# ON NUMERICAL APPROXIMATION OF AN AEROELASTIC PROBLEM

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## Introduction

The fluid-structure interaction play important role in many technical applications. In this paper the numerical approximation of fluid-structure interaction (FSI) problems is address. During last years, significant advances have been made in the development and use of computational methods for fluid flows with structural interactions. The arbitrary Lagrangian-Eulerian (ALE) formulations are widely used. The application of the ALE method is straightforward; however there are a number of important computational issues which needs to be properly addressed, cf. [1]. Here, the conservative ALE formulation of Navier-Stokes system of equations is employed (cf. [2]), weakly formulated and discretized by the stabilized finite element method (FEM). The structure model is described by the system of ordinary differential equations.

## Mathematical model

The flow of incompressible viscous fluid is described by the Navier Stokes equations. In order to treat the moving domain case we use the Navier-Stokes system of equations written in ALE conservative form, i.e. for $i = 1, 2$

$$\frac{1}{\mathcal{J}} \frac{D^{\mathcal{A}}}{Dt}\Big(\mathcal{J} v_i\Big) + \operatorname{div}\left((\mathbf{v} - \mathbf{w}_D) v_i\right) - \nu \triangle v_i + \frac{\partial p}{\partial x_i} = 0, \qquad \operatorname{div} \mathbf{v} = 0, \text{ on } \Omega_t \subset \mathbb{R}^2, \tag{1}$$

where $\mathcal{A} = \mathcal{A}(\xi, t)$ is the ALE mapping of $\xi \in \Omega_0$ onto $x \in \Omega_t$, $\mathbf{w}_D$ denotes the domain velocity given by

$$\mathbf{w}_D(x,t) = \frac{\partial \mathcal{A}}{\partial t}(\xi, t), \qquad \text{with } x = \mathcal{A}(\xi, t),$$

$\mathcal{J}$ denotes the Jacobian of the mapping $\mathcal{A}$, $\mathcal{J} = \frac{d\mathcal{A}}{d\xi}$ and $D^{\mathcal{A}}/Dt$ denotes the ALE derivative. The system (1) is equipped with boundary and initial conditions. Now, we take a test function $\mathbf{z} = \mathbf{z}(x,t)$ in the form $\mathbf{z} = \hat{\mathbf{z}} \circ \mathcal{A}_t^{-1}$ with $\hat{\mathbf{z}} \in \mathbf{H}^1(\Omega_0)$ and formulate equations (1) weakly, i.e.

$$\frac{d}{dt}\Big(\mathbf{v}, \mathbf{z}\Big)_{\Omega_t} + c(\tilde{\mathbf{w}}; \mathbf{v}, \mathbf{z}) + \nu(\nabla \mathbf{v}, \nabla \mathbf{z})_{\Omega_t} - \Big(p, \nabla^T \mathbf{z}\Big)_{\Omega_t} + \Big(\nabla \cdot \mathbf{w}_D\, \mathbf{v}, \mathbf{z}\Big)_{\Omega_t} = 0 \tag{2}$$

where $\tilde{\mathbf{w}} = \mathbf{v} - \mathbf{w}_D$, $(\cdot,\cdot)_{\Omega_t}$ denotes the dot product in $L^2(\Omega_t)$/$\mathbf{L}^2(\Omega_t)$ and the trilinear form $c(\cdot;\cdot,\cdot)$ is defined by $c(\tilde{\mathbf{w}_D}; \mathbf{v}, \mathbf{z}) = \int_{\Omega_t} (\tilde{\mathbf{w}_D} \cdot \nabla)\mathbf{v} \cdot \mathbf{z}\, \mathrm{dx}$. The structure model is described by a system of second order ordinary differential equations

$$\mathbb{M}\, \ddot{\mathbf{u}} + \mathbb{B}\dot{\mathbf{u}} + \mathbb{K}\, \mathbf{u} = \mathbf{f}, \tag{3}$$

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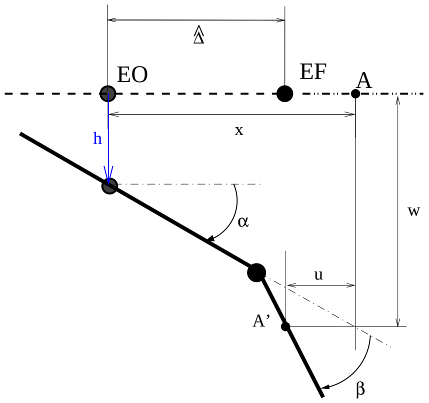
where $\mathbf{u} = (h, \alpha, \beta)^T$, $\mathbf{f} = (-L, M_\alpha, M_\beta)^T$, $\mathbb{K} = \text{diag}(k_{hh}, k_{\alpha\alpha}, k_{\beta\beta})$, $\mathbb{D} = \text{diag}(d_{hh}, d_{\alpha\alpha}, d_{\beta\beta})$, and

$$\mathbb{M} = \begin{pmatrix} m & S_\alpha & S_\beta \\ S_\alpha & I_\alpha & \widehat{\Delta} S_\beta + I_\beta \\ S_\beta & \widehat{\Delta} S_\beta + I_\beta & I_\beta \end{pmatrix},$$

where $m$ is the mass of the airfoil, $S_\alpha, I_\alpha$ are the static and inertia of the airfoil, $S_\beta, I_\beta$ are the static and inertia moment of the control section. Furthermore, $h$ denotes the vertical displacements, $\alpha$ denotes the rotation of the airfoil and $\beta$ denotes the rotation of the control section, see Figure 1. $L, M_\alpha, M_\beta$ then denotes the aerodynamical forces,



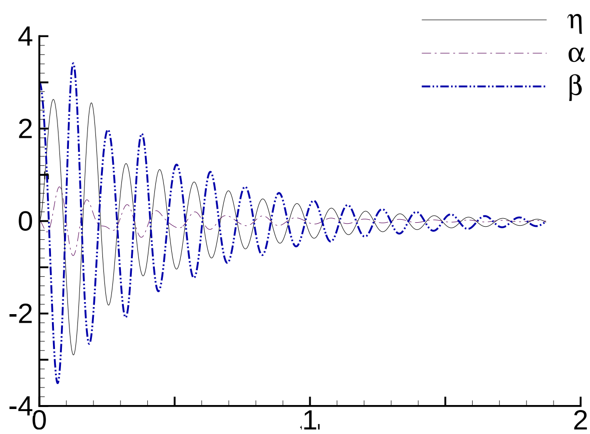




Figure 1: The 3dof structure model on the left. On the right the aeroelastic response $(h = \eta, \alpha, \beta)$ from numerical simulation for far field velocity $U_\infty = 17\,\text{m/s}$.

## Space-Time discretization

In order to discretize problem (2) we consider a time step $\tau > 0$, denote $t_k = k\tau$ and at every time step $t_k$ employ the approximation $\mathbf{v}^k \approx \mathbf{v}(\cdot, t_k)$ and $p^k \approx p(\cdot, t_k)$. The time derivative in the weak formulation (2) is approximated at time $t = t_{n+1}$ by the second order backward difference formula. The time disretized problem is then discretized with the aid of FEM: we seek for unknown functions $(\mathbf{v}, p)$ in the finite element spaces $\mathbf{v} \in \mathcal{W}_\triangle$ and $p \in \mathcal{Q}_\triangle$ such that for all test functions $\mathbf{z} \in \mathcal{X}_\triangle$ and $q \in \mathcal{Q}_\triangle$ holds

$$\Big(3\mathbf{v}/(2\tau), \mathbf{z}\Big)_{\Omega_{n+1}} + c(\tilde{\mathbf{w}}; \mathbf{v}, \mathbf{z}) + \nu(\nabla\mathbf{v}, \nabla\mathbf{z})_{\Omega_t} - \Big(p, \nabla^T\mathbf{z}\Big)_{\Omega_t} + \Big((\nabla\cdot\mathbf{w}_D^{n+1})\,\mathbf{v}, \mathbf{z}\Big)_{\Omega_t} = \mathcal{L}(\mathbf{z})/(2\tau)$$

where $\mathcal{L}(\mathbf{z}) = (4\mathbf{v}^n, \mathbf{z})_{\Omega_n} - (\mathbf{v}^{n-1}, \mathbf{z})_{\Omega_{n-1}}$ and $\mathbf{v}^{n+1} := \mathbf{v}$, $p^{n+1} := p$. In the practical implementation the stabilized finite element method is considered and Taylor-Hood couple of finite elements $(\mathcal{W}_\triangle, \mathcal{Q}_\triangle)$ is employed, which satisfies Babuška-Brezzi inf-sup condition.

## References

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[2] F. Nobile. *Numerical approximation of fluid-structure interaction problems with application to haemodynamics*. PhD thesis, Ecole Polytechnique Federale de Lausanne, 2001.