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Inverse problems for symmetric doubly stochastic matrices whose Suleĭmanova spectra are to be bounded below by $1/2$

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INVERSE PROBLEMS FOR SYMMETRIC DOUBLY STOCHASTIC MATRICES WHOSE SULEĬMANOVA SPECTRA ARE TO BE BOUNDED BELOW BY 1/2

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Abstract. A new sufficient condition for a list of real numbers to be the spectrum of a symmetric doubly stochastic matrix is presented; this is a contribution to the classical spectral inverse problem for symmetric doubly stochastic matrices that is still open in its full generality. It is proved that whenever $\lambda_2, \ldots, \lambda_n$ are non-positive real numbers with $1 + \lambda_2 + \ldots + \lambda_n \geqslant 1/2$, then there exists a symmetric, doubly stochastic matrix whose spectrum is precisely $(1, \lambda_2, \ldots, \lambda_n)$. We point out that this criterion is incomparable to the classical sufficent conditions due to Perfect–Mirsky, Soules, and their modern refinements due to Nadar *et al.* We also provide some examples and applications of our results.

1. Introduction

A square matrix with real entries is termed *stochastic* if all its terms are non-negative and all the rows add up to 1. Stochastic matrices are conveniently interpreted as transition matrices of finite-state Markov chains (hence the terminology). The aim of this note is to consider the inverse eigenvalue problem for doubly stochastic matrices (also called *bistochastic* in the literature; a stochastic matrix is *doubly stochastic* if its transpose is stochastic too). Doubly stochastic matrices may be interpreted as transition matrices of finite-state symmetric Markov chains. Permutation matrices are paradigm examples of doubly stochastic matrices; according to Birkhoff's theorem, the set of $n \times n$ doubly stochastic matrices is the convex hull of the set of permutation matrices of an n-element set.

Inverse eigenvalue problems for classes of matrices such as (symmetric or not) matrices with non-negative entries, stochastic, or doubly stochastic are well-rooted in the literature, having their origin in the works of Suleĭmanova [18, 19] and, independently, Perfect ([13, 14]) with an important subsequent continuation by Perfect and Mirsky [15]. Recently, the problem has gained new impetus as reflected by a plethora of new sufficient conditions ([2, 4, 5, 6, 9, 10]). We refer to Mourad's paper [8] for a good overview concerning the said problems.

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More explicitly, the symmetric doubly stochastic eigenvalue inverse problem (SDIEP) asks the following:

Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ be a list of real numbers. In what circumstances does there exist a symmetric doubly stochastic matrix whose spectrum consists of these numbers?

(Of course, the inverse problems for other classes of matrices are totally analogous.) Since any matrix A solving the above problem has non-negative entries, by the classical Frobenius–Perron theorem, A must have a non-negative eigenvalue λ_1 (that is called *the Perron eigenvalue* of A) such that $\lambda_1 \geqslant |\lambda|$ for any other eigenvalue λ of A (and the eigenvector associated to λ_1 has non-negative entries). It is to be noted that already for stochastic matrices, $\lambda_1 = 1$ is the Perron eigenvalue to which corresponds the eigenvector comprising only 1s. Consequently, without loss of generality we shall restrict ourselves to λs from the interval $[-1, 1]$.

Since the trace of (any power of) a square matrix with non-negative entries is non-negative, for a list $(1, \lambda_2, \ldots, \lambda_n)$ of real numbers to form a spectrum of a solution to SDIEP, it is necessary that $1 + \lambda_2^k + \ldots + \lambda_n^k \geqslant 0$ for any $k \in \mathbb{N}$. Perfect and Mirsky [15] provided a fairly general sufficient condition for the possibility of solving SDIEP for a list of real numbers $1 = \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant -1$. Namely, they proved that as long as

$$\frac{1}{n} + \frac{1}{n(n-1)}\lambda_2 + \ldots + \frac{1}{2 \cdot 1}\lambda_n \geqslant 0, \tag{1.1}$$

there exists a symmetric doubly stochastic $n \times n$ matrix whose spectrum is equal to $\{1, \lambda_2, \ldots, \lambda_n\}$. This condition was subsequently refined by Soules [16], who gave a finer condition depending on the remainder of n when divided by 2.

More specifically, let m be such that $n = 2m + 1$ in the case n is odd and $n = 2m + 2$ in the case where n is even. If

$$\frac{1}{n} + \frac{n-m-1}{n(m+1)}\lambda_2 + \sum_{k=1}^{n} \frac{1}{(k+1)k}\lambda_{n-2k+2} \geqslant 0, \tag{1.2}$$

then there exists a symmetric doubly stochastic $n \times n$ matrix whose spectrum coincides with $\{1, \lambda_2, \ldots, \lambda_n\}$.

Soules' condition was refined further by Nader *et al.* who arrived at a more complicated condition that depends on the remainder n mod 4 ([10, Theorem 5]) that covers more cases. Following the terminology introduced by Papardella, we call a list of real numbers $\sigma = (1, \lambda_2, \ldots, \lambda_n)$ a *normalised Suleĭmanova spectrum*, whenever $\lambda_j \leqslant 0$ for $j = 2, \ldots, n$ and $1 + \lambda_2 + \ldots + \lambda_n \geqslant 0$.

Nader *et al.* considered the problem of whether normalised Suleĭmanova spectrum may be realised as a spectrum of symmetric doubly stochastic matrix and proved, among other things, that this is so when $n = 2^k$ for some k, yet for any odd number n the list $(1, 0, 0, \ldots, 0, -1)$ cannot be a spectrum of a symmetric doubly stochastic matrix ([10, Corollary 1]).

For the sake of brevity, we call a Suleĭmanova spectrum $\sigma = (1, \lambda_2, \ldots, \lambda_n)$ *δ-normalised* ($\delta > 0$), whenever

$$1 + \lambda_2 + \ldots + \lambda_n \geqslant \delta. \tag{1.3}$$

The main result of this note is to prove that 1/2-normalised Suleĭmanova spectra may be indeed realised as spectra of symmetric, doubly stochastic matrices.

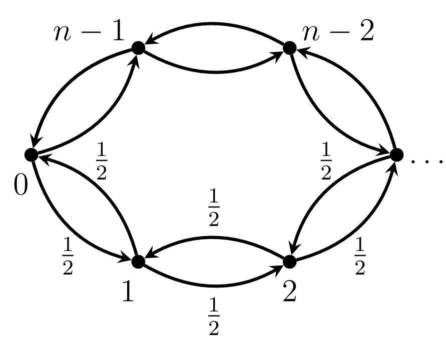
Theorem. *Let $\sigma = (1, \lambda_2, \ldots, \lambda_n)$ be a 1/2-normalised Suleĭmanova spectrum, that is, $\lambda_2, \ldots, \lambda_n \leqslant 0$ and $1 + \lambda_2 + \ldots + \lambda_n \geqslant 1/2$. Then there exists a symmetric, doubly stochastic matrix whose spectrum is precisely σ.*

Remark 1.1. It may be worth mentioning that the above result has a natural counterpart when all the eigenvalues are non-negative. That is, whenever $\lambda_j \geqslant 0$ for $j = 2, \ldots, n$ one may want to consider the condition

$$1 + \lambda_2 + \ldots + \lambda_n \leqslant \gamma, \tag{1.4}$$

where $\gamma \geqslant 1$. In this case, we show that if $\sigma = (1, \lambda_2, \ldots, \lambda_n)$ satisfies (1.4) with $\gamma = \frac{3}{2}$, then there is a symmetric, doubly stochastic matrix with σ as its spectrum.

The proof of the above result has a Markov-chain-theoretic flavour. Roughly speaking, we consider a simple (symmetric) random walk on the discrete torus $\mathbb{Z}/n\mathbb{Z}$ ($n \in \mathbb{N}$) represented by the following graph that we denote by S_n:



The crux of the proof is to use the transition probability matrix P_n of the above Markov chain as a *generator* for a class of new symmetric, doubly stochastic matrices. These matrices are defined in terms of suitably chosen eigenvectors of P_n. This idea is not specific to S_n as it offers a room for improvement—indeed, one may replace P_n with a transition matrix of another symmetric random walk on a possibly more complicated graph. By appealing to this heuristics and backed with some numerical evidence we raise the following problem.

Problem 1.2. Let $\delta > 0$. Can every δ-normalised Suleĭmanova spectrum be realised as the spectrum of a symmetric doubly stochastic matrix?

The main result of the note asserts that the answer is positive for $\delta \geqslant 1/2$.

2. Proof of the main result

We consider the symmetric random walk on the graph S_n described in the Introduction. The corresponding transition probability matrix P_n is then given by

$$P_n = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ & \ddots & \ddots & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Notation. In this section, we index the eigenvalues to start from 0 rather than 1 so that $\sigma = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ rather than $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ as done in the previous section. This assures that the formulae in this section are clearer and more compact.

A rote calculation shows that the eigenvalues λ_k and the corresponding (real) eigenvectors u_k ($k \in \{0, \ldots, n-1\}$) of P_n are given by

$$\lambda_k = \cos\Big(\frac{2\pi k}{n}\Big), \qquad u_k^{(j)} = \cos\Big(\frac{2\pi kj}{n}\Big) \qquad (k, j \in \{0, \ldots, n-1\}) \tag{2.1}$$

(for details see, *e.g.*, [7, 12.3.1]). Due to the symmetry, $\lambda_k = \lambda_{n-k}$ and $u_k = u_{n-k}$, so the eigenvectors of P_n fail to span $\mathbb{R}^n$; in the next lemma we find another set of eigenvectors corresponding to P_n that do indeed form an orthonormal basis of $\mathbb{R}^n$.

Lemma 2.1. *Let $\lambda_0, \ldots, \lambda_{n-1}$ be the eigenvalues of P_n as found in* (2.1)*. Set*

$$w_k^{(j)} = \sqrt{\frac{2}{n}} \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right) \qquad (k, j \in \{0, \ldots, n-1\}).$$

Then $w_k = [w_k^{(j)}]_{0 \leqslant j \leqslant n-1}$ are eigenvectors of P_n corresponding to $\lambda_0, \ldots, \lambda_{n-1}$. Furthermore, these eigenvectors form an orthonormal basis of $\mathbb{R}^n$.

Proof. It follows from [7, 12.3.1] that

$$\varphi_k = (1, e^{\frac{2\pi ik}{n}}, \ldots, e^{\frac{2\pi ikj}{n}} \ldots, e^{\frac{2\pi ik(n-1)}{n}})^{\mathrm{T}} \qquad (k \in \{0, \ldots, n-1\})$$

are (complex) eigenvectors corresponding to the eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$ accordingly. Since both the real and imaginary parts of φ_k are eigenvectors of P_n too, we conclude that $u_k^{(j)} = \cos\big(\frac{2\pi kj}{n}\big)$ and $v_k^{(j)} = \sin\big(\frac{2\pi kj}{n}\big)$ are the coordinates of the (real) eigenvectors u_k and v_k corresponding to λ_k for $k, j \in \{0, \ldots, n-1\}$. However, it is to be noted that neither the system $(u_k)_{k=0}^{n-1}$ nor $(v_k)_{k=0}^{n-1}$ spans $\mathbb{R}^n$. As sums of eigenvectors corresponding to the same eigenvalue, if non-zero, are still eigenvectors, let us consider the eigenvector $u_k + v_k$ corresponding to λ_k. Evidently,

$$u_k^{(j)} + v_k^{(j)} = \sqrt{2} \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right).$$

We then define the vectors w_k by

$$w_k^{(j)} = \sqrt{\frac{2}{n}} \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right) \qquad (k, j \in \{0, \dots, n-1\}).$$

Conspicuously, w_k is a (real) eigenvector of P_n corresponding to λ_k.

In order to show that $(w_k)_{k=0}^{\infty}$ are pairwise orthogonal unit vectors, we invoke the identitites $2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$ and $\cos\left(x + \frac{\pi}{2}\right) = -\sin(x)$. Using them we arrive at

$$\begin{aligned}\langle w_k, w_l\rangle =& \frac{2}{n}\sum_{j=0}^{n-1} \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right)\sin\left(\frac{2\pi lj}{n} + \frac{\pi}{4}\right)\\ =& \frac{1}{n}\left(-1 + \sum_{j=0}^{n} \cos\left(\frac{2\pi j(k-l)}{n}\right)\right) + \frac{1}{n}\sum_{j=0}^{n} \sin\left(\frac{2\pi j(k+l)}{n}\right).\end{aligned}$$

Let m be a positive integer. Then $\sum_{j=0}^{n} \cos\left(\frac{2\pi mj}{n}\right)$ and $\sum_{j=0}^{n} \sin\left(\frac{2\pi mj}{n}\right)$ are the real and imaginary part of the expression

$$\sum_{j=0}^{n} e^{i\frac{2\pi mj}{n}} = \frac{e^{2i\pi\left(m+\frac{m}{n}\right)} - 1}{e^{2i\pi\frac{m}{n}} - 1} = 1$$

respectively. Consequently, $\langle w_k, w_l\rangle = \delta_{k,l}$, where $\delta_{k,l}$ denotes the Kronecker delta, and thus $(w_k)_{k=0}^{n-1}$ forms an orthonormal basis of $\mathbb{R}^n$. □

Before we proceed, a piece of notation is required. Let Q be the matrix whose columns are precisely the eigenvectors from the statement of Lemma 2.1, that is, $Q = [w_0\ w_1\ \dots\ w_{n-1}]$. For a given diagonal matrix $\Lambda = \operatorname{diag}(1, \lambda_1, \dots, \lambda_{n-1})$ with $\lambda_k \in [-1, 1]$ $(k \in \{1, \dots, n-1\})$ we consider the product matrix

$$P(\Lambda) := Q\Lambda Q^{\mathrm{T}}. \tag{2.2}$$

Remark 2.2. We note that the matrix $P(\Lambda)$ is symmetric and every its row adds up to 1 since $P(\Lambda)w_0 = w_0$.

Let us record the form of the entries the matrix $P(\Lambda)$ has.

Lemma 2.3. *We have* $P(\Lambda) = [p_{kl}]_{k,l=0}^{n-1}$*, where*

$$p_{kl} = \frac{1}{n}\left(1 + 2\sum_{j=1}^{n-1} \lambda_j \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right)\sin\left(\frac{2\pi lj}{n} + \frac{\pi}{4}\right)\right) \qquad (k, l \in \{0, \dots n-1\}).$$

Proof. Note that $Q = [w_l^{(k)}]_{k,l=0}^{n-1}$ and $Q\Lambda = [\lambda_l w_l^{(k)}]_{k,l}$, where $\lambda_0 = 1$. Hence,

$$\begin{aligned}(Q\Lambda Q^T)_{k,l} =& \sum_{j=0}^{n-1} \lambda_j w_k^{(j)} w_j^{(l)} \\ =& \frac{1}{n}\left(1 + 2\sum_{j=1}^{n-1} \lambda_j \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right)\sin\left(\frac{2\pi lj}{n} + \frac{\pi}{4}\right)\right).\end{aligned}$$

□

Proposition 2.4. *Let $P(\Lambda)$ be as in (2.2). The matrix $P(\Lambda)$ is doubly stochastic if and only if*

$$\sum_{j=1}^{n-1} \lambda_j \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right)\sin\left(\frac{2\pi lj}{n} + \frac{\pi}{4}\right) \geqslant -\frac{1}{2} \tag{2.3}$$

for all $k \in \{0, \dots n-1\}$ and $l \in \{k, \dots, n-1\}$.

Proof. The matrix $P(\Lambda)$ is doubly-stochastic if and only if $p_{kl} \geqslant 0$ for all $k \in \{0, \dots n-1\}$ and $l \in \{k, \dots, n-1\}$ and this is indeed equivalent to (2.3) by Lemma 2.3. □

Finally, the main result and the statement from Remark 1.1 follows directly from the next corollary to Proposition 2.4.

Corollary 2.5. *Let $P(\Lambda)$ be as in (2.2).*

(1) *Suppose that $\lambda_i \leqslant 0$ for all $i \in \{1, \dots, n-1\}$. Then $P(\Lambda)$ is doubly stochastic as long as*

$$\sum_{i=1}^{n-1} \lambda_i \geqslant -\frac{1}{2}. \tag{2.4}$$

(2) *Suppose that $\lambda_i \geqslant 0$ for all $i \in \{1, \dots, n-1\}$. Then $P(\Lambda)$ is doubly stochastic as long as*

$$\sum_{i=1}^{n-1} \lambda_i \leqslant \frac{1}{2}. \tag{2.5}$$

Proof. First we show (1). Assume that (2.4) holds and set $S_j(k) = \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right)$ for any $j \in \{1, \dots, n-1\}$ and $k \in \{0, \dots, n-1\}$. Clearly, $S_j(k)S_j(l) \leqslant 1$ for all $k, l \in \{0, \dots n-1\}$ and since all $\lambda_j \leqslant 0$, we have $\lambda_j S_j(k) S_j(l) \geqslant \lambda_j$. Hence, we arrive at the estimate

$$\begin{aligned}\sum_{j=1}^{n-1} \lambda_j S_j(k) S_j(l) \geqslant& \sum_{j=1}^{n-1} \lambda_j \\ \geqslant& -\frac{1}{2}.\end{aligned}$$

To show (2), assume (2.5) and use the fact that $S_j(k)S_j(l) \geqslant -1$. □

3. Examples and applications

Examples. Let $n \in \mathbb{N}$. In this section we provide some examples of Suleĭmanova spectra, $\sigma_n = (1, \lambda_2, \ldots, \lambda_n)$, for which $\lambda_2, \ldots, \lambda_n$ add up to $-\frac{1}{2}$ and, thus, yield symmetric doubly stochastic matrices obtained via our construction (2.2), but do not satisfy known sufficient conditions (*e.g.*, (1.1), (1.2), *etc.*) to obtain symmetric doubly stochastic matrices.

To wit, neither (1.1) nor (1.2) is satisfied for

- $\sigma_5 = (1, -0.02, -0.03, -0.05, -0.4)$ (odd dimension);
- $\sigma_6 = (1, -0.01, -0.02, -0.06, -0.08, -0.33)$ (even dimension),

respectively.

Let $\sigma_n = (1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_2 \geqslant \lambda_3 \ldots \geqslant \lambda_n$. The *improved Soules' condition* when n is even, [10, Theorem 3, Notation 1, Observation 1], that is,

$$\frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{\frac{n}{2} - \left[\frac{n+2}{4}\right]}{\frac{n}{2}\left[\frac{n+2}{4}\right]}\lambda_4 + \sum_{k=1}^{\left[\frac{n+2}{4}\right]-1} \frac{\lambda_{n-4k+4}}{k(k+1)} \geqslant 0$$

is not satisfied as witnessed by

$$\sigma_{10} = (1, -0.01, -0.01, -0.025, -0.03, -0.035, -0.04, -0.05, -0.08, -0.22)$$

(the square brackets in the above formula denote the integral part of a real number).

Let n be odd, *new condition 1* ([10, Theorem 4, Notation 1]; we adapt the naming conventions from the said paper), that is,

$$\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{\frac{n+1}{2} - \left[\frac{n+3}{4}\right]}{\frac{n+1}{2}\left[\frac{n+3}{4}\right]}\lambda_4 + \sum_{k=1}^{\left[\frac{n+3}{4}\right]-1} \frac{\lambda_{n-4k+4}}{k(k+1)} \geqslant 0$$

is not satisfied as witnessed by the 1/2-normalised Suleĭmanova spectrum

$$\sigma_5 = (1, -0.03, -0.03, -0.04, -0.4).$$

Next we give examples of spectra that do not satisfy *New condition 2* ([10, Theorem 5, Notation 2]) which is given by (3.1, 3.2, 3.3, 3.4) depending on the remainder $n \mod 4$. Let m be an integer greater than 1. If

(1) $n = 4m$, then

$$\frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{2}{n}\lambda_4 + \frac{\frac{n}{4} - \left[\frac{n+4}{8}\right]}{\frac{n}{4}\left[\frac{n+4}{8}\right]}\lambda_8 + \sum_{k=1}^{\left[\frac{n+4}{8}\right]-1} \frac{\lambda_{n-8k+8}}{k(k+1)} \geqslant 0 \tag{3.1}$$

is not satisfied by the 1/2-normalised Suleĭmanova spectrum

$$\sigma_{16} = (1, -0.003, -0.003, -0.004, -0.007, -0.009, -0.02, -0.0209, -0.021, -0.024, -0.026, -0.035, -0.042, -0.076, -0.0811, -0.128);$$

(2) $n = 4m+2$, then

$$\frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{2(n-2)}{n(n+2)}\lambda_4 + \frac{\frac{n+2}{4} - \left[\frac{n+6}{8}\right]}{\frac{n+2}{4}\left[\frac{n+6}{8}\right]}\lambda_8 + \sum_{k=1}^{\left[\frac{n+6}{8}\right]-1} \frac{\lambda_{n-8k+8}}{k(k+1)} \geqslant 0 \tag{3.2}$$

is not satisfied by the 1/2-normalised Suleĭmanova spectrum

$$\sigma_{10} = (1, -0.01, -0.01, -0.01, -0.02, -0.02, -0.04, -0.07, -0.1, -0.22);$$

(3) $n = 4m+3$, then

$$\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2}{n+1}\lambda_4 + \frac{\frac{n+1}{4} - \left[\frac{n+5}{8}\right]}{\frac{n+1}{4}\left[\frac{n+5}{8}\right]}\lambda_8 + \sum_{k=1}^{\left[\frac{n+5}{8}\right]-1} \frac{\lambda_{n-8k+8}}{k(k+1)} \geqslant 0 \tag{3.3}$$

is not satisfied by the 1/2-normalised Suleĭmanova spectrum

$$\sigma_{11} = (1, -0.001, -0.004, -0.01, -0.01, -0.012, -0.013, -0.05, -0.09, -0.11, -0.2);$$

(4) $n = 4m+1$, then

$$\frac{1}{n} + \frac{n-1}{n(n+1)}\lambda_2 + \frac{2(n-1)}{(n+1)(n+3)}\lambda_4 + \frac{\frac{n+3}{4} - \left[\frac{n+7}{8}\right]}{\frac{n+3}{4}\left[\frac{n+7}{8}\right]}\lambda_8 + \sum_{k=1}^{\left[\frac{n+7}{8}\right]-1} \frac{\lambda_{n-8k+8}}{k(k+1)} \geqslant 0 \tag{3.4}$$

is not satisfied by the 1/2-normalised Suleĭmanova spectrum

$$\sigma_9 = (1, -0.006, -0.018, -0.02, -0.028, -0.028, -0.053, -0.105, -0.242).$$

New condition 3 ([10, Conjecture 1, Example 1]) that for $n = 26$ takes the form

$$\frac{1}{26} + \frac{1}{26}\lambda_2 + \frac{6}{13\cdot 7}\lambda_4 + \frac{3}{28}\lambda_8 + \frac{1}{4}\lambda_{16} + \frac{1}{2}\lambda_{26} \geqslant 0$$

is not satisfied by the 1/2-normalised Suleĭmanova spectrum

$$\sigma_{26} = (1, \underbrace{-0.004}_{\lambda_2}, -0.005, \underbrace{-0.006}_{\lambda_4}, -0.007, -0.01, -0.01, \underbrace{-0.011}_{\lambda_8}, -0.011, -0.011, -0.012,$$

$$-0.012, -0.015, -0.015, -0.016, \underbrace{-0.017}_{\lambda_{16}}, -0.019, -0.02, -0.022, -0.022, -0.025, -0.028,$$

$$-0.028, -0.032, -0.069, \underbrace{-0.073}_{\lambda_{26}}).$$

Applications to random generation of doubly stochastic matrices. Let $n \in \mathbb{N}$ and $\alpha \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Our construction provides a simple way to randomly generate symmetric doubly stochastic matrices via their spectrum. Namely, let $X_1, \ldots, X_{n-1}$ be independent random variables having probability distributions supported on $[0,1]$ and let us consider $S_n := X_1 + \ldots + X_{n-1}$. Set

$$\lambda_i = \alpha \frac{X_i}{S_n} \quad (i \in \{1, \ldots, n-1\}).$$

Then $\sigma = (1, \lambda_1, \ldots, \lambda_{n-1})$ is a spectrum of a symmetric doubly stochastic matrix and the corresponding matrix may be obtained via (2.2). More algorithms to generate doubly stochastic matrices (not necessarily symmetric) can be found in [1]. For an elaborate discussion on spectral properties of random doubly stochastic matrices we refer the reader to [11].

Supplementary material. We supplemnt the material with a Python code organised in a `JupyterNotebook` available at

`https://github.com/Nty24/DoublyStochasticMatricesGenerator`

that generates further examples and counterexamples in the spirit of Section 3.

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