SOLUTIONS OF NAVIER-STOKES EQUATIONS WITH NON-DIRICHLET BOUNDARY CONDITIONS

Petr Kučera, Jitka Píšová
Czech Technical University, Prague

Introduction

In this contribution we deal with the system of the non-steady Navier-Stokes equations with mixed boundary conditions. We present here survey of some recent results.

We suppose that Ω is a bounded domain in $\mathbf{R}^m$ with a Lipschitz boundary for $m = 2$ or $m = 3$. Γ_1 and Γ_2 are open disjoint subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$, $\Gamma_1 \neq \emptyset$ and the $(m-1)$–dimensional measure of $\partial\Omega - (\Gamma_1 \cup \Gamma_2)$ is zero. The domain Ω represents a channel, Γ_1 is a fixed wall and Γ_2 is both the input and the output of the channel. Let $T \in (0, \infty)$, $(0, T)$ is a time interval and $Q = \Omega \times (0, T)$. We consider the initial–boundary value problem

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\mathcal{P} = \boldsymbol{g} \qquad \text{in } Q, \tag{1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } Q, \tag{2}$$

$$\boldsymbol{u} = 0 \qquad \text{in } \Gamma_1 \times (0,T), \tag{3}$$

$$-\mathcal{P}\boldsymbol{n} + \nu\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} = \boldsymbol{\sigma} \qquad \text{in } \Gamma_2 \times (0,T), \tag{4}$$

$$\boldsymbol{u}(0) = \boldsymbol{\gamma} \qquad \text{in } \Omega. \tag{5}$$

Here $\boldsymbol{u} = (u_1, \dots, u_m)$ denotes the velocity, $\mathcal{P}$ represents the pressure, ν denotes the kinematic viscosity, $\boldsymbol{g}$ is a body force, $\boldsymbol{\sigma}$ is a prescribed vector function on Γ_2, $\boldsymbol{n} = (n_1, \dots, n_m)$ is the outer normal vector on $\partial\Omega$ and $\boldsymbol{\gamma}$ is an initial velocity. We suppose for simplicity that $\nu = 1$ throughout the whole paper.

The Navier–Stokes equations have been mostly considered with the Dirichlet boundary condition. The theory of the Navier–Stokes equations with this boundary condition is relatively deeply elaborated (the results on the global in time existence of weak solutions, uniqueness of weak solutions in an appropriate function space, local in time existence of strong solutions, global in time existence of strong solutions). However, the homogeneous Dirichlet boundary condition (i.e. the no–slip boundary condition) is not natural in some situations – for example on the output of a channel. Some authors therefore use boundary condition (4) or

$$-\mathcal{P}\boldsymbol{n} + \nu\,\boldsymbol{e}(\boldsymbol{u})\cdot\boldsymbol{n} = \boldsymbol{\sigma} \tag{6}$$

on the input and on the output. By $\boldsymbol{e}(\boldsymbol{u})$ we denote the symmetric part of the velocity gradient with components $e_{ij} = \left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right)/2$.

Boundary condition (4) do not exclude the possibility of backward flows that could eventually bring back to the channel an uncontrollable amount of kinetic energy. Consequently, they do not enable us to obtain a priori estimate of a weak solution. Due to this fact, the problem of the global in time existence of our problem is still open.

In [3] and [4], Kračmar and Neustupa prescribed an additional condition on the output (which bounds the kinetic energy of the backward flow), formulated steady and evolutionary

Navier–Stokes problems by means of certain variational inequalities of the Navier–Stokes type and proved the existence of a weak solution of these inequalities.

In [5], Kučera and Skalák proved the local–in–time existence of a strong solution of the non–steady Navier–Stokes problem with boundary condition (4) on the part of the boundary.

In [6], Kučera proved the global–in–time existence and uniqueness of a strong solution in a small neighbourhood of another known solution.

In [1], Beneš proved the local–in–time existence of the strong solution (in the sense that the solution possess second spatial derivatives) to our system.

In [2], Beneš and Kučera proved the global–in–time existence and uniqueness of a strong solution in a small neighbourhood of another known solution in the sense that the solution possess second spatial derivatives.

In [7], Kučera solved problem in which we prescribe condition

$$\boldsymbol{u}(0) = \boldsymbol{u}(T) \tag{7}$$

instead of (5). He defined a solution $\boldsymbol{u}$ of this problem to be regular if the problem is uniquely solvable in its neighbourhood. He proved that a set of solution which are not regular is small in a specific sence.

Acknowledgements: The research was supported by the project SGS12/100/OHK1/2T/11.

Reference

[1] M. Beneš, Mixed Initial-Boundary Value Problem for the Three-dimensional Navier–Stokes Equations in Polyhedral Domains. DCDS suppl. **1**, 135–144, (2011).

[2] M. Beneš, P. Kučera, Solutions of the Navier–Stokes equations with various types of boundary conditions. Arch. Math. **98**, 487-497, (2012).

[3] S. Kračmar, J. Neustupa, Modelling of flows of a viscous incompressible fluid through a channel by means of variational inequalities. ZAMM **74**, 637-639, (1994).

[4] S. Kračmar, J. Neustupa, A weak solvability of a steady variational inequality of the Navier-Stokes type with mixed boundary conditions, Proceedings of the Third World Congress of Nonlinear Analysis, Nonlinear Anal. **47**, 4169–4180, (2001).

[5] P. Kučera, Z. Skalák, Solutions to the Navier-Stokes Equations with Mixed Boundary Conditions, Acta Applicandae Mathematicae, **54**, Kluwer Academic Publishers, 275-288, (1998).

[6] P. Kučera, Basic properties of solution of the non-steady Navier-Stokes equations with mixed boundary conditions in a bounded domain. Ann. Univ. Ferrara **55**, 289–308, (2009).

[7] P. Kučera, The time periodic solutions of the Navier-Stokes Equations with mixed boundary conditions. Discrete and continuous dznamical systems series S. **3**, 325–337, (2010).