A REMARK ON THE DEVIATORIC DECOMPOSITION OF OLDROYD TYPE MODELS

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1 Introduction

Certain viscoelastic fluids can be described by the class of Oldroyd type models. This has been for example shown in our previous publications [3], [5] or [1]. In these models the viscoelastic extra stress tensor is computed from a tensorial evolution equation. In contrast to classical Newtonian fluids, this viscoelastic stress tensor has non-vanishing trace. It means that it contains not only deviatoric, but also spherical part. The aim of the present paper is to develop a reformulation of the Odroyd type model, where the deviatoric and spherical components are computed separately. It will be shown how such a deviatoric reformulation of an Oldroyd model can be derived and how it can be used.

2 Mathematical model

2.1 Basic conservation laws

Conservation of mass (incompressibility constraint):

$$\mathrm{div}\boldsymbol{u} = 0 \tag{1}$$

Conservation of momentum:

$$\rho\dot{\boldsymbol{u}} = \mathrm{div}\mathbf{T} - \nabla p \tag{2}$$

Here $\boldsymbol{u}$ stands for the velocity vector, ρ is density, p is pressure. The stress tensor is denoted by $\mathbf{T}$.

2.2 Rheological model - classical formulation

The rheological model is based on the Oldroyd-type model often referred to as the *Johnson-Segalmann* model. The well known upper-, lower- and co-rotational Maxwell models as well as the Oldroyd-A and Oldroyd-B models are just special sub-cases of the Johnson-Segalmann class of models.

Stress tensor $\mathbf{T}$ consists of the Newtonian (solvent) part $\mathbf{T}_s$ and the viscoelastic part $\mathbf{T}_e$.

$$\mathbf{T} = \mathbf{T}_s + \mathbf{T}_e \tag{3}$$

These two stress components $\mathbf{T}_s$ and $\mathbf{T}_e$ are defined as follows.

$$\mathbf{T}_s = 2\mu_s \mathbf{D} \tag{4}$$

$$\mathbf{T}_e + \lambda \frac{\delta \mathbf{T}_e}{\delta t} = 2\mu_e \mathbf{D} \tag{5}$$

The symbol $\mathbf{D}$ denotes the symmetric part of the velocity gradient. The physical parameters in this model are the solvent and elastic viscosities μ_s, resp. μ_e and the relaxation time λ.

The *convected derivative* $\frac{\delta \mathbf{T}_e}{\delta t}$ in the equation (5) can be chosen from the one-parametric family of *Gordon-Schowalter* derivatives given by :

$$\left(\frac{\delta \mathbf{T}_e}{\delta t}\right)_a = \dot{\mathbf{T}}_e - \mathbf{W}\mathbf{T}_e + \mathbf{T}_e\mathbf{W} + a(\mathbf{D}\mathbf{T}_e + \mathbf{T}_e\mathbf{D}) \qquad a \in \langle -1; 1 \rangle \tag{6}$$

For $a = -1$, this leads to upper convected derivative, $a = 0$ gives co-rotational (or Jaumann) derivative and for $a = 1$ we get the lower convected derivative. The most commonly used Oldroyd-B (upper convected Maxwell) model is obtained for $a = -1$.

$$\frac{\partial \mathbf{T}_e}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \mathbf{T}_e = \frac{2\mu_e}{\lambda}\mathbf{D} - \frac{1}{\lambda}\mathbf{T}_e + (\mathbf{W}\mathbf{T}_e - \mathbf{T}_e\mathbf{W}) - a(\mathbf{D}\mathbf{T}_e + \mathbf{T}_e\mathbf{D}) \qquad a \in \langle -1; 1 \rangle \tag{7}$$

3 Deviatoric reformulation of the model

The elastic stress tensor $\mathbf{T}_e$ is described by the following evolution equation:

$$\frac{\partial \mathbf{T}_e}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \mathbf{T}_e = \frac{1}{\lambda}\left[2\mu_e \mathbf{D} - \mathbf{T}_e\right] + \underbrace{(\mathbf{W}\mathbf{T}_e - \mathbf{T}_e\mathbf{W})}_{\mathrm{Tr}=0} - a\,\underbrace{(\mathbf{D}\mathbf{T}_e + \mathbf{T}_e\mathbf{D})}_{\mathrm{Tr}\neq 0} \qquad a \in \langle -1; 1 \rangle \tag{8}$$

If the $\mathbf{T}_e$ is initialized by the traceless tensor field (e.g. $\mathbf{T}_e = c\mathbf{D}$), then only the last term on the right hand side of (8) will contribute to the $\mathrm{Tr}(\mathbf{T}_e)$. The elastic stress tensor $\mathbf{T}_e$ can be decomposed in its deviatoric (traceless) part $\mathbf{T}_e^{\square}$ and the spherical, diagonal part $\mathbf{T}_e^{\blacksquare}$.

$$\mathbf{T}_e = \mathbf{T}_e^{\square} + \mathbf{T}_e^{\blacksquare} \qquad \text{where} \qquad \mathbf{T}_e^{\blacksquare} = \frac{1}{3}\mathrm{Tr}(\mathbf{T}_e)\mathbf{I} \tag{9}$$

This decomposition (9) is introduced into the constitutive relation (8). The tensor products on the right hand side are decomposed in a similar way following the rule for general tensor $\mathbf{M}$:

$$\mathbf{M} = \mathbf{M}^{\square} + \mathbf{M}^{\blacksquare} \qquad \text{where} \qquad \mathbf{M}^{\blacksquare} = \frac{1}{3}\mathrm{Tr}(\mathbf{M})\mathbf{I} \tag{10}$$

The terms on the right hand side of (8) should be decomposed as follows:

$$(\mathbf{D}\mathbf{T}_e + \mathbf{T}_e\mathbf{D}) = \underbrace{(\mathbf{D}\mathbf{T}_e^{\square} + \mathbf{T}_e^{\square}\mathbf{D})}_{\mathrm{Tr}\neq 0} + \underbrace{(\mathbf{D}\mathbf{T}_e^{\blacksquare} + \mathbf{T}_e^{\blacksquare}\mathbf{D})}_{=2\mathbf{T}_e^{\blacksquare}\mathbf{D};\ \mathrm{Tr}=0} \tag{11}$$

The term $(\mathbf{D}\mathbf{T}_e^{\square} + \mathbf{T}_e^{\square}\mathbf{D})$ is not traceless and thus it should be further decomposed:

$$(\mathbf{D}\mathbf{T}_e^{\square} + \mathbf{T}_e^{\square}\mathbf{D}) = (\mathbf{D}\mathbf{T}_e^{\square} + \mathbf{T}_e^{\square}\mathbf{D})^{\square} + \underbrace{(\mathbf{D}\mathbf{T}_e^{\square} + \mathbf{T}_e^{\square}\mathbf{D})^{\blacksquare}}_{=2(\mathbf{T}_e^{\square}\mathbf{D})^{\blacksquare}} \tag{12}$$

In a similar way (for $\mathbf{W}$ being skew-symmetric and $\mathbf{T}_e^{\blacksquare}$ symmetric) the expression $(\mathbf{W}\mathbf{T}_e - \mathbf{T}_e\mathbf{W})$ can further be simplified:

$$(\mathbf{W}\mathbf{T}_e - \mathbf{T}_e\mathbf{W}) = (\mathbf{W}\mathbf{T}_e^{\square} - \mathbf{T}_e^{\square}\mathbf{W}) + \underbrace{(\mathbf{W}\mathbf{T}_e^{\blacksquare} - \mathbf{T}_e^{\blacksquare}\mathbf{W})}_{0} = (\mathbf{W}\mathbf{T}_e - \mathbf{T}_e\mathbf{W})^{\square} \tag{13}$$

The evolution of the deviatoric (traceless) part $\mathbf{T}_e^{\square}$ is then governed by:

$$\frac{\partial \mathbf{T}_e^{\square}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \mathbf{T}_e^{\square} = \frac{1}{\lambda}\left[2\mu_e \mathbf{D} - \mathbf{T}_e^{\square}\right] + (\mathbf{W}\mathbf{T}_e - \mathbf{T}_e\mathbf{W})^{\square} - a(\mathbf{D}\mathbf{T}_e + \mathbf{T}_e\mathbf{D})^{\square} \tag{14}$$

Using the relations (11), (13) and (12), the equation (14) can be rewritten into its (almost) final form:

$$\frac{\partial \mathbf{T}_e^{\square}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \mathbf{T}_e^{\square} = \frac{1}{\lambda}\left[2\mu_e \mathbf{D} - \mathbf{T}_e^{\square}\right] + (\mathbf{W}\mathbf{T}_e^{\square} - \mathbf{T}_e^{\square}\mathbf{W}) - a(\mathbf{D}\mathbf{T}_e^{\square} + \mathbf{T}_e^{\square}\mathbf{D})^{\square} - 2a\mathbf{T}_e^{\blacksquare}\mathbf{D} \tag{15}$$

The equation for the spherical part $\mathbf{T}_e^{\blacksquare}$ is simpler, because $(\mathbf{W}\mathbf{T}_e - \mathbf{T}_e\mathbf{W})^{\blacksquare} = \mathbf{0}$.

$$\frac{\partial \mathbf{T}_e^{\blacksquare}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \mathbf{T}_e^{\blacksquare} = -\frac{1}{\lambda}\mathbf{T}_e^{\blacksquare} - a\,\underbrace{(\mathbf{D}\mathbf{T}_e^{\square} + \mathbf{T}_e^{\square}\mathbf{D})^{\blacksquare}}_{=2(\mathbf{T}_e^{\square}\mathbf{D})^{\blacksquare}} \tag{16}$$

The decomposed system of tensor equations for the elastic stress can be written as:

$$\frac{\partial \mathbf{T}_e^{\square}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \mathbf{T}_e^{\square} = \frac{1}{\lambda}\left[2\mu_e \mathbf{D} - \mathbf{T}_e^{\square}\right] + (\mathbf{W}\mathbf{T}_e^{\square} - \mathbf{T}_e^{\square}\mathbf{W}) - a(\mathbf{D}\mathbf{T}_e^{\square} + \mathbf{T}_e^{\square}\mathbf{D})^{\square} - 2a\mathbf{T}_e^{\blacksquare}\mathbf{D} \tag{17}$$

$$\frac{\partial \mathbf{T}_e^{\blacksquare}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \mathbf{T}_e^{\blacksquare} = -\frac{1}{\lambda}\mathbf{T}_e^{\blacksquare} - 2a(\mathbf{T}_e^{\square}\mathbf{D})^{\blacksquare} \tag{18}$$

In this system of equations the stress components $\mathbf{T}_e^{\square}$ and $\mathbf{T}_e^{\blacksquare}$ only appear separately. The separate systems governing their evolution are linked to each other (also to momentum equations) via the last term on the RHS. It seems we have now twice the number of original tensor equations, but:

- From the system (17) only *5 components* (in 3D) need to be computed, while the last (diagonal) component can be obtained by applying the definition $\mathrm{Tr}(\mathbf{T}_e^{\square}) = 0$.
- The equation (18) is in fact only a scalar evolution equation for the trace of $\mathbf{T}_e$. This is due to specific structure of $\mathbf{T}_e^{\blacksquare}$ given by the definition $\mathbf{T}_e^{\blacksquare} = \frac{1}{3}\mathrm{Tr}(\mathbf{T}_e)\mathbf{I}$.

The $\mathrm{Tr}(\mathbf{T}_e)$ plays a role similar to pressure. Based on this observation a new *elastic pressure* variable p_e can be defined as:

$$\mathbf{T}_e^{\blacksquare} = -p_e\mathbf{I} \qquad \Longrightarrow \qquad p_e = -\frac{1}{3}\mathrm{Tr}(\mathbf{T}_e) = -\frac{1}{3}\mathrm{Tr}(\mathbf{T}_e^{\blacksquare}) \tag{19}$$

From (18) follows the evolution equation for elastic pressure:

$$\frac{\partial p_e}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, p_e = -\frac{1}{\lambda}p_e + \frac{2a}{3}\mathrm{Tr}(\mathbf{T}_e^{\square}\mathbf{D}) \tag{20}$$

Finally instead of the original governing system for (6 components of) T_e, we end-up with a system of equations for (5 components of) $\mathsf{T}_e^\square$ and (1 scalar) equation for elastic pressure p_e.

The full new model can be written as

$$\begin{aligned} \operatorname{div} \boldsymbol{u} &= 0 && (21)\\ \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} &= \frac{1}{\rho}[\operatorname{div}(\mathsf{T}_s + \mathsf{T}_e^\square) - \nabla(p+p_e)] && (22)\\ \mathsf{T}_s &= 2\mu_s \mathsf{D} && (23)\\ \frac{\partial \mathsf{T}_e^\square}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\mathsf{T}_e^\square &= \frac{1}{\lambda}[2\mu_e\mathsf{D} - \mathsf{T}_e^\square] + (\mathsf{W}\mathsf{T}_e^\square - \mathsf{T}_e^\square\mathsf{W}) - a(\mathsf{D}\mathsf{T}_e^\square + \mathsf{T}_e^\square\mathsf{D})^\square + 2a(p_e\mathsf{D}) && (24)\\ \frac{\partial p_e}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,p_e &= -\frac{1}{\lambda}p_e + \frac{2a}{3}\operatorname{Tr}(\mathsf{T}_e^\square\mathsf{D}) && (25) \end{aligned}$$

Conclusions, remarks

It has been demonstrated that it is possible to reformulate the Oldroyd type models in such a way that the deviatoric and spherical components of the viscoelastic stress tensor can be computed separately. The elastic pressure has been introduced as a new variable which makes the new formulation easier to implement. This formulation can be useful for numerical simulations as well as for mathematical analysis due to its resemblance to the mathematical description of compressible fluids flows.

Further work will focus on numerical implementation of this newly formulated model and its comparison with standard formulation.

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