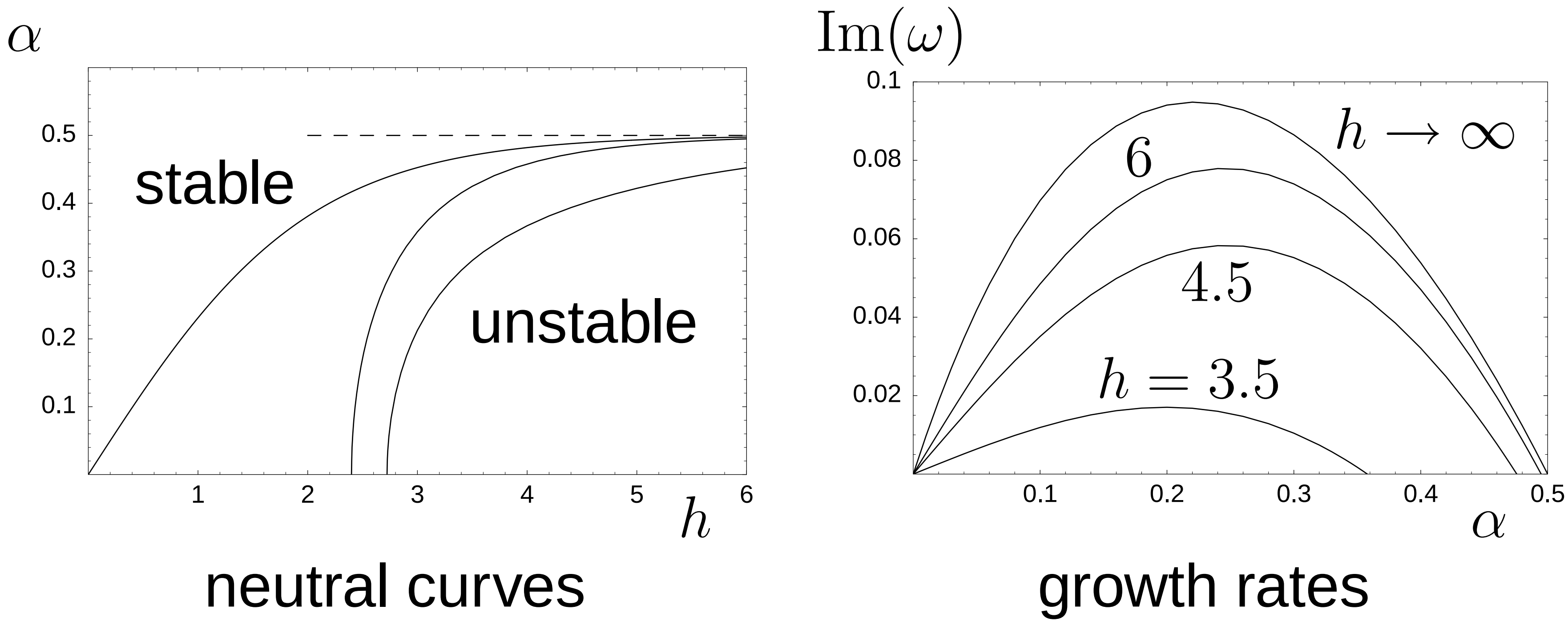
Temporal instability

- A spatially localized initial condition can be expressed as a superposition of normal modes with real α.
- Each normal mode evolves with an ω satisfying the dispersion relation.
- If there exists a real α with $\omega_i > 0$, then there is growth in time:

$$\exp(-\mathrm{i}\omega t) = (\exp -\mathrm{i}\omega_r t)(\exp \omega_i t).$$

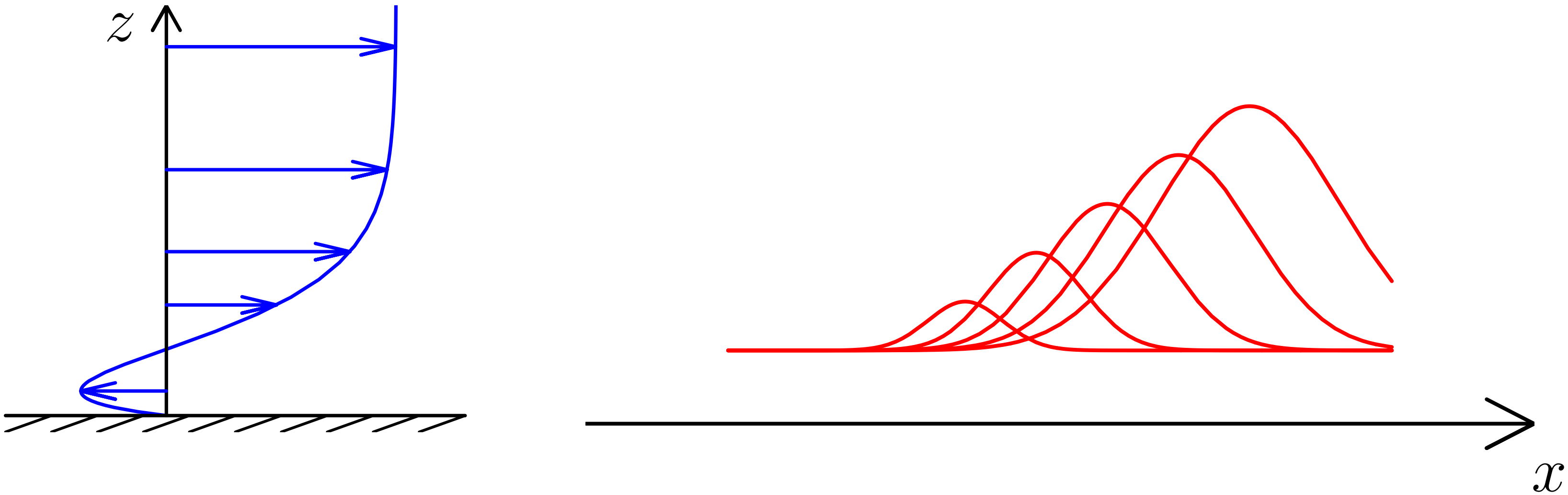
But isn't confinement stabilizing?

- Boundary conditions for Rayleigh equation:
  - Unconfined flow: $v \to 0$ as $y \to \pm\infty$.
  - Confined flow with plates at $y = \pm h$: $v(\pm h) = 0$.

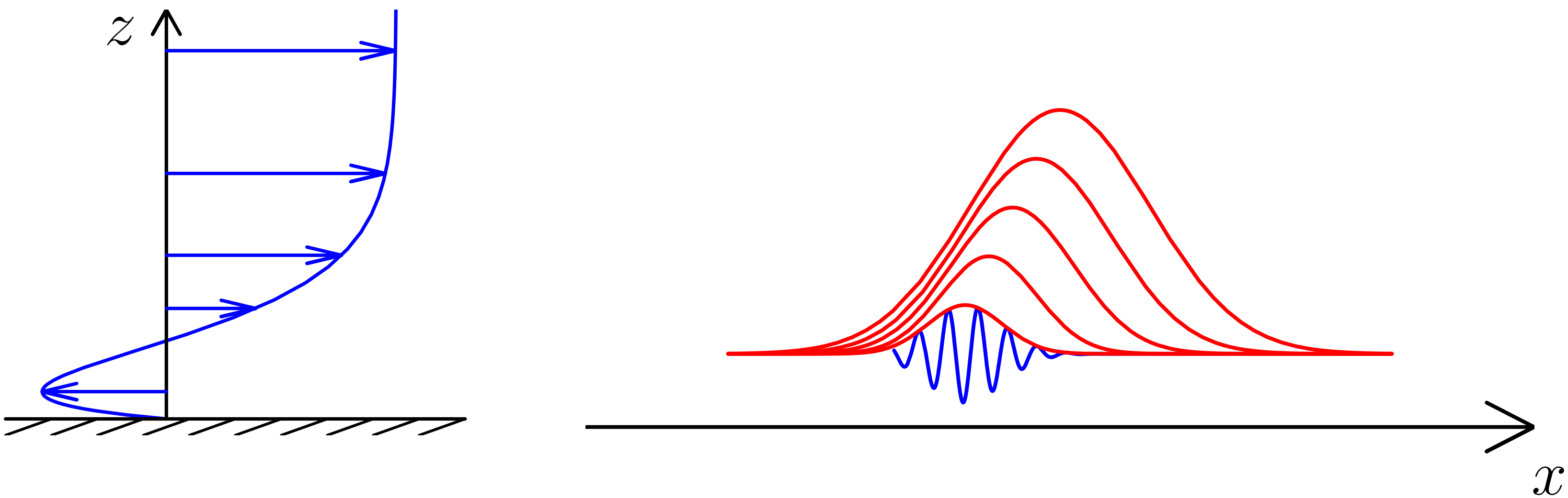


Absolute and convective instabilities

Convective instability:

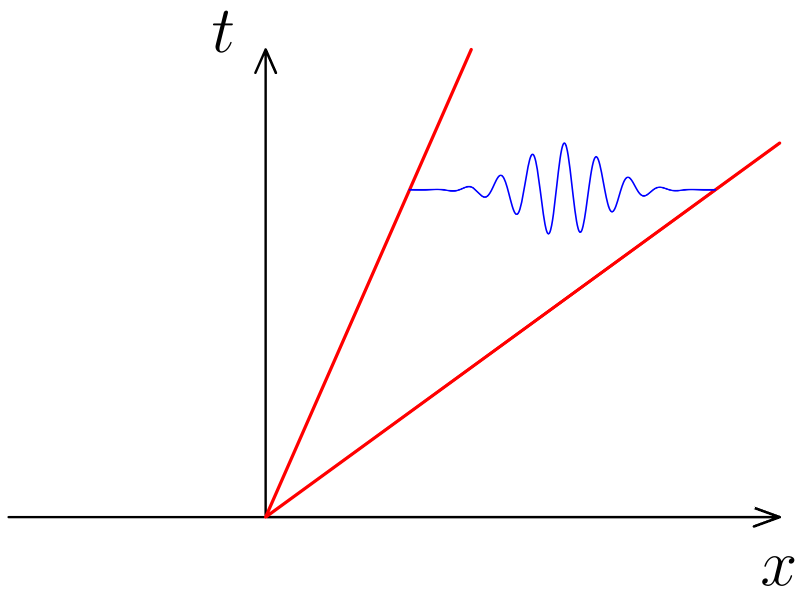


Absolute instability:



Space-time diagrams

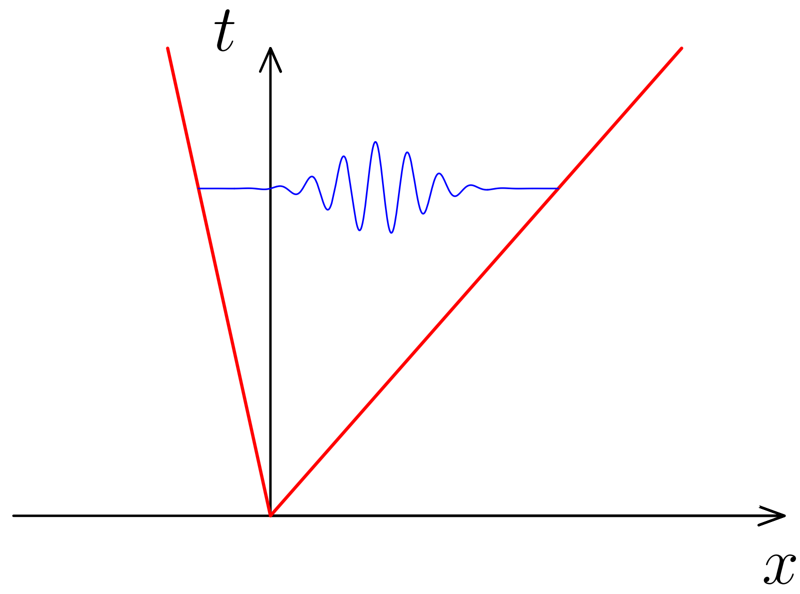
Convective instability



Disturbance grows as it propagates away, eventually leaving flow undisturbed.

Flow acts as spatial amplifier of transients.

Absolute instability



Disturbance grows in time everywhere.

Flow can act as self-excited oscillator.

Impulse calculations

- A wavepacket is constructed from a superposition of normal modes $v(y)\exp\mathrm{i}[\alpha x - \omega(\alpha)t]$ of the form:

$$\hat{v}(x,y,t) = \int_A v(y)\exp\phi t \,\mathrm{d}\alpha$$

where

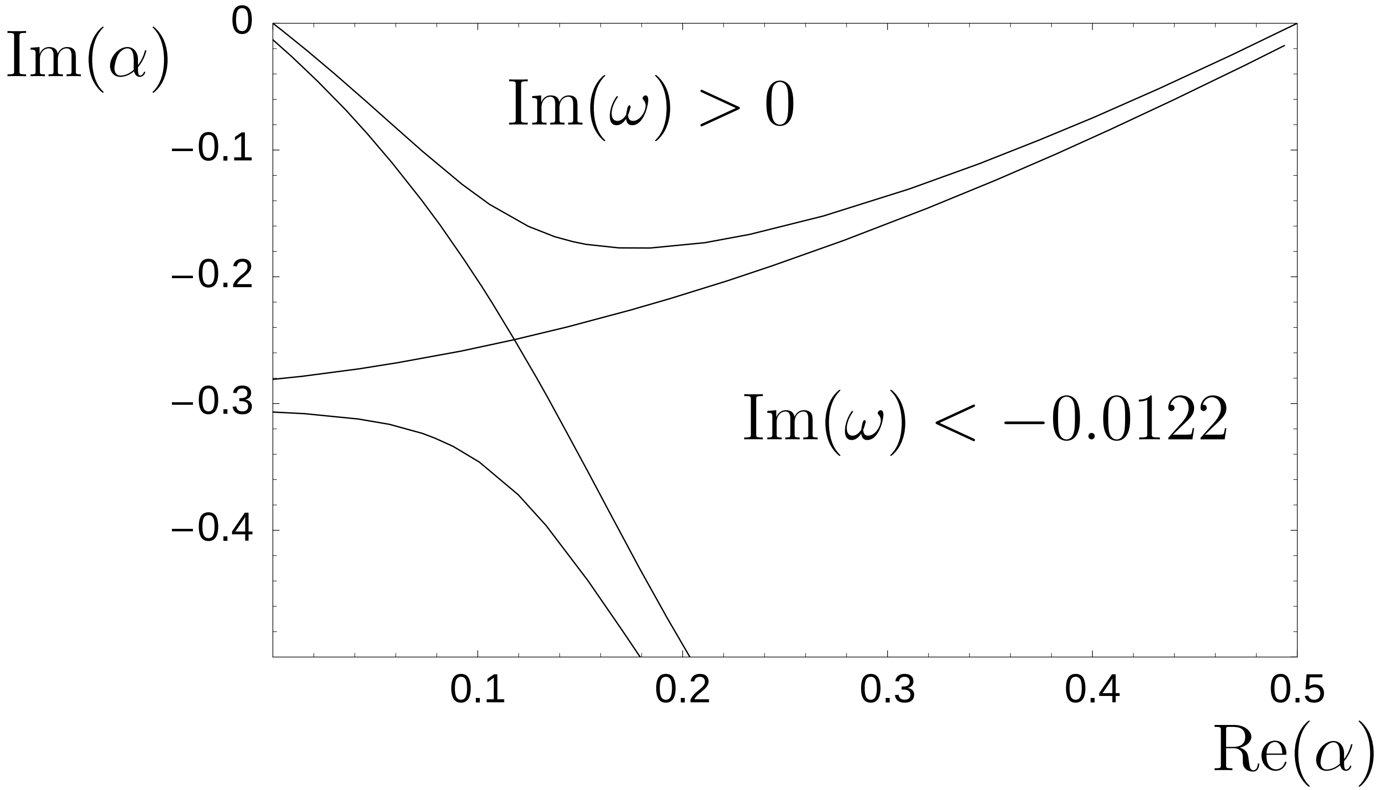
$$\phi(\alpha) = \mathrm{i}\left[\alpha\frac{x}{t} - \omega(\alpha)\right].$$

- In the limit $t \to \infty$ this integral is dominated by the contribution from a saddle-point, at which

$$\frac{\mathrm{d}\phi}{\mathrm{d}\alpha} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\omega}{\mathrm{d}\alpha} = \frac{x}{t}.$$

- There is absolute instability if $\mathrm{Im}(\omega) > 0$ at the dominant saddle (pinch-point) for $x/t = 0$.

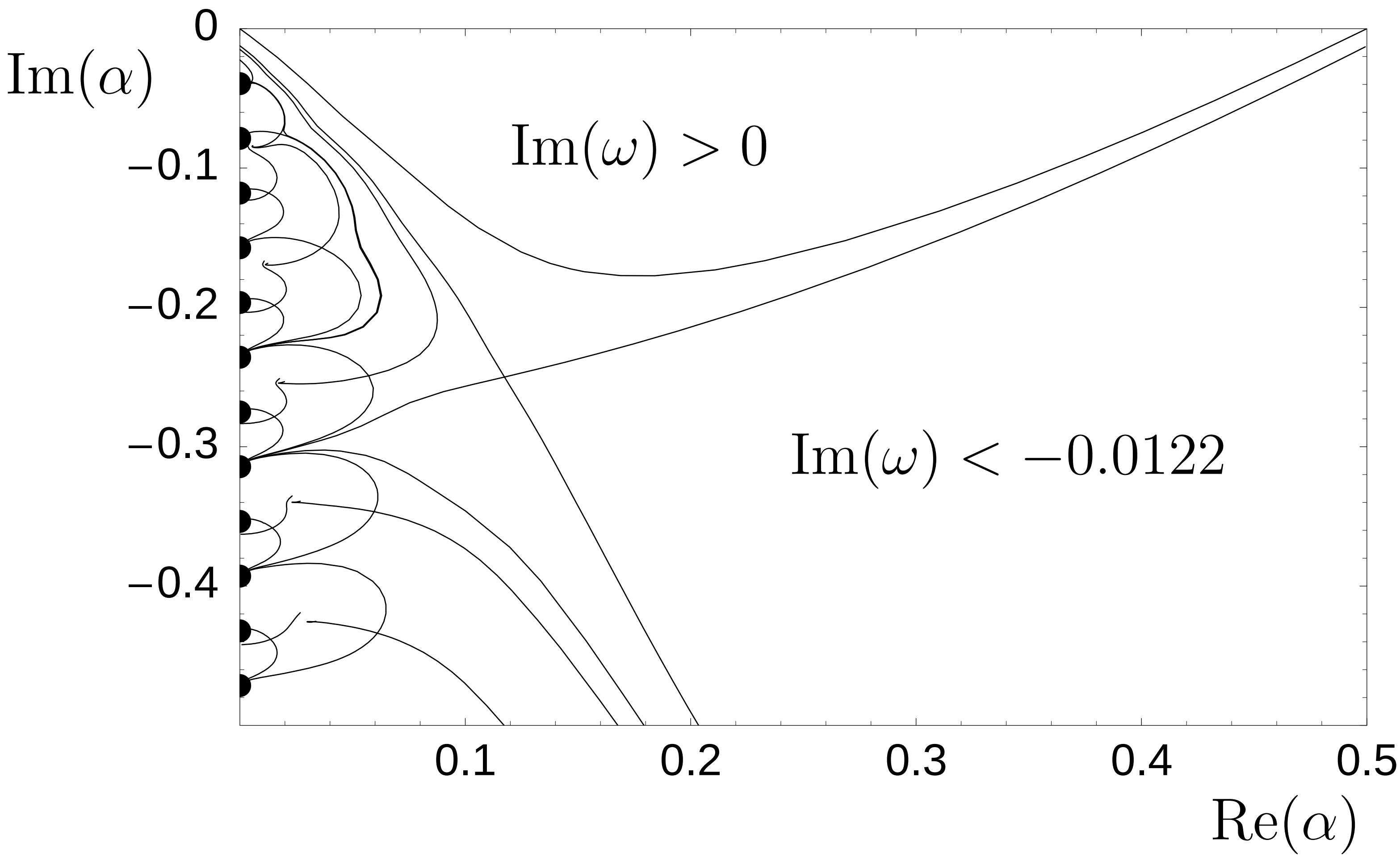
Saddle point for an unconfined flow



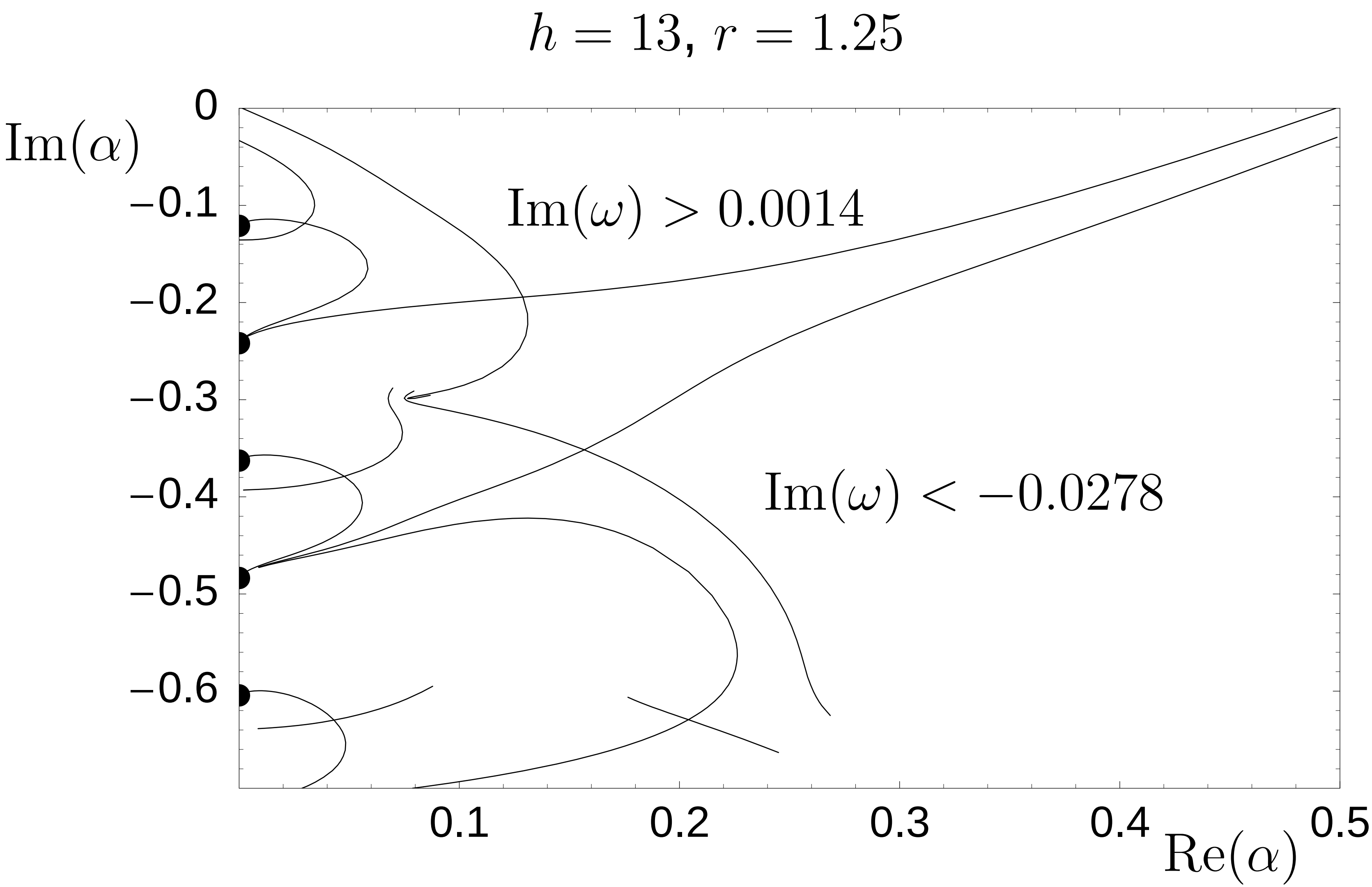
- Basic flow: $U = 1 + r\tanh(y/2)$, with $r = 1.25$.
- Huerre & Monkewitz (1985) found absolute instability for $r > 1.315$.

Confinement saddle points

Confinement creates an infinite number of saddle points near imaginary axis, e.g. at $h = 40$ and $r = 1.25$:



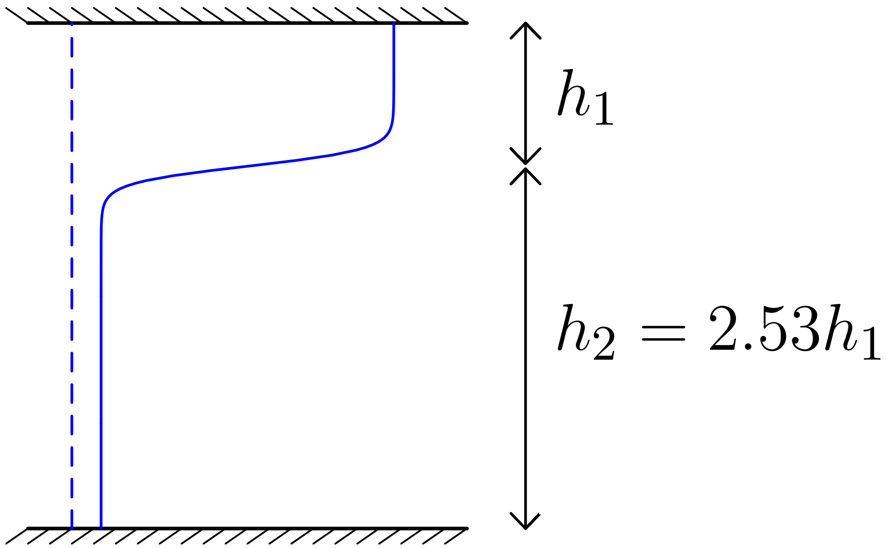
A more confined flow



Flow has been made absolutely unstable by confinement.

Asymmetric confinement

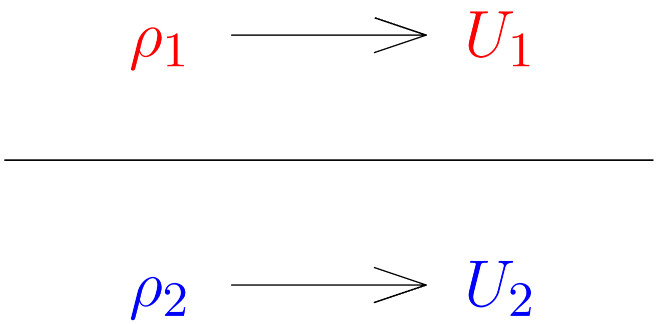
- Asymmetric confinement can create co-flow absolute instability:



- Substantial destabilization of absolute instability can also occur with only a single plate.

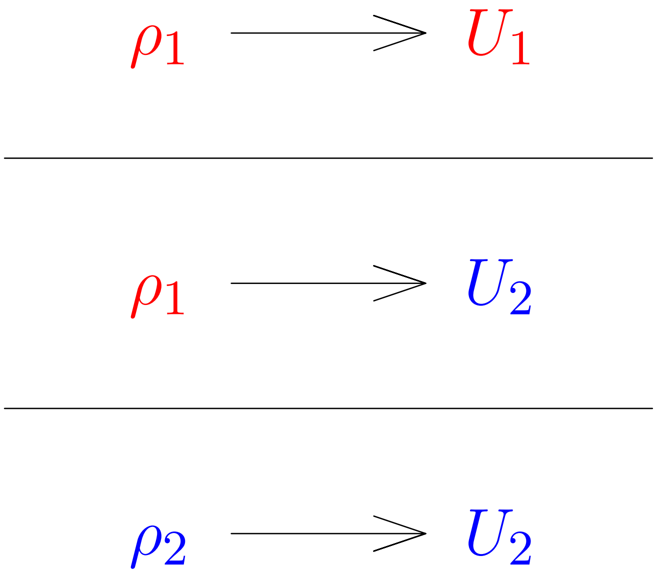
The basic flow

Classic two-layered stratified Kelvin-Helmholtz flow:



Coincident velocity and density interfaces.

A three-layered model:



Distinct velocity and density interfaces.

Stratified Kelvin-Helmholtz instability

- We shall only consider stable stratification: $\rho_2 \geq \rho_1$.
- Special cases:
  - Internal gravity waves: $U_1 = U_2$,

$$c = U_2 \pm \sqrt{\frac{g(\rho_2 - \rho_1)}{\alpha(\rho_1 + \rho_2)}}.$$

  - Homogeneous Kelvin-Helmholtz instability: $\rho_1 = \rho_2$,

$$c = \frac{1}{2}(U_1 + U_2) \pm \frac{\mathrm{i}}{2}(U_1 - U_2).$$

Stratified Kelvin-Helmholtz instability

- General case:

$$c = \frac{(\rho_1 U_1 + \rho_2 U_2)}{(\rho_1 + \rho_2)} \pm \sqrt{\frac{g(\rho_2 - \rho_1)}{\alpha(\rho_1 + \rho_2)} - \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)^2}(U_1 - U_2)^2}.$$

- Instability for short enough waves.
- Short waves, long waves,

$$c \sim \bar{U} \pm \mathrm{i}\frac{\sqrt{\rho_1 \rho_2}}{(\rho_1 + \rho_2)}(U_1 - U_2), \qquad c \sim \bar{U} \pm \sqrt{\frac{g(\rho_2 - \rho_1)}{\alpha(\rho_1 + \rho_2)}}.$$

where

$$\bar{U} = \frac{(\rho_1 U_1 + \rho_2 U_2)}{(\rho_1 + \rho_2)}.$$

Three-layered model

- Dispersion relation is fourth order in c.
- Nondimensionalize so that limiting cases can be examined:

  lengths by h, velocities by $(U_1 + U_2)/2$, density by ρ_1.

- Introduce dimensionless parameters:

$$r = \frac{(U_1 - U_2)}{(U_1 + U_2)}, \quad b = \frac{4gh(\rho_2 - \rho_1)}{(U_1 + U_2)^2(\rho_1 + \rho_2)} = F^{-2},$$

$$\rho = \frac{\rho_2}{\rho_1} \geq 1.$$

Three-layered model

- Short waves, $\alpha \gg 1$, (thick middle layer):

$$c \sim 1 \pm \mathrm{i}r, \quad 1 - r \pm \sqrt{\frac{b}{\alpha}}$$

or in dimensional form:

$$c \sim \frac{1}{2}(U_1 + U_2) \pm \frac{\mathrm{i}}{2}(U_1 - U_2), \quad U_2 \pm \sqrt{\frac{g(\rho_2 - \rho_1)}{\alpha(\rho_1 + \rho_2)}}.$$

- Homogeneous Kelvin-Holmholtz instability on the velocity interface.
- Internal gravity waves on the density interface.
- (As expected).

Three-layered model

- Long waves, $\alpha \ll 1$, (thin middle layer):

$$c \sim 1 - r\frac{(\rho - 1)}{(\rho + 1)} \pm \sqrt{\frac{b}{\alpha}}, \quad 1 - r \pm 2\mathrm{i}r\sqrt{\alpha},$$

  or in dimensional form:

$$c \sim \bar{U} \pm \sqrt{\frac{g(\rho_2 - \rho_1)}{\alpha(\rho_1 + \rho_2)}}, \quad U_2 \pm \mathrm{i}(U_1 - U_2)\sqrt{h\alpha}.$$

- Internal gravity waves, like the two-layered case.
- Instability, NOT like the two-layered case!

Three-layered model

- Long waves, $\alpha \ll 1$, in zero buoyancy limit, $b = 0$:

$$c \sim 1 - r\frac{(\rho - 1)}{(\rho + 1)} \pm 2\mathrm{i}\frac{\sqrt{\rho}}{(\rho + 1)}r, \quad c = 1 - r, \quad 1 - r,$$

  or in dimensional form:

$$c \sim \bar{U} \pm \mathrm{i}\frac{\sqrt{\rho_1 \rho_2}}{(\rho_1 + \rho_2)}(U_1 - U_2), \quad c = U_2, \quad U_2,$$

- as in short-wave limit of two-layered case.

Three-layered model

- Long waves, $\alpha \ll 1$, and small buoyancy, $b = b_0\alpha$:

$$c \sim 1 - r\frac{(\rho-1)}{(\rho+1)} \pm \sqrt{b_0 - \frac{4\rho r^2}{(1+\rho)^2}}, \quad 1 - r \pm 2\sqrt{\frac{b_0(\rho+1)r^2\alpha}{4r^2 - b_0(\rho+1)}}$$

  or in dimensional form:

$$c \sim \bar{U} \pm \sqrt{\frac{g(\rho_2-\rho_1)}{\alpha(\rho_1+\rho_2)} - \frac{\rho_1\rho_2}{(\rho_1+\rho_2)^2}(U_1-U_2)^2},$$

$$c \sim U_2 \pm (U_1-U_2)\sqrt{\frac{\alpha h g(\rho_2-\rho_1)}{\alpha\rho_1(U_1-U_2)^2 - g(\rho_2-\rho_1)}}.$$

- New mode is unstable for strong enough stable stratification!

Three-layered model

- For long waves, $\alpha \ll 1$, and small buoyancy, $b = b_0\alpha$:
  - The K-H mode is unstable for short enough waves:

$$\alpha > \frac{g(\rho_2^2 - \rho_1^2)}{\rho_1\rho_2(U_1 - U_2)^2}.$$

  - The new mode is unstable for long enough waves:

$$\alpha < \frac{g(\rho_2 - \rho_1)}{\rho_1(U_1 - U_2)^2}.$$

  - Waves are stable for

$$1 < \frac{\rho_1(U_1 - U_2)^2}{g(\rho_2 - \rho_1)}\alpha < 1 + \frac{\rho_1}{\rho_2}$$

- But for $b = O(1)$, this stable interval closes up, e.g. at $r = 1$ and $\rho = 2$, all waves are unstable for $b > 0.0073$.

Smooth profiles

- The generalization of the Rayleigh equation for inviscid disturbances to a stratified flow is the Taylor-Goldstein equation:

$$(U-c)(v''-\alpha^2 v) - U''v - \frac{b\rho_B'}{\rho_B(U-c)}v + \frac{\rho_B'}{\rho_B}[(U-c)v' - U'v] = 0$$

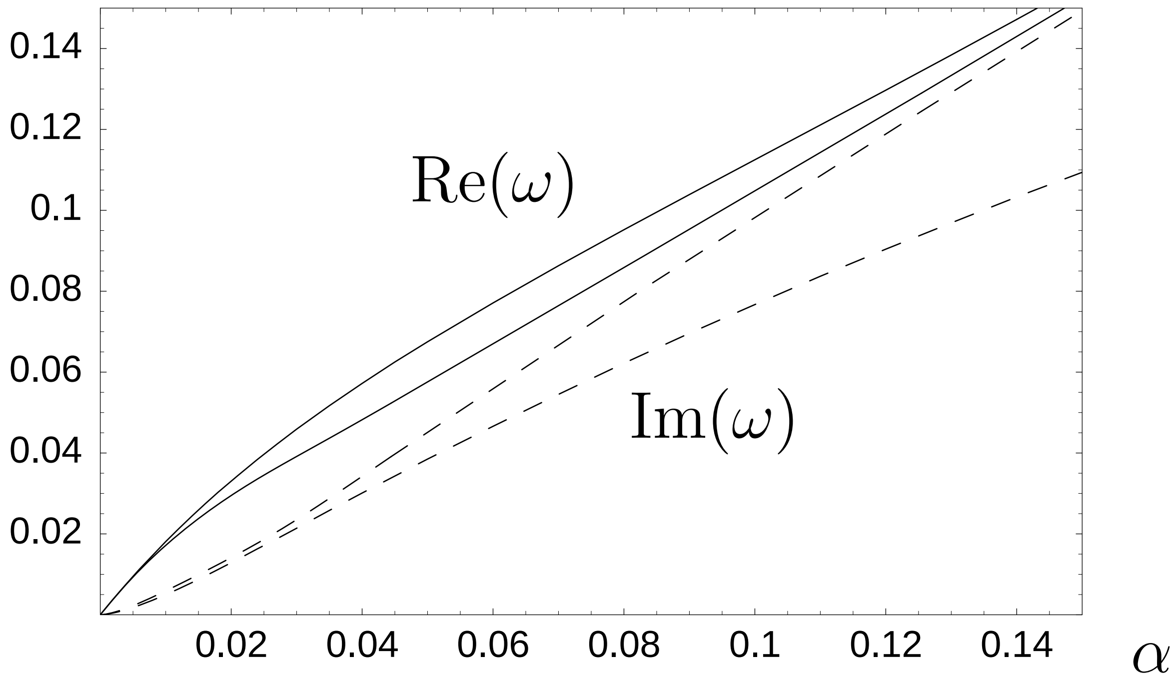
(including density variation in the inertia terms).

- Consider basic velocity U and basic density ρ_B:

$$\begin{aligned} U &= 1 + r\tanh y, \quad r = \frac{U_1 - U_2}{U_1 + U_2} \\ \rho_B &= 1 + \delta\tanh(y+h), \quad \delta = \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}. \end{aligned}$$

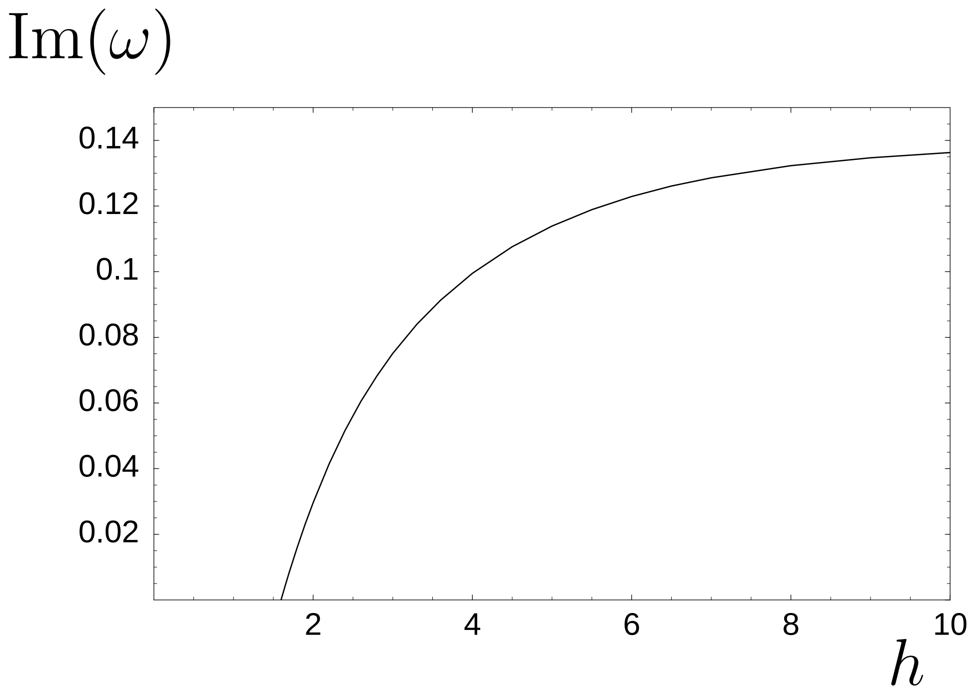
Numerical Taylor-Goldstein solutions

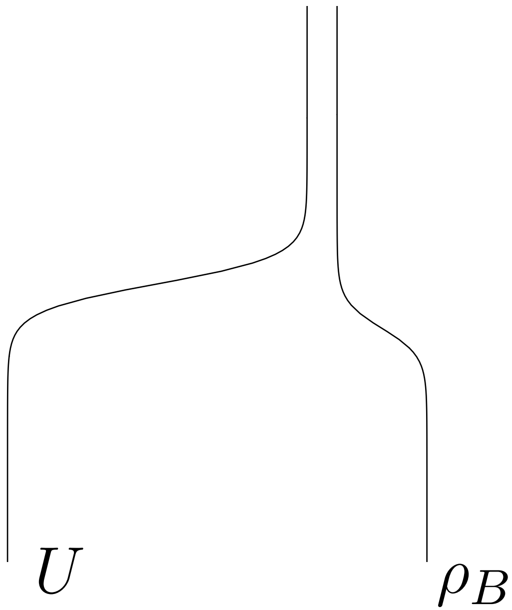
- Comparison between numerical Taylor-Goldstein solutions and analytic results of 3-layered model for $r = -1$, $b = 1$, $\rho_2/\rho_1 = 2$:



- Independent confirmation of long-wave instability.

Merging density and velocity layers

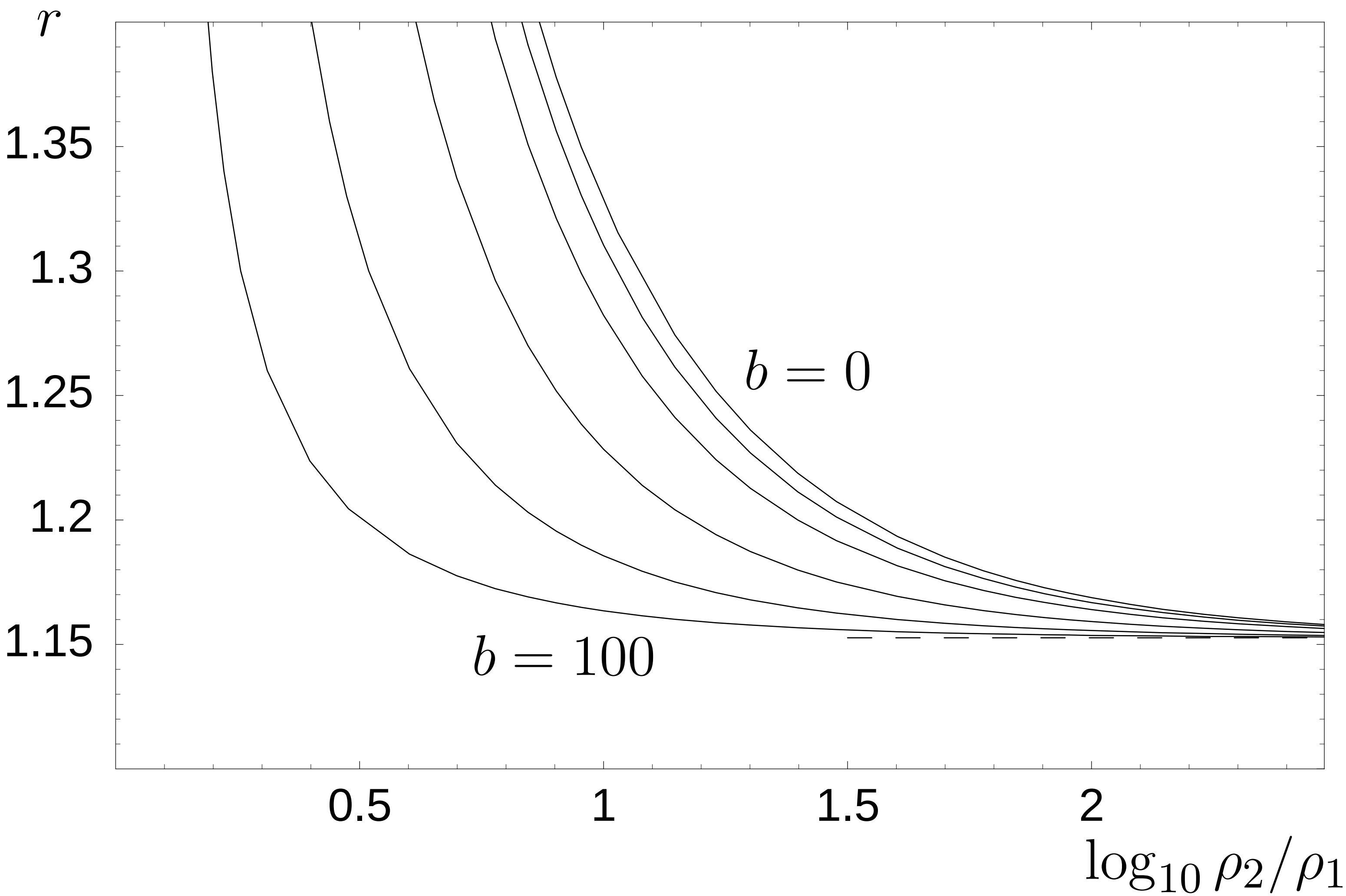




$\alpha = 0.2$, $r = -1$, $b = 1$, $\rho_2/\rho_1 = 2$

Layer separation
$h = 1.6$

Absolute instability of 3-layered flow



$b = 0,\ 1,\ 3,\ 10,\ 30,\ 100.$

Conclusions I: Absolute instability

- A change in density of the fluid has the same destabilizing effect on absolute instability as found previously for confinement by a rigid plate.
- Increasing buoyancy (i.e. stable stratification) causes the effect to occur at smaller density ratios.
- This corresponds to a reduction in Froud number, which enhances upstream propagation of internal gravity waves.
- Required density ratios probably rather large for terrestial oceanic/atmospheric flows.

Conclusions II: Temporal instability

- If the velocity jump and density jump do not coincide, then there is qualitatively different behaviour to the K-H case where they do coincide.
- Stably stratified K-H flow is stable for long waves.
- But three-layered flow is unstable for long waves.
- The new mode is destabilized by increasing stable stratification, and stabilized by increasing shear.
- Could be important, e.g., in wave generation when there is a shear layer in the air above a body of water.
- Results have been confirmed by numerical solution for smooth profiles.