Solving ill–posed problems

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Dynamics Days Europe, Rostock 2 September - 6 September 2019

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Euler system for a barotropic inviscid fluid

Equation of continuity: $\varrho = \varrho(t,x)$ **- mass density**

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Momentum equation: $\mathbf{m} = \mathbf{m}(t,x) = (\varrho \mathbf{u})$**- momentum**

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0,\ \ p(\varrho) = a\varrho^\gamma,\ \ a > 0,\ \ \gamma > 1$$

Impermeability boundary conditions or periodic boundary conditions

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \text{ or } \Omega = \left([-1,1]|_{\{-1,1\}} \right)^d,\ \ d = 2,3$$

Initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ \ \mathbf{m}(0,\cdot) = \mathbf{m}_0$$

Admissible solutions – energy dissipation

Energy

$$\mathcal{E} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

$$p' \geq 0 \Rightarrow [\varrho, \mathbf{m}] \mapsto \left\{ \begin{array}{l} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0 \\ P(\varrho) \text{ if } |\mathbf{m}| = 0,\ \ \varrho \geq 0 \\ \infty \text{ otherwise} \end{array} \right. \quad \text{is convex l.s.c}$$

Energy balance (conservation)

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E}\frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p\frac{\mathbf{m}}{\varrho} \right) = 0$$

Energy dissipation

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E}\frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p\frac{\mathbf{m}}{\varrho} \right) \boxed{\leq} 0$$

$$E = \int_\Omega \mathcal{E}\ \mathrm{d}x,\ \partial_t E \leq 0,\ E(0+) = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right]\ \mathrm{d}x$$

Known facts about Euler equations

Well/ill posedness

- Local in time existence of unique smooth solutions for smooth initial data
- Blow–up (shock wave) in a finite time for a generic class of initial data
- Existence of infinitely many weak solution for any continuous initial data (Chiodaroli, DeLellis–Széhelyhidi, EF...)
- Existence of "many" initial data that give rise to infinitely many weak solutions satisfying the energy inequality (Chiodaroli, EF, Luo, Xie, Xin...)
- Existence of smooth initial data that ultimately give rise to infinitely many weak solutions satisfying the energy inequality (Kreml et al)
- Weak–strong uniqueness in the class of admissible weak solutions (Dafermos)

Wild solutions?

Charles Hermite [1822-1901]

In a letter to Stieltjes

I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives

- **Past:** What is not allowed is forbidden
- **Present:** What is not forbidden is allowed

Oscillations

Oscillatory sequence

$$g(x+a) = g(x) \text{ for all } x \in R, \ \int_0^a g(x)\mathrm{d}x = 0,$$

$$g_n(x) = g(nx), \ \ n = 1, 2, \ldots$$

Weak convergence (convergence in integral averages)

$$\int_R g_n(x)\varphi(x)\ \mathrm{d}x, \ \text{ where } \varphi \in C_c^\infty(R).$$

$$G(x) = \int_0^x g(z)\ \mathrm{d}z$$

$$\int_R g_n(x)\varphi(x)\ \mathrm{d}x = \int_R g(nx)\varphi(x)\ \mathrm{d}x = -\frac{1}{n}\int_R G(nx)\partial_x\varphi(x)\ \mathrm{d}x \to 0$$

Beware

$$g_n \rightharpoonup g \ \boxed{\textit{does not}} \ \text{imply } H(g_n) \rightharpoonup H(g) \text{ if } H \text{ is not linear.}$$

Concentrations

Concentrating sequence

$$g_n(x) = ng(nx)$$

$$g \in C_c^\infty(-1,1),\ g(-x) = g(x),\ g \geq 0,\ \int_R g(x)\ \mathrm{d}x = 1.$$

- $$g_n(x) \to 0 \text{ as } n \to \infty \text{ for any } x \neq 0,\ \text{in particular } g_n \to 0 \text{ a.a. in } R;$$

- $$\|g_n\|_{L^1(R)} = \int_R g_n(x)\ \mathrm{d}x = \int_R g(x)\ \mathrm{d}x = 1 \text{ for any } n = 1, 2, \dots.$$

Convergence in the space of measures

$$\int_R g_n(x)\varphi(x)\ \mathrm{d}x = \int_{-1/n}^{1/n} g_n(x)\varphi(x)\ \mathrm{d}x$$

$$\in \left[\min_{x\in[-1/n,1/n]} \varphi(x),\ \max_{x\in[-1/n,1/n]} \varphi(x)\right] \to \varphi(0)\ \Rightarrow\ g_n \to \delta_0$$

Ill posedness

Theorem [A.Abbatiello, EF 2019]

Let $d = 2, 3$. Let ϱ_0, $\mathbf{m}_0$ be given such that

$$\varrho_0 \in \mathcal{R},\ 0 \le \underline{\varrho} \le \varrho_0 \le \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R},\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R},\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^{\infty} \subset (0, T)$ be an arbitrary (countable dense) set of times. Then the Euler problem admits infinitely many weak solutions ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that

$$\varrho \in C_{\rm weak}([0, T]; L^{\gamma}(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

but

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)]\ \boxed{\text{is not}}\ \text{strongly continuous at any } \tau_i,\ i = 1, 2, \dots .$$

Consistent approximation

Equation of continuity

$$\int_0^T \int_\Omega \left[\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi \right] \mathrm{d}x\mathrm{d}t = e_{1,n}[\varphi]$$

Momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + 1_{\varrho_n > 0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \varphi + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t = e_{2,n}[\boldsymbol{\varphi}]$$

Stability - bounded energy

$$\mathcal{E}(\varrho_n, \mathbf{m}_n) \equiv \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] \mathrm{d}x \lesssim 1$$

Consistency

$$e_{1,n}[\varphi] \to 0, \ \ e_{2,n}[\boldsymbol{\varphi}] \to 0 \text{ as } n \to \infty$$

Weak vs strong convergence

Weak convergence

$$\varrho_n \to \varrho \text{ weakly-(*) } L^\infty(0,T;(L^\gamma)(\Omega))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ weakly-(*) } L^\infty(0,T;(L^{\frac{2\gamma}{\gamma+1}})(\Omega))$$

Strong convergence (Theorem EF, M.Hofmanová)

$$K \subset [0,T] \times \overline{\Omega} \text{ compact}$$

$$\varrho_n \to \varrho,\ \mathbf{m}_n \to \mathbf{m} \text{ strongly (pointwise) in } \mathcal{U} \text{ open},\ \partial K \subset \mathcal{U}$$

$$\varrho, \mathbf{m} \text{ weak solution to the Euler system}$$

$$\Rightarrow$$

$$\varrho_n \to \varrho,\ \mathbf{m}_n \to \mathbf{m} \text{ strongly (pointwise) in } K$$

Should we go beyond weak solutions?

However beautiful the strategy, you should occasionally look at the results...
Sir Winston Churchill
[1874-1965]

Dissipative solutions – limits of numerical schemes

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0,\ \ \varrho(0,\cdot) = \varrho_0$$

Momentum balance

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \left(\mathfrak{R}_v + \mathfrak{R}_p \mathbb{I} \right),\ \ \mathbf{m}(0,\cdot) = \mathbf{m}_0$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq 0,\ \ E(t) \leq E_0,\ \ E_0 = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$

$$E \equiv \left(\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d} \frac{1}{2} \mathrm{trace}[\mathfrak{R}_v] + \int_{\overline{\Omega}} \mathrm{d} \frac{1}{\gamma - 1} \mathfrak{R}_p \right)$$

Turbulent defect measures

$$\mathfrak{R}_v \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}; R^{d\times d}_{\mathrm{sym}})),\ \ \mathfrak{R}_p \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}))$$

Basic properties of dissipative solutions

Well posedness, weak strong uniqueness

- **Existence.** Dissipative solutions exist globally in time for any finite energy initial data
- **Limits of consistent approximations** Limits of consisten approximations are dissipative solutions, in particular limits of consistent numerical schemes.
- **Compatibility.** Any C^1 dissipative solution $[\varrho, \mathbf{m}]$, $\varrho > 0$ is a classical solution of the Euler system
- **Weak–strong uniqueness.** If $[\widetilde{\varrho}, \widetilde{\mathbf{m}}]$ is a classical solution and $[\varrho, \mathbf{m}]$ a dissipative solution starting from the same initial data, then $\mathfrak{R}_v = \mathfrak{R}_p = 0$ and $\varrho = \widetilde{\varrho}$, $\mathbf{m} = \widetilde{\mathbf{m}}$.
- **Semiflow selection.** There exists a measurable selection of dissipative solution that forms a semigroup

Komlos ($\mathcal{K}$) convergence

Komlos theorem

$$\{U_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q)$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} U_{n_k} \to \overline{U} \text{ a.a. in } Q \text{ as } N \to \infty$$

Conclusion for the approximate solutions

$$\frac{1}{N}\sum_{k=1}^{N} \varrho_{n_k} \to \varrho \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{m}_{n_k} \to \mathbf{m} \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \left[\frac{1}{2}\frac{|\mathbf{m}_{n,k}|^2}{\varrho_{n,k}} + P(\varrho_{n,k})\right] \to \overline{\mathcal{E}} \in L^1((0,T)\times\Omega) \text{ a.a. in } (0,T)\times\Omega$$

Visualising oscillations – Young measures

Limits of compositions

$$\varrho_n \to \varrho,\ \mathbf{m}_n \to \mathbf{m},\ B(\varrho_n, \mathbf{m}_n) \to \overline{B(\varrho, \mathbf{m})} \neq B(\varrho, \mathbf{m}),\ B \in BC(R^{d+1})$$

Young measure

$$\langle \nu_{t,x} B(\varrho, \mathbf{m}) \rangle = \overline{B(\varrho, \mathbf{m})}(t, x)$$

Compactness

$$\text{tightness} \ \Leftrightarrow \ \text{uniform } L^1 - \text{bound}$$

$\mathcal{K}$–convergence of Young measures [Balder]

Young measure

$$\{U_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q) \approx \nu_{t,x}^n = \delta_{U_n(t,x)}$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} \nu_{t,x}^{n_k} \to \nu_{t,x} \text{ narrowly a.a. in } Q \text{ as } N \to \infty$$

Monge–Kantorowich (Wasserstein) distance

$$\left\| \operatorname{dist}\left(\frac{1}{N}\sum_{k=1}^{N} \nu_{t,x}^{n_k}; \nu_{t,x} \right) \right\|_{L^q(Q)} \to 0$$

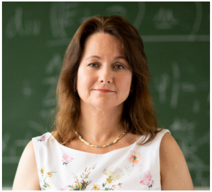
for some $q > 1$.

Thanks to ...

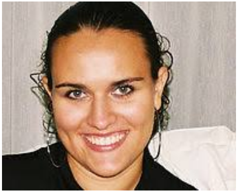
Anna Abbatiello (TU Berlin)

Dominic Breit (Heriot-Watt, Edinburgh)

Martina Hofmanová (Bielefeld)



Mária Lukáčová (JGU Mainz)



Hana Mizerová (CU Bratislava)

Bangwei She (IM Praha)