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PROPERTY (T), FINITE-DIMENSIONAL REPRESENTATIONS, AND GENERIC REPRESENTATIONS

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Abstract. Let G be a discrete group with property (T). It is a standard fact that, in a unitary representation of G on a Hilbert space $\mathcal{H}$, almost invariant vectors are close to invariant vectors, in a quantitative way. We begin by showing that, if a unitary representation has some vector whose coefficient function is close to a coefficient function of some finite-dimensional unitary representation σ, then the vector is close to a subrepresentation isomorphic to σ: this makes quantitative a result of P.S. Wang [12]. We use that to give a new proof of a result by D. Kerr, H. Li and M. Pichot [9], that a group G with property (T) and such that $C^*(G)$ is residually finite-dimensional, admits a unitary representation which is generic (i.e. the orbit of this representation in $Rep(G, \mathcal{H})$ under the unitary group $U(\mathcal{H})$ is comeager). We also show that, under the same assumptions, the set of representations equivalent to a Koopman representation, is comeager in $Rep(G, \mathcal{H})$.

1. Introduction

Let G be a discrete group and π be a unitary representation of G on some Hilbert space $\mathcal{H}$. For a finite set $F \subset G$ and $\varepsilon > 0$, a vector $\xi \in \mathcal{H}$ is (F, ε)-invariant if $\max_{g \in F} \|\pi(g)\xi - \xi\| < \varepsilon$. Recall that π almost has invariant vectors if, for every pair (F, ε), the group G has (F, ε)-vectors; and that the group G has *Kazhdan's property (T)* or is a *Kazhdan group* if every unitary representation of G almost having invariant vectors, has non-zero invariant vectors; see e.g. [2] for Property (T). The definition can be reformulated in terms of weak containment of representations: G has Property (T) if every unitary representation weakly containing the trivial representation of G, contains it strongly (see Remark 1.1.2 in [2]). Crucial for us is an equivalent characterization due to P.S. Wang (Corollary 1.9 and Theorem 2.1 in [12]): the group G has property (T) if and only if for some (hence every) irreducible finite-dimensional unitary representation σ of G, every unitary representation π of G that contains σ weakly, contains it strongly.

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It is a simple but useful fact that, if G has property (T) and π is a unitary representation almost having invariant vectors, "almost invariant vectors are close to invariant vectors". More precisely:

Proposition 1.1 (Proposition 1.1.9 in [2]). *Let G be a Kazhdan group. If S is a finite generating set of G and ε_0 is the corresponding Kazhdan constant, then for every $\delta \in]0,1[$ and every unitary representation π of G, any $(S,\varepsilon_0\delta)$-invariant vector ξ satisfies $\|\xi - P\xi\| \leq \delta\|\xi\|$, where P is the orthogonal projection on the subspace of $\pi(G)$-invariant vectors.* □

For a Kazhdan group G and a unitary representation π of G, fix a unit vector ξ and look at the coefficient function

$$\phi_{\pi,\xi}(g) = \langle \pi(g)\xi, \xi\rangle \ (g \in G).$$

The question we first address in this paper is: if $\phi_{\pi,\xi}$ is close to some coefficient of an irreducible finite-dimensional unitary representation σ of G, must ξ be close to a finite-dimensional invariant subspace of π carrying a sub-representation isomorphic to σ? We will see that, in analogy to Proposition 1.1, the answer is positive - with some care.

Definition 1.2. *Let G be a finitely generated group with a symmetric finite generating set $S \subseteq G$ and let ϕ be some normalized positive definite function on G associated with a unitary irreducible representation σ, of finite dimension d. Let π be some unitary representation of G on $\mathcal{H}$. Let $\varepsilon > 0$. Say that a unit vector $\xi \in \mathcal{H}$ is (π,ϕ,ε)-good if for every $s \in S^{2d^2+1}$ we have $|\phi_{\pi,\xi}(s) - \phi(s)| < \varepsilon$.*

Note that S^k is just the ball of radius k centered at the identity in G. So there is a certain lack of uniformity in Definition 1.2: we require an approximation of $\phi_{\pi,\xi}$ by ϕ on a ball whose size depends on the dimension of the representation d. Our main result, proved in section 2, can be viewed as a quantitative version of Wang's result.

Theorem 1.3. *Let G be a discrete Kazhdan group, S a finite symmetric generating set with $e \in S$, and let ϕ be a normalized positive definite function on G associated with a finite-dimensional unitary irreducible representation σ of G. For every $0 < \delta < 1$ there exists $\varepsilon_{\phi,\delta} > 0$ such that for every unitary representation π of G on a Hilbert space $\mathcal{H}$, and every unit vector $x \in \mathcal{H}$ that is $(\pi,\phi,\varepsilon_{\phi,\delta})$-good, there exists a unit vector $x' \in \mathcal{H}$ with $\|x - x'\| \leq \delta$ such that the restriction of π to the span of $\pi(G)x'$ is isomorphic to σ.*

In section 3, we apply Theorem 1.3 to the study of the global structure of the space of unitary representations of Kazhdan groups. Let us start with the notation. Let G be an arbitary countable group and let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. The set $\mathrm{Rep}(G,\mathcal{H})$ of all homomorphisms from G into the unitary group $U(\mathcal{H})$ can be viewed as a closed subset of the product space $U(\mathcal{H})^G$, when we equip $U(\mathcal{H})$ with the strong operator topology. With this identification, $\mathrm{Rep}(G,\mathcal{H})$ is a Polish (i.e. separable and completely metrizable) space. We refer the reader to the monograph [8], especially to the section on the spaces of unitary representations, for more information about this point of view on unitary representations. Recall that two unitary representations $\pi_1, \pi_2 \in \mathrm{Rep}(G, \mathcal{H})$ are *isomorphic*, or *unitarily equivalent* if there is a unitary operator $\phi \in U(\mathcal{H})$ such that $\pi_1(g) = \phi\pi_2(g)\phi^*$, for every $g \in G$. Notice that this is an orbit equivalence relation given by the action of the unitary group $U(\mathcal{H})$ on the space $\mathrm{Rep}(G, \mathcal{H})$ by conjugation. Kechris raised a question (see again the section on the space of unitary representations in [8]) if there are countable groups with a *generic unitary representation*, where "generic" here means its conjugacy class is large in the sense of Baire category, i.e. a representation whose class under the unitary equivalence contains a dense G_δ subset. As a matter of fact, we mention that it follows from the topological zero-one law that for every countable group G either there is a generic representation in $\mathrm{Rep}(G, \mathcal{H})$, or all conjugacy classes are meager (see e.g. Theorem 8.46 in [7]; to apply it, note that there is a dense conjugacy class in $\mathrm{Rep}(G, \mathcal{H})$ — indeed, take some countable dense set of representations from $\mathrm{Rep}(G, \mathcal{H})$ and consider their direct sum).

Here as an application of Theorem 1.3 we prove the following result.

Theorem 1.4. *Let G be a discrete Kazhdan group such that finite-dimensional representations are dense in the unitary dual $\hat{G}$. Then there is a generic unitary representation of G.*

We note that, although not explicitly stated there, this result already follows from a more general result of Kerr, Li and Pichot from [9], where they prove (see Theorem 2.5 there) that if A is a separable C*-algebra where finite-dimensional representations are dense in $\hat{A}$, then there is a dense G_δ class in $\mathrm{Rep}(A, \mathcal{H})$. Theorem 1.4 is then a special case for $A = C^*(G)$. Our proof is nevertheless done by more elementary means, in particular it does not invoke Voiculescu's theorem (see the proof of Theorem 2.5 in [9] for details).

Another open question posed by Kechris as Problem H.16 in [8] is whether the subset of those representations $\pi \in \mathrm{Rep}(G, \mathcal{H})$, where G is still a countable group, that are equivalent to Koopman representations is meager in $\mathrm{Rep}(G, \mathcal{H})$. Such representations are called *realizable by an action* in [8]. Let us recall the terminology first. Let (X, μ) be a standard probability space (i.e. a space isomorphic to the unit interval $[0, 1]$ equipped with the Lebesgue measure). Let $\alpha : G \curvearrowright (X, \mu)$ be an action of a countable group G on X by measure preserving measurable transformations. Consider the unitary representation $\pi_\alpha : G \to L^2(X, \mu)$ defined by $\pi_\alpha(g)f(x) = f(\alpha(g^{-1}, x))$, for every $f \in L^2(X, \mu)$. The *Koopman representation* of α is the restriction of π_α to the invariant subspace $L^2_0(X, \mu)$, which is the orthogonal complement of the invariant subspace of constant functions.

In section 4 we prove the following result addressing the question of Kechris.

Theorem 1.5. *Let G be a discrete Kazhdan group such that finite-dimensional representations are dense in the unitary dual $\hat{G}$. Then the set of representations realizable by an action is comeager in* $\mathrm{Rep}(G, \mathcal{H})$.

Let us mention that the condition that finite-dimensional representations are dense in the unitary dual $\hat{G}$ is, by the result of Archbold from [1], equivalent with the statement that the full group C*-algebra $C^*(G)$ is residually finite-dimensional. That is in turn, by the result of Exel and Loring from [6] (see also [11]), equivalent with the statement that finite-dimensional representations are dense in $\mathrm{Rep}(G, \mathcal{H})$, which we shall use in the proof. Note that we call a representation $\pi \in \mathrm{Rep}(G, \mathcal{H})$ finite-dimensional if the subalgebra $\pi(G)$ generates in $B(\mathcal{H})$ is finite-dimensional.

The existence of infinite discrete Kazhdan groups with residually finite-dimensional C*-algebras seems to be open — see Question 7.10 in [2] and also Question 6.5 of Lubotzky and Shalom in [10] where they ask if there are infinite discrete Kazhdan groups with property FD, which is strictly stronger than having a residually finite-dimensional C*-algebra (a group has *property FD* if representations factoring through finite groups are dense in the unitary dual).

Question 1.6. *It is known that being residually finite is not a sufficient condition to have a residually finite-dimensional C*-algebra by a result of Bekka* [3]. *However how about being LERF? (Recall that a finitely generated group is LERF if any finitely generated subgroup is the intersection of the finite index subgroups containing it). Ershov and Jaikin-Zapirain constructed in* [5] *a Kazhdan group which is LERF. Is its group C*-algebra residually finite-dimensional?*

Remark 1.7. We note that on the other hand we cannot exclude that it is possible to prove by a different argument that for every infinite group G, all classes in $\mathrm{Rep}(G, \mathcal{H})$ are meager. That would together with Theorem 1.4 give that there are no infinite Kazhdan groups with a residually finite-dimensional C*-algebra.

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2. A quantitative version of Wang's theorem

Let G be an infinite, finitely generated group. Let S be a finite, symmetric, generating set of G, with $e \in S$. Let $\mathbb{C}G$ be the complex group ring of G.

2.1. **Quantifying the Burnside theorem.** Let σ be an irreducible unitary representation of dimension d, i.e. a homomorphism $\sigma : G \to U_d(\mathbb{C})$

such that $\sigma(G)$ has no proper invariant subspace. The classical Burnside theorem says that $\sigma(\mathbb{C}G) = M_d(\mathbb{C})$, i.e. $\sigma(G)$ contains a basis of $M_d(\mathbb{C})$.

Definition 2.1. *Set* $k(\sigma) = \min\{k > 0 : \dim_{\mathbb{C}} span\, \sigma(S^k) = d^2\}$.

Lemma 2.2. *There is a constant* $C > 0$ *(only depending on* S*) such that* $C \log d \leq k(\sigma) \leq d^2$.

Proof. We have

$$d^2 = \dim_{\mathbb{C}} span\, \sigma(S^{k(\sigma)}) \leq |\sigma(S^{k(\sigma)})| \leq |S^{k(\sigma)}| \leq |S|^{k(\sigma)}.$$

Taking logarithms: $\frac{2}{\log |S|} \log d \leq k(\sigma)$. To prove the upper bound, observe that the sequence $span\, \sigma(S^k)$ of subspaces of $M_d(\mathbb{C})$, is strictly increasing for $k < k(\sigma)$. Indeed, assume that k is such that $span\, \sigma(S^k) = span\, \sigma(S^{k+1})$: this means that $span\, \sigma(S^k)$ is stable by left multiplication by $\sigma(S)$, hence by $\sigma(G)$ as S is generating. Since the identity matrix is in $\sigma(S^k)$, we have $\sigma(G) \subset span\, \sigma(S^k)$, hence $k \geq k(\sigma)$. From this it is clear that $k(\sigma) \leq d^2$. □

Let v be a unit vector in $\mathbb{C}^d$. Since v is cyclic for $\sigma(G)$, the map:

$$T_v : \mathbb{C}S^{k(\sigma)} \to \mathbb{C}^d : f \mapsto \sigma(f)v$$

is onto. Let $(\ker T_v)^{\perp}$ denote the orthogonal of $\ker T_v$ in $\mathbb{C}S^{k(\sigma)}$, let U_v be the inverse of the map $T_v|_{(\ker T_v)^{\perp}}$. Endow $\mathbb{C}S^{k(\sigma)}$ with the ℓ^1-norm, and let $\|U_v\|_{2\to 1}$ be the corresponding operator norm of U_v. So for every w a unit vector in $\mathbb{C}^d$, there exists $f \in \mathbb{C}S^{k(\sigma)}$ with $\|f\|_1 \leq \|U_v\|_{2\to 1}$, such that $\sigma(f)v = w$.

Lemma 2.3. *There exists* $M > 0$ *such that for every two unit vectors* $v, w \in \mathbb{C}^d$*, there exists* $f \in \mathbb{C}S^{k(\sigma)}$ *with* $\|f\|_1 \leq M$*, such that* $\sigma(f)v = w$.

Proof. This is the preceding observation plus compactness of the unit sphere in $\mathbb{C}^d$: the constant $M = \max_{\|v\|=1} \|U_v\|_{2\to 1}$ does the job. □

2.2. **From weak containment to weak containment à la Zimmer.** Recall that, if π, ρ are unitary representations of a discrete group G, the representation π is weakly contained in the representation ρ (i.e. $\pi \preceq \rho$) if every function of positive type associated with π can be pointwise approximated by finite sums of positive definite type associated with ρ. If π is irreducible, this is equivalent to require that every normalized function of positive type associated with π can be pointwise approximated by normalized functions of positive type associated with ρ (see Proposition F.1.4 in [2]).

Zimmer introduced in Definition 7.3.5 of [13] a different, inequivalent notion of weak containment. A $n \times n$-submatrix of π is a function

$$G \to M_n(\mathbb{C}) : g \mapsto (\langle \pi(g)e_i, e_j \rangle)_{1 \leq i,j \leq n}$$

where $\{e_1, ..., e_n\}$ is an orthonormal family in $\mathcal{H}_\pi$. Say that π is weakly contained in ρ in Zimmer's sense (i.e. $\pi \preceq_Z \rho$) if, for every $n > 0$, every

$n\times n$-submatrix of π can be pointwise approximated by $n\times n$-submatrices of ρ. The exactly relation with the classical notion recalled above, is worked out in Remark F.1.2(ix) in [2]; in particular, when π is irreducible, $\pi\preceq\rho$ implies $\pi\preceq_Z\rho$. Our first goal will be to make the latter statement quantitative. For this we need a definition.

Let ϕ be associated with σ, as in Definition 1.2. Let v be a unit vector in $\mathcal{H}_\sigma$ such that $\phi=\phi_{\sigma,v}$. Let $e_1,...,e_d$ be an orthonormal basis of $\mathbb{C}^d$; by lemma 2.3, we find functions $f_1,...,f_d\in\mathbb{C}S^{k(\sigma)}$, with $\max_i\|f_i\|_1\leq M$, such that $\sigma(f_i)v=e_i\,(i=1,...,d)$.

Lemma 2.4. *Let $\pi\in\mathrm{Rep}(G,\mathcal{H})$ be a unitary representation. Assume there is $\varepsilon>0$ and a unit vector $\eta\in\mathcal{H}$ such that for $s\in S^{2k(\sigma)+1}$ we have $|\langle\pi(s)\eta,\eta\rangle-\langle\sigma(s)v,v\rangle|<\varepsilon$. Set $\eta_i=\pi(f_i)\eta$. Then for $i,j=1,...,d$ and $g\in S$:*

$$|\langle\sigma(g)e_i,e_j\rangle-\langle\pi(g)\eta_i,\eta_j\rangle|\leq\varepsilon M^2.$$

Proof. For $g\in S$:

$$\begin{aligned}|\langle\sigma(g)e_i,e_j\rangle-\langle\pi(g)\eta_i,\eta_j\rangle|&=|\langle\sigma(g)\sigma(f_i)v,\sigma(f_j)v\rangle-\langle\pi(g)\pi(f_i)\eta,\pi(f_j)\eta\rangle|\\&=|\sum_{s,t\in G}f_i(s)\overline{f_j(t)}(\langle\sigma(t^{-1}gs)v,v\rangle-\langle\pi(t^{-1}gs)\eta,\eta\rangle)|\\&\leq\sum_{s,t\in G}|f_i(s)||f_j(s)||\langle\sigma(t^{-1}gs)v,v\rangle-\langle\pi(t^{-1}gs)\eta,\eta\rangle|.\end{aligned}$$

Since the supports of the f_i's are contained in $S^{k(\sigma)}$, and $t^{-1}gs\in S^{2k(\sigma)+1}$ for $s,t\in S^{k(\sigma)}$, we get using the assumption:

$$|\langle\sigma(g)e_i,e_j\rangle-\langle\pi(g)\eta_i,\eta_j\rangle|\leq\varepsilon\sum_{s,t\in G}|f_i(s)||f_j(t)|=\varepsilon\|f_i\|_1\|f_j\|_1\leq\varepsilon M^2.$$

□

In the previous proof, by applying the Gram-Schmidt orthonormalization process to the η_i's, it is possible to show that the $d\times d$-submatrix $(\langle\sigma(\cdot)e_i,e_j\rangle)_{1\leq i,j\leq d}$ of σ, is close on S to some $d\times d$-submatrix of α, with an explicit bound; but we don't need it at this point.

2.3. **Quantifying Wang's theorem.** Let $\mathcal{H}_\sigma$ be the (d-dimensional) Hilbert space of σ, and let $\mathcal{H}_{\overline{\sigma}}$ be the conjugate Hilbert space (with complex conjugate scalar multiplication and complex conjugate inner product), equipped with the conjugate representation $\overline{\sigma}$. Form the tensor product $\mathcal{H}_{\overline{\sigma}}\otimes\mathcal{H}_\pi$, carrying the representation $\overline{\sigma}\otimes\pi$. Set $\xi_i=e_i\otimes\eta_i$ and $\xi=\sum_{i=1}^d\xi_i\in\mathcal{H}_{\overline{\sigma}}\otimes\mathcal{H}_\pi$, where the e_i's and η_i's are as in the section above; observe that the ξ_i's are pairwise orthogonal. We need an estimate on how ξ is moved by $\overline{\sigma}\otimes\pi$.

$$\begin{aligned}\|\xi-(\overline{\sigma}\otimes\pi)(g)\xi\|^2&=2\|\xi\|^2-2\mathrm{Re}\langle(\overline{\sigma}\otimes\pi)(g)\xi,\xi\rangle\\&=2\sum_{i=1}^d\|\xi_i\|^2-2\sum_{i,j=1}^d\mathrm{Re}\langle(\overline{\sigma}\otimes\pi)(g)\xi_i,\xi_j\rangle\end{aligned}$$

$$= 2\sum_{i=1}^{d}\|\eta_i\|^2 - 2\sum_{i,j=1}^{d}\mathrm{Re}\langle e_j,\sigma(g)e_i\rangle\langle\pi(g)\eta_i,\eta_j\rangle.$$

Observe that for every $g \in G$ we have: $d = \sum_{i,j=1}^{d}\langle\sigma(g)e_i,e_j\rangle\langle e_j,\sigma(g)e_i\rangle$ as the e_i's are an orthonormal basis. Subtracting and adding $2d$ to the previous formula we get:

$$\|\xi-(\overline{\sigma}\otimes\pi)(g)\xi\|^2 = 2[\sum_{i=1}^{d}(\|\eta_i\|^2-1)]-2\sum_{i,j=1}^{d}\mathrm{Re}\langle e_j,\sigma(g)e_i\rangle(\langle\pi(g)\eta_i,\eta_j\rangle-\langle\sigma(g)e_i,e_j\rangle)$$

hence, using Cauchy-Schwarz:

$$\|\xi-(\overline{\sigma}\otimes\pi)(g)\xi\|^2 \leq 2\sum_{i=1}^{d}|\|\eta_i\|^2-1|+2\sum_{i,j=1}^{d}|\langle\pi(g)\eta_i,\eta_j\rangle-\langle\sigma(g)e_i,e_j\rangle| \quad (2.1)$$

Theorem 1.3 will follow immediately form the next Proposition, together with lemma 2.2

Proposition 2.5. *Let G be a discrete Kazhdan group, S a finite symmetric generating set with $e \in S$, and let ϕ be a normalized positive definite function on G associated with a finite-dimensional unitary irreducible representation σ of G. For every $0 < \delta < 1$ there exists $\varepsilon_{\phi,\delta} > 0$ such that for every $\pi \in \mathrm{Rep}(G,\mathcal{H})$, and every unit vector $x \in \mathcal{H}$ such that $|\phi(s)-\phi_{\pi,x}(s)| < \varepsilon_{\phi,\delta}$ for $s \in S^{2k(\sigma)+1}$, there exists a unit vector $x' \in \mathcal{H}$ with $\|x-x'\| \leq \delta$ such that the restriction of π to the span of $\pi(G)x'$ is isomorphic to σ.*

Proof. Set $d = \dim\sigma$, let v be a unit vector in $\mathcal{H}_\sigma$ such that $\phi(g) = \langle\sigma(g)v,v\rangle$ for every $g \in G$. As in section 2.2, for an orthonormal basis $e_1,...,e_d$ of $\mathbb{C}^d$, we find functions $f_1,...,f_d \in \mathbb{C}S^{k(\sigma)}$, with $\max_i\|f_i\|_1 \leq M$, such that $\sigma(f_i)v = e_i\,(i = 1,...,d)$.

Let $0 < \varepsilon_0 < 2$ be such that (S,ε_0) is a Kazhdan pair for G. Fix δ with $0 < \delta < 1$, and set

$$\varepsilon_{\phi,\delta} = \varepsilon = \frac{\delta^2\varepsilon_0^2}{24d(d+1)M^2}.$$

Let $\pi \in \mathrm{Rep}(G,\mathcal{H})$ and $x \in \mathcal{H}$ be a unit vector with $|\phi_{\pi,x}(s)-\phi(s)| < \varepsilon$ for $s \in S^{2k(\sigma)+1}$. Set $\eta_i = \pi(f_i)x$. We may assume that $e_1 = v$ and the function f_1 is δ_e, so that $\eta_1 = x$. We want to prove that the vector $\xi = \sum_{i=1}^{d}(e_i\otimes\eta_i) \in \mathcal{H}_{\overline{\sigma}}\otimes\mathcal{H}$ is $(S,t\varepsilon_0)$-invariant for some $0 < t < 1$, in order to apply Proposition 1.1.

For $g \in S$ we have, by lemma 2.4 and the inequality 2.1:

$$\|\xi-(\overline{\sigma}\otimes\pi)(g)\xi\|^2 \leq 2d\varepsilon M^2 + 2d^2\varepsilon M^2 = 2d(d+1)\varepsilon M^2 = \frac{\delta^2\varepsilon_0^2}{12}$$

Again by lemma 2.4, evaluated at $g = e$, we have: $|\|\eta_i\|^2-1| \leq \varepsilon M^2 < \frac{1}{2}$, hence $\frac{1}{2} \leq \|\eta_i\|^2 \leq \frac{3}{2}$ and $\frac{d}{2} \leq \|\xi\|^2 = \sum_{i=1}^{d}\|\eta_i\|^2 \leq \frac{3d}{2}$. So that, for $g \in S$:

$$\|\xi-(\overline{\sigma}\otimes\pi)(g)\xi\|^2 \leq \frac{\delta^2\varepsilon_0^2}{6d}\|\xi\|^2.$$

By Proposition 1.1, there exists a G-fixed $\xi' \in \mathcal{H}_{\overline{\sigma}} \otimes \mathcal{H}$ such that $\|\xi - \xi'\|^2 \leq \frac{\delta^2}{6d}\|\xi\|^2$. Write $\xi' = \sum_{i=1}^d e_i \otimes \zeta_i$, so that $\|\xi - \xi'\|^2 = \sum_{i=1}^d \|\eta_i - \zeta_i\|^2$. Identify $\mathcal{H}_{\overline{\sigma}} \otimes \mathcal{H}$ with the space of linear operators from $\mathcal{H}_\sigma$ to $\mathcal{H}$ (endowed with the Hilbert-Schmidt norm), via $u \otimes y \mapsto (w \mapsto \langle w, u\rangle y)$. Then ξ' identifies with the operator $w \mapsto \sum_{i=1}^d \langle w, e_i\rangle \zeta_i$, which is therefore an intertwining operator between σ and π. The image of this operator, which is $span\{\zeta_1, ..., \zeta_d\}$, carries a sub-representation of π unitarily equivalent to σ (by Schur's lemma). Set $x'' = \zeta_1$, then:

$$\|x - x''\|^2 = \|\eta_1 - \zeta_1\|^2 \leq \sum_{i=1}^d \|\eta_i - \zeta_i\|^2 = \|\xi - \xi'\|^2 \leq \frac{\delta^2}{6d}\|\xi\|^2 \leq \frac{\delta^2}{6d}\frac{3d}{2} = \frac{\delta^2}{4},$$

i.e. $\|x - x''\| \leq \frac{\delta}{2}$. Finally, set $x' = \frac{x''}{\|x''\|}$, a unit vector in $\mathcal{H}$. Then by the triangle inequality:

$$\begin{aligned}\|x - x'\| &\leq \|x - x''\| + \|x'' - x'\| = \|x - x''\| + \|x''\||1 - \frac{1}{\|x''\|}| \\ &= \|x - x''\| + |\|x''\| - \|x\|| \leq 2\|x - x''\| \leq \delta.\end{aligned}$$

This concludes the proof. □

Question 2.6. *In the previous proof, the constant $\varepsilon_{\phi,\delta}$ depends on σ through the dimension d and the constant M from lemma 2.3. By Theorem 2.6 in* [12]*, a discrete Kazhdan group has finitely many unitary irreducible representations of a given finite dimension (up to unitary equivalence), so Theorem 1.3 can be made uniform over all unitary irreducible representations σ with dimension less than a given dimension. Can it be made uniform over all finite-dimensional unitary representations?*

3. Proof of Theorem 1.4

Let $\{U_n\}$ be a countable basis of open sets in the unit sphere K of $\mathcal{H}$, and let Φ be the set of all positive definite functions on G defining irreducible finite dimensional representations. Notice that the set $X' \subseteq \mathrm{Rep}(G, \mathcal{H})$ of all representations π such that for every $n \in \mathbb{N}$ and every $\delta > 0$ there exist $m > 0$, $x \in U_n$, $x_i \in K$, $c_i \in \mathbb{C} \setminus \{0\}$, and $\phi_i \in \Phi$, $i \leq m$, such that the x_i's are pairwise orthogonal, $x = \sum c_i x_i$, and each x_i is $(\pi, \phi_i, \varepsilon_{\phi_i,\delta'_i})$-good, where $\delta'_i = \frac{\delta}{|c_i|.m}$, and $\varepsilon_{\phi_i,\delta'_i}$ is given by Theorem 1.3, is a G_δ set. Indeed, for fixed n, δ, m, x, $\bar{x} = (x_1, \ldots, x_m)$, $\bar{c} = (c_1, \ldots, c_m)$, $\bar{\phi} = (\phi_1, \ldots, \phi_m)$ as above, the set

$$V^{n,\delta,m}_{x,\bar{x},\bar{c},\bar{\phi}} = \{\pi \in \mathrm{Rep}(G, \mathcal{H}) : \text{ each } x_i \text{ is } (\pi, \phi_i, \varepsilon_{\phi_i,\delta'_i})\text{-good}\}$$

is clearly open. We also put $V^{n,\delta,m}_{x,\bar{x},\bar{c},\bar{\phi}}$ to be the empty set if the x_i's are not pairwise orthogonal or $x \neq \sum c_i x_i$. Now we can define X' by

$$X' = \bigcap_{n\in\mathbb{N}} \bigcap_{\delta\in\mathbb{Q}^+} \bigcup_{m\in\mathbb{N}} \bigcup_{x\in U_n} \bigcup_{\bar{x}\in K^m} \bigcup_{\bar{c}\in\mathbb{C}^m} \bigcup_{\bar{\phi}\in\Phi^m} V^{n,\epsilon,m}_{x,\bar{x},\bar{c},\bar{\phi}},$$

which is a G_δ condition.

Moreover, X' is dense in $\mathrm{Rep}(G,\mathcal{H})$ as it contains all direct sums of finite-dimensional representations, which, by our assumption, are dense in $\mathrm{Rep}(G,\mathcal{H})$. This is because it is easy to see that for every such sum π there are densely many elements $x \in K$ of the form $\sum c_i x_i$, where x_i are pairwise orthogonal unit vectors, and each x_i is (π, ϕ_i, δ)-good for some ϕ_i and every $\delta > 0$.

Now we show that every representation in X' is a direct sum of finite-dimensional representations. Fix $\pi \in X'$. Using Zorn's lemma, we can decompose $\mathcal{H}$ into $\mathcal{H}_0$ and $\mathcal{H}_1$ such that $\mathcal{H}_0$ is the direct sum of all finite-dimensional representations contained in π. For $i = 0, 1$, let $P_{\mathcal{H}_i}$ be the orthogonal projection of $\mathcal{H}$ on $\mathcal{H}_i$. Suppose that $\mathcal{H}_1$ is not trivial, and fix $0 < \delta < 1$, $x \in K$, pairwise orthogonal $x_i \in K$ and $c_i \in \mathbb{C} \setminus \{0\}$, $i \le m$, such that $x = \sum c_i x_i$, each x_i is $(\pi, \phi_i, \varepsilon_{\phi_i, \frac{\delta}{|c_i|.m}})$-good for some $\phi_i \in \Phi$, and $\|x - P_{\mathcal{H}_0} x\| > \delta$ (the last condition can be satisfied by choosing x in an appropriate U_n.) By Theorem 1.3, there exist $x'_i \in K$, $i \le m$, inducing irreducible finite-dimensional representations, and such that $\|x_i - x'_i\| < \frac{\delta}{|c_i|.m}$, that is, $\|x - \sum c_i x'_i\| < \delta$. But then, clearly, $x'_{i_0} \notin \mathcal{H}_0$ for some $i_0 \le m$, as if it was not the case, we would get that $\|x - \sum c_i x'_i\| \ge \|x - P_{\mathcal{H}_0} x\| > \delta$. Since $P_{\mathcal{H}_1}$ is a G-intertwiner, the image under $P_{\mathcal{H}_1}$ of the linear span of $\pi(G)x'_{i_0}$, is an invariant subspace of $\mathcal{H}_1$, which is a contradiction.

Now let X'' be the set of all those representations that contain every finite dimensional representation with infinite multiplicity. As G is a Kazhdan group, we can see that X'' is given by a G_δ condition. Indeed, for $[\sigma]$ the isomorphism class of a finite-dimensional unitary irreducible representation of G, and $n > 0$, let $V_{[\sigma],n}$ be the set of representations $\pi \in \mathrm{Rep}(G,\mathcal{H})$ such that $[\sigma]$ appears in π with multiplicity at least n. Clearly $V_{[\sigma],n}$ is open and

$$X'' = \bigcap_{[\sigma]} \bigcap_{n} V_{[\sigma],n},$$

where the intersection is countable because there are countably many $[\sigma]$'s.

By our assumption on $C^*(G)$, the set X'' is dense. Thus, $X = X' \cap X''$ is a dense G_δ set, all the representations of which are direct sums of finite dimensional representations, each appearing with infinite multiplicity. Clearly, all elements in X are conjugate. □

Remark 3.1. The converse of Theorem 1.4 also follows from Theorem 2.5 in [9]. That is, if either G does not have property (T), or $C^*(G)$ is not residually finite-dimensional, then all classes in $\mathrm{Rep}(G,\mathcal{H})$ are meager. Indeed, Theorem 2.5 from [9] says: if for a separable C*-algebra A the set of isolated points in $\hat{A}$ is not dense, then the restriction of the action of $U(\mathcal{H})$ by conjugation on a dense G_δ invariant subset of $\mathrm{Rep}(A,\mathcal{H})$ is turbulent. That, by the definition of turbulence, in particular implies that every class in $\mathrm{Rep}(G,\mathcal{H})$ is meager. Now take $A = C^*(G)$: as isolated points in

$\hat{G}$ correspond to finite-dimensional representations, it follows that when G does not have property (T), $\hat{G}$ does not have isolated points, by Theorem 2.1 in Wang [12]; when $C^*(G)$ is not residually finite-dimensional, then the isolated points in $\hat{G}$ are not dense by Archbold's main result in [1].

4. Proof of Theorem 1.5

For a unitary representation π, we denote by $\infty \cdot \pi$ the ℓ^2-direct sum of countably many copies of π.

Lemma 4.1. *Let H be a locally compact group. Assume that H has (up to unitary equivalence) countably many finite-dimensional irreducible unitary representations $\sigma_1, \sigma_2, \ldots$. Then the representation $\oplus_{n=1}^{\infty} \infty \cdot \sigma_n$ is unitarily equivalent to a Koopman representation.*

Proof. View σ_n as a continuous homomorphism $H \to U(N_n)$. Let K_n denote the closure of $\sigma_n(H)$ in $U(N_n)$, so that K_n is a compact group (on which H acts by left translations by elements of $\sigma_n(H)$). Let m_n denote normalized Haar measure on K_n, and let λ_n denote the regular representation of K_n on $L^2(K_n, m_m)$. For $p \geq 1$, let $K_{n,p}$ denote a copy of K_n endowed with the measure $2^{-n-p} m_n$. Set $X = \coprod_{n,p} K_{n,p}$, endowed with the H-invariant probability measure $\mu = \oplus_{n,p} 2^{-n-p} m_n$. Note that the H-representations on $L^2(X, \mu)$ and on $L^2_0(X, \mu)$ are equivalent, as $L^2(X, \mu)$ contains the trivial representation with infinite multiplicity.

So it is enough to prove that the H-representation π on $L^2(X, \mu)$ is equivalent to $\oplus_{n=1}^{\infty} \infty \cdot \sigma_n$. To see this, first observe that π is equivalent to $\oplus_n \infty \cdot \pi_n$, where $\pi_n = \lambda_n \circ \sigma_n$. By Peter-Weyl, π_n decomposes as a direct sum of finite-dimensional irreducible representations of H, hence of certain σ_k's, and moreover σ_n is a sub-representation of π_n (because the natural representation of K_n on $\mathbb{C}^{N_n}$ is irreducible, hence appears as a sub-representation of λ_n). This shows that $\oplus_n \infty \cdot \pi_n$ is equivalent to $\oplus_n \infty \cdot \sigma_n$. □

To prove Theorem 1.5, observe that a discrete Kazhdan group G satisfies the assumption of lemma 4.1 (by Theorem 2.6 in [12]). Let $(\sigma_n)_{n \in \mathbb{N}}$ be an enumeration of all finite-dimensional irreducible unitary representations of G. By Theorem 1.4 and its proof, the representation $\bigoplus_{n=1}^{\infty} \infty \cdot \sigma_n$ has a comeager conjugacy class.

In particular, we get the following statement which was proved in [4] only for finite abelian groups.

Corollary 4.2. *Let G be a finite group. Then the set of unitary representations realizable by an action is comeager in* $\mathrm{Rep}(G, \mathcal{H})$.

Remark 4.3. Kechris proves (see section (F) in Appendix H of [8]) that, if G is torsion-free abelian, then the set of representations realizable by an action is meager in $\mathrm{Rep}(G, \mathcal{H})$.

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