Numerical Solution of 2D Flows in Atmospheric Boundary Layer

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The work deals with numerical solution of the 2D incompressible laminar flows over the profile DCA 10% for Reynold's numbers 10^5 and 10^6 and stratified flows in atmospheric boundary layer over the "sinus hill" with Reynolds numbers 10^8 and $5 \cdot 10^8$. Mathematical model for the 2D laminar flows over the profile DCA 10% is the system of Navier-Stokes equations for incompressible laminar flow and the Reynolds averaged Navier-Stokes equations (RANS) for incompressible turbulent flow with addition of the equation of density change (Boussinesq model) was used as a mathematical model for stratified flows in ABL. The artificial compressibility method and the finite volume method was used in all cases and the Lax-Wendroff scheme (Richtmyer form) was used in laminar cases and Lax-Wendroff scheme (MacCormack form) was used to compute turbulent stratified flows in ABL using Cebeci-Smith algebraic turbulence model.

1 Mathematical model

Navier-Stokes equations for 2D incompressible laminar flow were used as a mathematical model for flows over the profile DCA 10%:

$$\begin{aligned} u_x + v_y &= 0 && (1)\\ u_t + (u^2+p)_x + (u \cdot v)_y &= \nu \cdot (u_{xx} + u_{yy}) && (2)\\ v_t + (u \cdot v)_x + (v^2+p)_y &= \nu \cdot (v_{xx} + v_{yy}), && (3) \end{aligned}$$

where $(u,\ v)$ is a velocity vector, $p = \frac{P}{\rho}$ (P - static pressure), ρ - density, ν - kinematic viscosity. Using artificial compressibility method, continuity equation is completed by term $\frac{p_t}{\beta^2},\ \beta^2 \in \mathbb{R}^+$.

Reynolds averaged Navier-Stokes equations for 2D incompressible flows with addition of the equation of density change (Boussinesq model) were used as a mathematical model for flows over the "sinus hill" in ABL:

$$\begin{aligned} u_x + v_y &= 0 && (4)\\ u_t + (u^2+p)_x + (u \cdot v)_y &= (\nu + \nu_T) \cdot (u_{xx} + u_{yy}) && (5)\\ v_t + (u \cdot v)_x + (v^2+p)_y &= (\nu + \nu_T) \cdot (v_{xx} + v_{yy}) - \frac{\rho}{\rho_0} g && (6)\\ \rho_t + u \cdot \rho_x + v \cdot \rho_y &= 0, && (7) \end{aligned}$$

where $(u,\ v)$ is a velocity vector, $p = \frac{P}{\rho_0}$ (P- static pressure, ρ_0 - initial maximal density), ρ - density, ν - kinematic viscosity, ν_T - turbulent kinematic viscosity computed by Cebeci-Smith algebraic turbulence model and g - gravity acceleration. Using artificial compressibility method, continuity equation is completed by term $\frac{p_t}{\beta^2},\ \beta^2 \in \mathbb{R}^+$.
Density and pressure are changing depending on height (y-axis) as follows:

$$\begin{aligned} \rho_\infty(y) &= -\frac{\rho_0 - \rho_h}{h} \cdot y + \rho_0 && (8)\\ \frac{\partial p_\infty}{\partial y} &= -\frac{\rho_\infty(y)}{\rho_0} \cdot g && (9) \end{aligned}$$

The (8) is the linear decreasing function of density and the (9) is the hydrostatic pressure function.
It is possible to separate $p = p_\infty + p'$ and $\rho = \rho_\infty + \rho'$, where the therm p_∞ is the initial state of pressure, the term p' is the pressure disturbance, the therm ρ_∞ is the initial state of density and the therm ρ' is the density disturbance. After the substitution to (5) (6) we obtain following system of RANS:

$$\begin{aligned} u_x + v_y &= 0 && (10)\\ u_t + (u^2 + p')_x + (u \cdot v)_y &= (\nu + \nu_T) \cdot (u_{xx} + u_{yy}) && (11)\\ v_t + (u \cdot v)_x + (v^2 + p')_y &= (\nu + \nu_T) \cdot (v_{xx} + v_{yy}) - \frac{\rho'}{\rho_0} g && (12)\\ \rho_t + u \cdot \rho_x + v \cdot \rho_y &= 0 && (13) \end{aligned}$$

1.1 Boundary conditions for laminar flows over the profile DCA 10%

Inlet boundary condition has been set as follows: $u = u_\infty = 1.0,\ v = v_\infty = 0$ and the pressure term p has been extrapolated.
Outlet boundary conditions: $p = p_\infty$ and the velocity vector $(u,\ v)$ has been extrapolated.
Boundary conditions on the wall: $u = 0,\ v = 0,\ \frac{\partial p}{\partial n} = 0$
Boundary conditions on the upper domain boundary: symmetry: $\frac{\partial p}{\partial n} = 0,\ \frac{\partial u}{\partial n} = 0,\ v = 0$

1.2 Boundary conditions for stratified turbulent flows over the "sinus hill"

Inlet boundary condition has been set as follows: $u = u_\infty = 1.0,\ v = v_\infty = 0,\ \rho = \rho_\infty(y)$, where $\rho_\infty(y)$ is a linear function which is decreasing with increasing y:

$$\rho_\infty(y) = -\frac{\rho_0 - \rho_h}{h} \cdot y + \rho_0,$$

where ρ_0 is a lower (maximal) density and ρ_h is a upper (minimal) density (both are constants). Pressure change term p' has been extrapolated.
Outlet boundary conditions: $p' = 0$ and $(u,\ v)$ and density ρ have been extrapolated.
Boundary conditions on the wall: $u = 0,\ v = 0,\ \frac{\partial p}{\partial n} = \frac{\partial p_\infty}{\partial n} + \frac{\partial p'}{\partial n} = 0$ i.e. $\frac{\partial p'}{\partial n} = -\frac{\partial p_\infty}{\partial n}$ and $\frac{\partial \rho}{\partial n} = 0$.
Boundary conditions on the upper domain boundary: $p' = 0,\ \frac{\partial u}{\partial n} = 0,\ \frac{\partial v}{\partial n} = 0,\ \rho = \rho_h$

1.3 Turbulence model (Cebeci-Smith)

Domain Ω is divided into two subdomains. In the inner subdomain (near walls) the inner turbulent viscosity ν_{Ti} is computed. In the outer subdomain the outer turbulent viscosity ν_{To} is computed. Most common procedure is to compute both turbulent viscosities and use the minimal one:

$$\nu_T = \min\left(\nu_{Ti},\ \nu_{To}\right).$$

For turbulent viscosity computing is necessary to use local systems of coordinates $(X,\ Y)$, where X is parallel with the profile and Y is normal of the profile.
In inner subdomain the turbulent viscosity is defined as follows:

$$\nu_{Ti} = \rho l^2 \left| \frac{\partial U}{\partial Y} \right|,$$

where ρ is the density of fluid, $(U,\ V)$ are components of velocity vector in direction of $(X,\ Y)$ and l is given by equation:

$$l = \kappa \cdot Y \cdot \left[1 - \exp\left(-\frac{1}{A^+} u_r \cdot Y \cdot Re \right) \right],\ where\ u_r = \left(\nu \left| \frac{\partial U}{\partial Y} \right| \right)_\omega^{\frac{1}{2}}$$

In outer subdomain the turbulent viscosity is defined by Clauser's equation:

$$\nu_{To} = \frac{\rho\alpha\delta^* U_e}{1 + 5.5\ \left(\frac{Y}{\delta}\right)^6},$$

$U_e = U(\delta)$ where δ is the thickness of boundary layer and $\delta^* = \int_0^\delta \left(1 - \frac{U}{U_e}\right) dY.$
Following values of the constants were used: $\kappa = 0.4,\ \alpha = 0.0168,\ A^+ = 26.$

2 Numerical solution

In all cases the artificial compressibility method and the finite volume method have been used at structured grid of quadrilateral cells (in x direction uniform, in y direction refined near walls).
Term $\frac{p_t}{\beta^2}$ is added to the continuity equation (1) also (4). Other equations in both Navier-Stokes system (laminar) and RANS (turbulent with stratification) are without any changes. The new system of Navier-Stokes equations is:

$$\begin{aligned} u_x + v_y &= 0 && (14)\\ u_t + (u^2 + p)_x + (u \cdot v)_y &= \nu \cdot (u_{xx} + u_{yy}) && (15)\\ v_t + (u \cdot v)_x + (v^2 + p)_y &= \nu \cdot (v_{xx} + v_{yy}), && (16) \end{aligned}$$

in the vector form:

$$W_t + F_x + G_y = R_x + S_y \tag{17}$$

$$W = \left\| \begin{array}{c} \frac{p}{\beta^2} \\ u \\ v \end{array} \right\|,\ F = \left\| \begin{array}{c} u \\ u^2 + p \\ u \cdot v \end{array} \right\|,\ G = \left\| \begin{array}{c} v \\ u \cdot v \\ v^2 + p \end{array} \right\|,\ R = \nu \cdot \left\| \begin{array}{c} 0 \\ u_x \\ v_x \end{array} \right\|,\ S = \nu \cdot \left\| \begin{array}{c} 0 \\ u_y \\ v_y \end{array} \right\|, \tag{18}$$

and the new RANS system is:

$$\begin{aligned} \frac{p'_t}{\beta^2} + u_x + v_y &= 0 && (19)\\ u_t + (u^2 + p')_x + (u \cdot v)_y &= (\nu + \nu_T) \cdot (u_{xx} + u_{yy}) && (20)\\ v_t + (u \cdot v)_x + (v^2 + p')_y &= (\nu + \nu_T) \cdot (v_{xx} + v_{yy}) - \frac{\rho'}{\rho_0} g && (21)\\ \rho_t + u \cdot \rho_x + v \cdot \rho_y &= 0, && (22) \end{aligned}$$

in vector form:

$$W_t + F_x + G_y = (R_x + S_y) + K \tag{23}$$

where:

$$\begin{array}{c} W = \left\| \begin{array}{c} \frac{p'}{\beta^2} \\ u \\ v \\ \rho \end{array} \right\|, \qquad F = \left\| \begin{array}{c} u \\ u^2 + p' \\ u \cdot v \\ u \cdot \rho \end{array} \right\|, \qquad G = \left\| \begin{array}{c} v \\ u \cdot v \\ v^2 + p' \\ v \cdot \rho \end{array} \right\|, \\ R = (\nu + \nu_T) \left\| \begin{array}{c} 0 \\ u_x \\ v_x \\ 0 \end{array} \right\|, \quad S = (\nu + \nu_T) \cdot \left\| \begin{array}{c} 0 \\ u_y \\ v_y \\ 0 \end{array} \right\|, \quad K = \left\| \begin{array}{c} 0 \\ 0 \\ -\frac{\rho'}{\rho_0} g \\ 0 \end{array} \right\| \end{array} \tag{24}$$

The new Navier-Stokes and the new RANS system is parabolic for every $W = \left\| \frac{p}{\beta^2},\ u,\ v \right\|^T$ and $W = \left\| \frac{p'}{\beta^2},\ u,\ v,\ \rho \right\|^T$. It is necessary to use stable boundary conditions to govern when $t \to +\infty$ then $\left\| \frac{p}{\beta^2},\ u,\ v \right\|_t^T \to 0$ and $\left\| \frac{p'}{\beta^2},\ u,\ v,\ \rho \right\|_t^T \to 0$.

Lax-Wendroff scheme (Richtmyer form) was used to compute laminar flows over the profile DCA 10% for Reynolds numbers 10^5 and 10^6 in following form:

$$
\begin{aligned}
W_{ij}^{n+\frac{1}{2}} &= W_{ij}^n - \frac{\Delta t}{2\mu_{ij}} \sum_{k=1}^{4} \left[\left(\tilde{F}_k^n - \tilde{R}_k^n \right) \Delta y_k - \left(\tilde{G}_k^n - \tilde{S}_k^n \right) \Delta x_k \right] + \frac{\epsilon}{4} \sum_{k=1}^{4} \left(W_k^n - W_{ij}^n \right) \\
W_{ij}^{n+1} &= W_{ij}^n - \frac{\Delta t}{\mu_{ij}} \sum_{k=1}^{4} \left[\left(\tilde{F}_k^{n+\frac{1}{2}} - \tilde{R}_k^{n+\frac{1}{2}} \right) \Delta y_k - \left(\tilde{G}_k^{n+\frac{1}{2}} - \tilde{S}_k^{n+\frac{1}{2}} \right) \Delta x_k \right] + AD_{ij}^n,
\end{aligned}
$$

where AD_{ij}^n is the Jameson's artificial disipation which has been used to stabilize numerical solution and:

$$
\tilde{F}_1^n = \frac{1}{2} \left(F_{i,j} + F_{i,j-1} \right),\ \tilde{F}_2^n = \frac{1}{2} \left(F_{i,j} + F_{i+1,j} \right),\ \tilde{F}_3^n = \frac{1}{2} \left(F_{i,j} + F_{i,j+1} \right),\ \tilde{F}_4^n = \frac{1}{2} \left(F_{i,j} + F_{i-1,j} \right).
$$

$\tilde{G}_k^n$ were computed in the same way as $\tilde{F}_k^n$. $\tilde{R}_k^n,\ \tilde{S}_k^n$ were computed on dual grid using finite volume method.

Lax-Wendroff scheme (MacCormack form) was used to compute stratified flows over the "sinus hill" for Reynolds numbers 10^8 and $5 \cdot 10^8$ in the following form (published in (5)):

$$
\begin{aligned}
W_{ij}^{n+\frac{1}{2}} &= W_{ij}^n - \frac{\Delta t}{\mu_{ij}} \left(\left\{ \sum_{k=1}^{4} \left[\left(\widehat{F}_k^n - \widehat{R}_k^n \right) \Delta y_k - \left(\widehat{G}_k^n - \widehat{S}_k^n \right) \Delta x_k \right] \right\} - \widehat{K}_{ij}^n \cdot \mu_{ij} \right) \\
W_{ij}^{n+1} &= \frac{1}{2} (W_{ij}^n + W_{ij}^{n+\frac{1}{2}}) - \\
&- \frac{\Delta t}{2\mu_{ij}} \left(\left\{ \sum_{k=1}^{4} \left[\left(\widehat{F}_k^{n+\frac{1}{2}} - \widehat{R}_k^{n+\frac{1}{2}} \right) \Delta y_k - \left(\widehat{G}_k^{n+\frac{1}{2}} - \widehat{S}_k^{n+\frac{1}{2}} \right) \Delta x_k \right] \right\} - \widehat{K}_{ij}^{n+\frac{1}{2}} \cdot \mu_{ij} \right) + AD_{ij}^n,
\end{aligned}
\tag{25}
$$

where AD_{ij}^n is the Jameson's artificial disipation which has been used to stabilize numerical solution and:

$$
\widehat{F}_1^n = F_{i-1,j}^n,\ \ \widehat{G}_1^n = G_{i-1,,j}^n,\ \widehat{F}_1^{n+\frac{1}{2}} = F_{i,j}^{n+\frac{1}{2}},\ \widehat{G}_1^{n+\frac{1}{2}} = G_{i,,j}^{n+\frac{1}{2}}
$$

$$
\widehat{F}_2^n = F_{i,j-1}^n,\ \ \widehat{G}_2^n = G_{i,,j-1}^n,\ \widehat{F}_2^{n+\frac{1}{2}} = F_{i,j}^{n+\frac{1}{2}},\ \widehat{G}_2^{n+\frac{1}{2}} = G_{i,,j}^{n+\frac{1}{2}}
$$

$$
\widehat{F}_3^n = F_{i,j}^n,\ \ \widehat{G}_3^n = G_{i,,j}^n,\ \widehat{F}_3^{n+\frac{1}{2}} = F_{i+1,j}^{n+\frac{1}{2}},\ \widehat{G}_3^{n+\frac{1}{2}} = G_{i,+1,j}^{n+\frac{1}{2}}
$$

$$
\widehat{F}_4^n = F_{i,j}^n,\ \ \widehat{G}_4^n = G_{i,,j}^n,\ \widehat{F}_4^{n+\frac{1}{2}} = F_{i,j+1}^{n+\frac{1}{2}},\ \widehat{G}_4^{n+\frac{1}{2}} = G_{i,,j+1}^{n+\frac{1}{2}}
$$

$R,\ S$ are computed in the same way in both n and $n+1$ time layer:

$$
\widehat{R}_1 = \frac{1}{2} \left(R_{i,j} + R_{i-1,j} \right),\ \widehat{R}_2 = \frac{1}{2} \left(R_{i,j} + R_{i,j-1} \right)\ \widehat{R}_3 = \frac{1}{2} \left(R_{i+1,j} + R_{i,j} \right),\ \widehat{R}_4 = \frac{1}{2} \left(R_{i,j+1} + R_{i,j} \right)
$$

$$
\widehat{K}_{ij}^n = \left\| 0,\ f \cdot v_{ij}^n,\ \frac{\rho_{ij}^n}{\rho_0} g,\ 0 \right\|^T,\ \widehat{K}_{ij}^{n+\frac{1}{2}} = \left\| 0,\ f \cdot v_{ij}^{n+\frac{1}{2}},\ \frac{\rho_{ij}^{n+\frac{1}{2}}}{\rho_0} g,\ 0 \right\|^T
$$

3 Numerical results

The following cases of laminar neutrally stratified flows and stratified turbulent flows were computed. Authors consider flows over the profile DCA 10% (laminar neutrally stratified flows) with $Re = 10^5$ and $Re = 10^6$ and the "sinus hill" (10% of domain height - stratified turbulent flows) and the figures show results with $Re = 10^8$ and $Re = 5 \cdot 10^8$ with density change $\rho_\infty \in [1.2;\ 1.1]$. For the future also higher "sinus hill" and greater range of density will be considered.

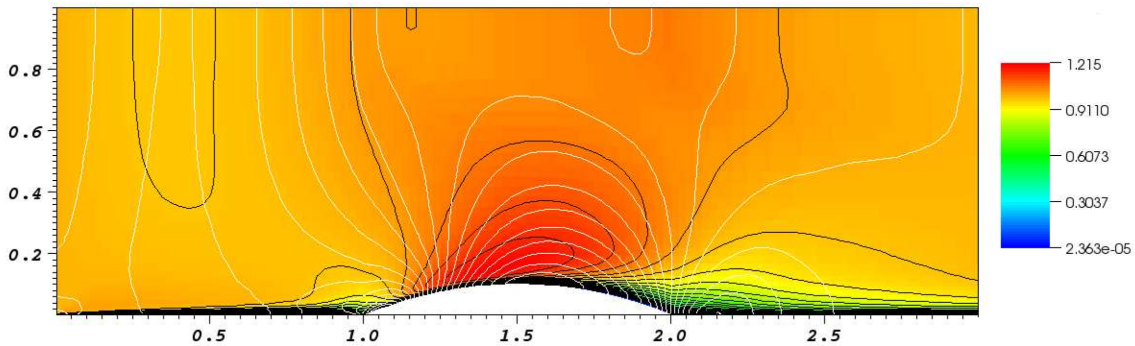

Figure 1: DCA 10% - incompressible viscous laminar flow, $Re = 10^5,\ u_\infty = 1.0$, white lines - contours of pressure, black lines - contours of velocity magnitude

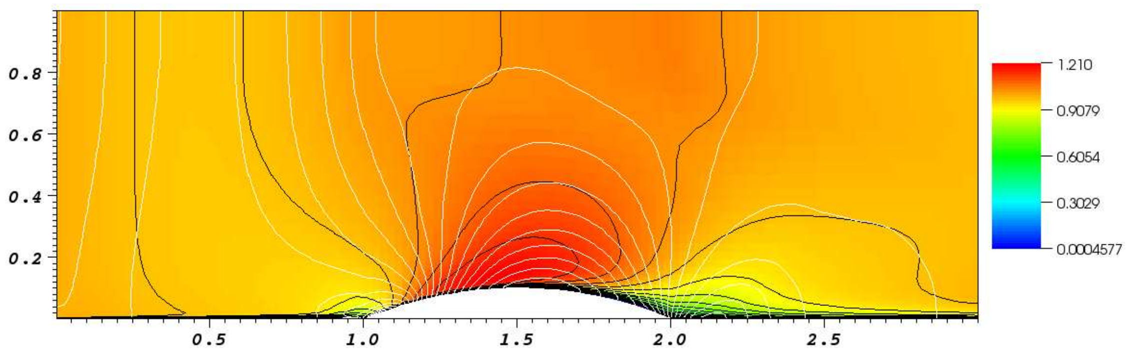

Figure 2: DCA 10% - incompressible viscous laminar flow, $Re = 10^6,\ u_\infty = 1.0$, white lines - contours of pressure, black lines - contours of velocity magnitude

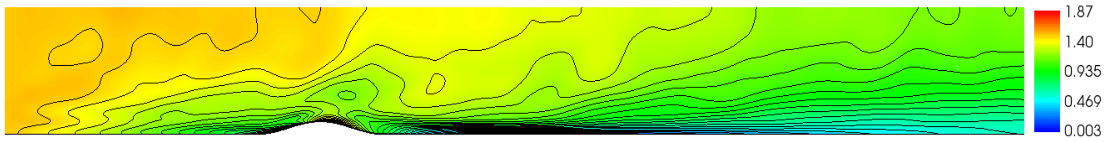

Figure 3: "Sinus hill 10%" - incompressible viscous turbulent stratified flow, $Re = 10^8,\ u_\infty = 1.5\, m{\cdot}s^{-1}$, contours of velocity magnitude $[m \cdot s^{-1}]$

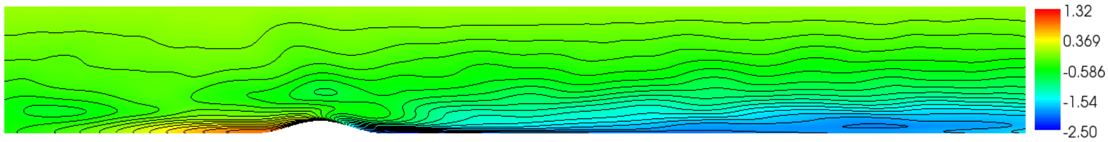

Figure 4: "Sinus hill 10%" - incompressible viscous turbulent stratified flow, $Re = 10^8,\ u_\infty = 1.5\, m{\cdot}s^{-1}$, contours of pressure disturbances $[pa]$

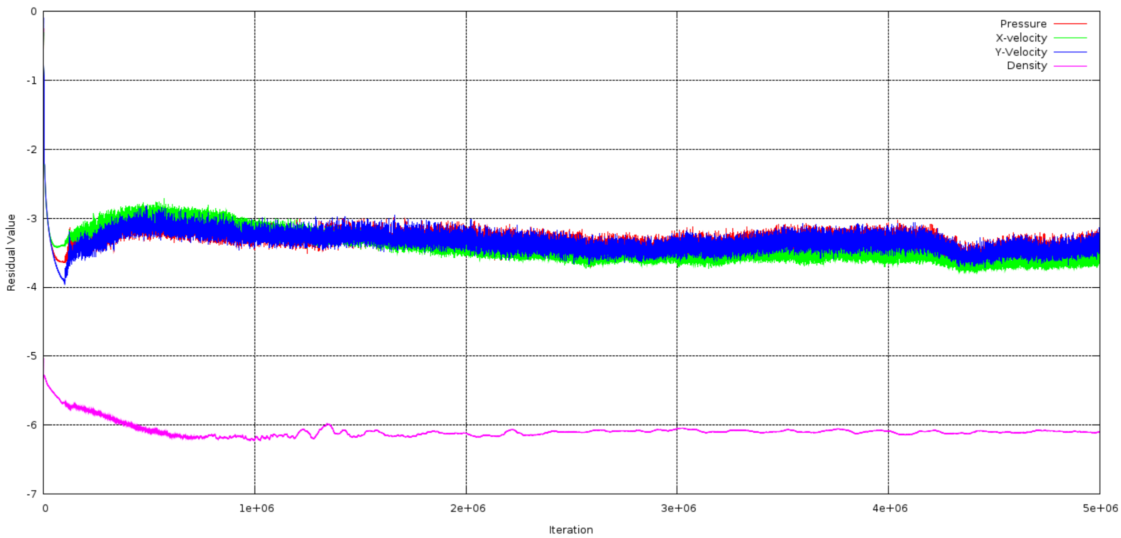

Figure 5: "Sinus hill 10%" - incompressible viscous turbulent stratified flow, $Re = 10^8,\ u_\infty = 1.5\, m{\cdot}s^{-1}$, Residuals

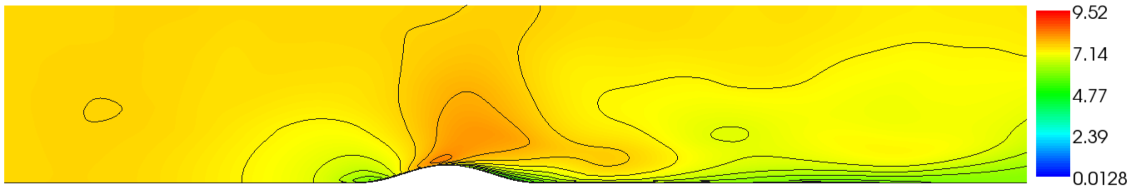

Figure 6: "Sinus hill 10%" - incompressible viscous turbulent stratified flow, $Re = 5 \cdot 10^8$, $u_\infty = 7.5\ m \cdot s^{-1}$, contours of velocity magnitude $[m \cdot s^{-1}]$

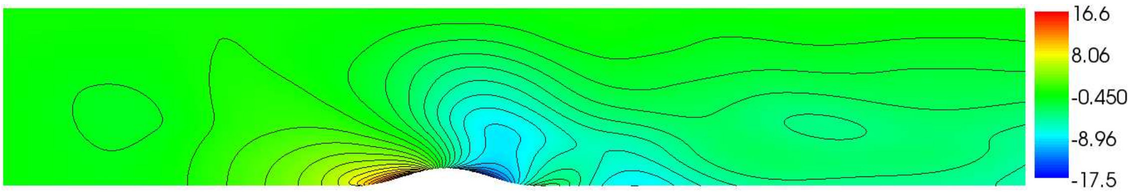

Figure 7: "Sinus hill 10%" - incompressible viscous turbulent stratified flow, $Re = 5 \cdot 10^8$, $u_\infty = 7.5\ m \cdot s^{-1}$, contours of pressure disturbances $[pa]$

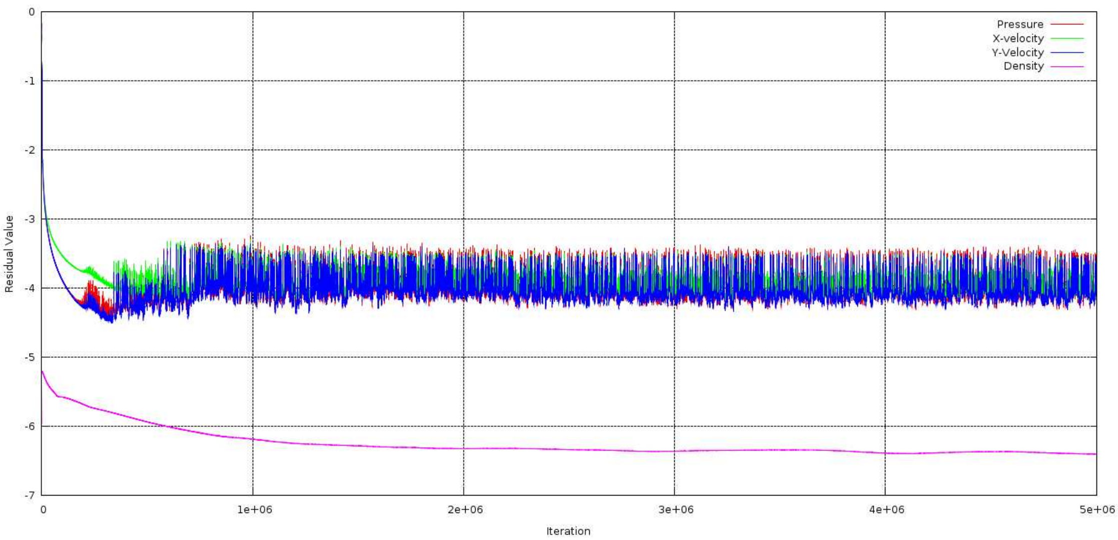

Figure 8: "Sinus hill 10%" - incompressible viscous turbulent stratified flow, $Re = 5 \cdot 10^8$, $u_\infty = 7.5\ m \cdot s^{-1}$, Residuals

4 Closure

Numerical method solving incompressible laminar viscous flow and turbulent stratified viscous flow near ground has been developed and applied to the flow over the profile DCA 10% (laminar flows) and over the "sinus hill" with good results. Continuation of our work expects using two-equation turbulence model, for test case with a higher hill as well as an implicit scheme.

5 Acknowledgement

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References

[1] Dvořák R., Kozel K.: Mathematical modeling in aerodynamics, CTU, Prague 1996, (in Czech)

[2] Feistauer M., Felcman J., Straškraba I.: Mathematical and computational methods for compressible flow, Clarendon Press, Oxford 2003

[3] Hirsh C.: Numerical computation of internal and external flows I. and II., John Willey and Sons, New York 1991

[4] Fletcher C. A. J.: Computational techniques for fluid dynamics I. and II., Springer Verlag, Berlin 1996

[5] Šimonek J., Kozel K., Trefilík J.: Numerical solution of inviscid and viscous flow with application, Topical problems of fluid machanics 2007, Prague 2007

[6] Šimonek J., Kozel K., Fraunié Ph., Jaňour Z.: Numerical Solution of 2D Stratified Flows in Atmospheric Boundary Layer, Topical problems of fluid machanics 2008, Prague 2008

[7] Uchida T., Ohya Y.: Numerical study of stably stratified flows over a two dimensional hill in a channel of finite depth, Fluid dynamics research 29/2001 (p. 227 - 250)