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Preprint No. 53-2018

PRAHA 2018

LIPSCHITZ FREE p-SPACES FOR $0 < p < 1$

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Abstract. This paper initiates the study of the structure of a new class of p-Banach spaces, $0 < p < 1$, namely the Lipschitz free p-spaces (alternatively called Arens-Eells p-spaces) $\mathcal{F}_p(\mathcal{M})$ over p-metric spaces. We systematically develop the theory and show that some results hold as in the case of $p = 1$, while some new interesting phenomena appear in the case $0 < p < 1$ which have no analogue in the classical setting. For the former, we, e.g., show that the Lipschitz free p-space over a separable ultrametric space is isomorphic to ℓ_p for all $0 < p \leq 1$, or that ℓ_p isomorphically embeds into $\mathcal{F}_p(\mathcal{M})$ for any p-metric space $\mathcal{M}$. On the other hand, solving a problem by the first author and N. Kalton, there are metric spaces $\mathcal{N} \subset \mathcal{M}$ such that the natural embedding from $\mathcal{F}_p(\mathcal{N})$ to $\mathcal{F}_p(\mathcal{M})$ is not an isometry.

1. Introduction

It is safe to say that most of the research in functional analysis is done in the framework of Banach spaces. While the theory of the geometry of these spaces has evolved very rapidly over the past sixty years, by contrast, the study of the more general case of quasi-Banach spaces has lagged far behind despite the fact that the first papers in the subject appeared in the early 1940's ([5,10]). The neglect of non-locally convex spaces within functional analysis is easily understood. Even when they are complete and metrizable, working with them requires doing without one of the most powerful tools in Banach spaces: the

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2010 *Mathematics Subject Classification.* 26A16; 46A16; 46B20; 46B80; 46B85.

Key words and phrases. Quasimetric space, quasi-Banach space, metric envelope, Lipschitz Free p-space, Arens-Eells p-space.

F. Albiac acknowledges the support of the Spanish Ministry for Economy and Competitivity Grants MTM2014-53009-P for *Análisis Vectorial, Multilineal y Aplicaciones*, and MTM2016-76808-P for *Operators, lattices, and structure of Banach spaces*. J.L. Ansorena acknowledges the support of the Spanish Ministry for Economy and Competitivity Grant MTM2014-53009-P for *Análisis Vectorial, Multilineal y Aplicaciones* M. Cúth has been supported by Charles University Research program No. UNCE/SCI/023 and by the Research grant GAČR 17-04197Y. M. Doucha was supported by the GAČR project 16-34860L and RVO: 67985840.

Hahn-Banach theorem and the duality techniques that rely on it. This difficulty in even making the simplest initial steps has led some to regard quasi-Banach spaces as too challenging and consequently they have been assigned a secondary role in the theory. However, these challenges have been accepted by some researchers and the number of fresh techniques available in this general setting is now increasing (see a summary in [15]). We emphasize that proving new results in p-Banach spaces for $0 < p < 1$ often provides an alternative proof even for the limit case $p = 1$. Hence, quasi-Banach spaces help us appreciate better and also shed new light on regular Banach spaces. Taking into account that more analysts find that quasi-Banach spaces have uses in their research, the task to know more about their structure seems to be urgent and important.

Every family of classical Banach spaces, like the sequence spaces ℓ_p, the function spaces L_p, the Hardy spaces $H_p(\mathbb{T})$, the Lorentz sequence spaces $d(w,p)$, or the Orlicz sequence spaces l_φ, have a non-locally convex counterpart corresponding to the values of $0 < p < 1$, and φ non-convex. In the present paper we study Lipschitz free p-spaces over quasimetric spaces. These new class of p-Banach spaces for $0 < p < 1$ are an analogy of the Lipschitz free spaces $\mathcal{F}(\mathcal{M})$, whose study has become a very active research field within Banach space theory since the appearance in 1999 of the important book [23] by Weaver and, more notably, after the seminal paper [14] by Godefroy and Kalton in 2003. We note that quasimetric spaces are generalizations of not only the classical Euclidean setting, but of quasi-Banach spaces and ultrametric spaces. For instance, quasimetrics play an important role in the study of Gromov's hyperbolic spaces. Indeed, the boundary $\partial_\infty X$ of a Gromov hyperbolic space X carries a natural quasi-metric $\rho(x,y) = e^{\langle x,y\rangle_0}$, where $\langle\cdot,\cdot\rangle_0$ is the Gromov product with respect to some base point $0 \in X$.

Lipschitz free p-spaces were introduced in [3] with the sole instrumental purpose to build examples for each $0 < p < 1$ of two *separable* p-Banach spaces which are Lipschitz-isomorphic but fail to be linearly isomorphic. Whether this is possible or not for $p = 1$ remains as of today the single most important open problem in the theory of non-linear classification of Banach spaces. However, even though Lipschitz free p-spaces were proved to be of substantial utility in functional analysis, the structure of those spaces has not been investigated ever since. Our goal in this paper is to fill this gap in the theory, encourage further research in this direction, and help those who want to contribute to this widely unexplored topic.

To that end, after the preliminary Section 2 on the basics in quasimetric and quasi-Banach spaces, in Section 3 we introduce the notion of metric envelope of a quasimetric space $\mathcal{M}$ and relate it to the existence of non-constant Lipschitz maps on $\mathcal{M}$ as well as to the Banach envelope when $\mathcal{M}$ is a quasi-Banach space. In Section 4 we recall the definition of Lipschitz free p-space and bring up to light the main differences and setbacks of this theory with respect to the case $p = 1$. We also settle a question that was raised in [3] and use molecules and atoms in order to give an alternative equivalent definition of Lispchitz free p-spaces which will be very useful in order to provide examples of Lipschitz free p-spaces isometrically isomorphic to ℓ_p and L_p for $0 < p < 1$. In Section 5 we completely characterise Lipschitz free p-spaces over separable ultrametric spaces, showing that for $0 < p \leq 1$ they are isomorphic to ℓ_p.

The most important results are perhaps the ones in Sect. 6, where we study the relation between the subset structure of a quasimetric space $\mathcal{M}$ and the subspace structure of $\mathcal{F}_p(\mathcal{M})$. To be precise, for each $p < 1$ and each $p < q \leq 1$ we provide an example of a subset $\mathcal{N}$ of a q-metric space $\mathcal{M}$ such that $\mathcal{F}_p(\mathcal{N})$ is not naturally a subspace of $\mathcal{F}_p(\mathcal{M})$. This fact evinces a very important dissimilarity with respect to the case $p = 1$ and solves another problem raised in [3]. We close the section and the article exhibiting a method to linearly embed ℓ_p in $\mathcal{F}_p(\mathcal{M})$ for $\mathcal{M}$ quasimetric and $p \leq 1$. The complementability of ℓ_p in Lipschitz free p-spaces over quasimetric spaces will be the subject of a further publication.

Throughout this note we use standard terminology and notation in Banach space theory as can be found in [4]. We refer the reader to [23] for basic facts on Lipschitz free spaces and some of their uses, and to [16] for background on quasi-Banach spaces.

2. Preliminaries

There are two main goals in this preliminary section. First we review the notion of quasi-metric space along with the related notion of quasi-Banach space and their main topological features. Second, we lay out the notation and terminology used in this article.

2.1. **Quasimetric spaces and Lipschitz maps.**

Given an arbitrary nonempty set $\mathcal{M}$, a *quasimetric* on $\mathcal{M}$ is a symmetric map $\rho\colon \mathcal{M} \times \mathcal{M} \to [0, \infty)$ such that $\rho(x, y) = 0$ if and only if $x = y$, and for some constant $\kappa \geq 1$, ρ satisfies the quasi-triangle inequality

$$\rho(x, z) \leq \kappa(\rho(x, y) + \rho(y, z)), \qquad x, y, z \in \mathcal{M}. \tag{2.1}$$

The space $(\mathcal{M},\rho)$ is then called a *quasimetric space* (see [18, p. 109]). A quasimetric ρ on a set $\mathcal{M}$ is said to be a *p-metric*, $0<p\le 1$, if ρ^p is a metric, i.e.,

$$\rho^p(x,y)\le \rho^p(x,z)+\rho^p(z,y), \qquad x,y,z\in\mathcal{M},$$

in which case we call $(\mathcal{M},\rho)$ a *p-metric space.* An analogue of the Aoki-Rolewicz theorem holds in this context: every quasimetric space can be endowed with an equivalent p-metric τ for some $0<p\le 1$, i.e., there is a constant $C=C(\kappa)\ge 1$ such that

$$C^{-1}\tau(x,y)\le \rho(x,y)\le C\tau(x,y), \qquad x,y\in\mathcal{M}$$

(see [18, Proposition 14.5]).

If $(\mathcal{M},\rho)$ and $(\mathcal{N},\tau)$ are quasimetric spaces we shall say that a map $f\colon\mathcal{M}\to\mathcal{N}$ is *Lipschitz* if there exists a constant $C\ge 0$ so that

$$\tau(f(x),f(y))\le C\rho(x,y), \qquad x,y\in\mathcal{M}. \tag{2.2}$$

We denote by $\operatorname{Lip}(f)$ the smallest constant which can play the role of C in the last inequality (2.2), i.e.,

$$\operatorname{Lip}(f)=\sup\left\{\frac{\tau(f(x)f(y))}{\rho(x,y)}\colon x,y\in\mathcal{M}, x\neq y\right\}\in[0,\infty).$$

If f is injective, and both f and f^{-1} are Lipschitz, then we say that f is *bi-Lipschitz* and that $\mathcal{M}$ Lipschitz-embeds into $\mathcal{N}$. If there is a bi-Lipschitz map from $\mathcal{M}$ onto $\mathcal{N}$, the spaces $\mathcal{M}$ and $\mathcal{N}$ are said to be *Lipschitz isomorphic.* A map f from a quasimetric space $(\mathcal{M},\rho)$ into a quasimetric space $(\mathcal{N},\tau)$ is an *isometry* if

$$\tau(f(x),f(y))=\rho(x,y), \qquad x,y\in\mathcal{M}.$$

We shall say that $(\mathcal{M},\rho)$ is a *pointed quasimetric space (or a pointed p-metric space, or a pointed metric space)*, if it has a distinguished point that we call the *origin* and denote 0. The assumption of an origin is convenient to normalize Lipschitz functions.

The *Lipschitz dual* of a quasimetric space $(\mathcal{M},\rho)$, denoted $\operatorname{Lip}_0(\mathcal{M})$, is the (possibly trivial) vector space of all real-valued Lipschitz functions f defined on $\mathcal{M}$ such that $f(0)=0$, endowed with the Lipschitz norm

$$\|f\|_{\operatorname{Lip}}=\sup\left\{\frac{|f(x)-f(y)|}{\rho(x,y)}\colon x,y\in\mathcal{M}, x\neq y\right\}.$$

It can be readily checked that $(\operatorname{Lip}_0(\mathcal{M}),\|\cdot\|_{\operatorname{Lip}})$ is a Banach space.

Example 2.1. Let $\mathcal{M}$ be the real line equipped with the quasimetric $\rho(x,y) = |x-y|^{1/p}$, where $0<p<1$. Then $\operatorname{Lip}_0(\mathcal{M}) = \{0\}$. Indeed, suppose $f\colon \mathbb{R} \to \mathbb{R}$ verifies

$$|f(x)-f(y)| \le C|x-y|^{1/p}, \quad \forall x,y \in \mathbb{R},$$

for some $C>0$. Since $1/p>1$,

$$\frac{|f(x)-f(y)|}{|x-y|} \le C|x-y|^{1/p-1} \to 0 \;\text{ if }\; |x-y| \to 0.$$

That is, $f'(x) = 0$ for all $x \in \mathbb{R}$ and so f must be constant. Since $f(0)=0$ it follows that $f=0$.

2.2. **Quasi-normed spaces and their Banach envelopes.** Recall that a *quasi-normed space* is a (real) vector space X equipped with a map $\|\cdot\|_X \colon X \to [0,\infty)$ with the properties:

(i) $\|x\|_X > 0$ for all $x \neq 0$,
(ii) $\|\alpha x\|_X = |\alpha|\|x\|_X$ for all $\alpha \in \mathbb{R}$ and all $x \in X$,
(iii) there is a constant $\kappa \ge 1$ so that for all x and $y \in X$ we have

$$\|x+y\|_X \le \kappa(\|x\|_X + \|y\|_X). \tag{2.3}$$

A quasi-norm $\|\cdot\|_X$ induces a linear metric topology. X is called a *quasi-Banach space* if X is complete for this metric. Given $0<p\le 1$, X is said to be a *p-normed* space if the quasi-norm $\|\cdot\|_X$ verifies (i), (ii) and it is p-subadditive, i.e.,

(iv) $\|x+y\|_X^p \le \|x\|_X^p + \|y\|_X^p$ for all $x,y \in X$.

Of course, (iv) implies (iii). In case $\|\cdot\|_X$ is p-normed, a metric inducing the topology can be defined by $d(x,y) = \|x-y\|_X^p$. A quasi-Banach space with an associated p-norm is also called a *p-Banach space.*

A map $\|\cdot\|_X \colon X \to [0,\infty)$ that verifies properties (ii) and (iv) is called a *p-seminorm* on X. Given a p-seminorm $\|\cdot\|_X$ on a vector space X it is standard to construct a p-Banach space from the pair $(X, \|\cdot\|_X)$ following the so-called *completion method.* For that we consider the vector subset $N = \{x \in X\colon \|x\|_X = 0\}$ and form the quotient space X/N, which is p-normed when endowed with $\|\cdot\|_X$. Now we just need to complete $(X/N, \|\cdot\|_X)$. The reader should be acquainted with the fact that completeness and completion for quasi-metric spaces are completely analogous to such notions for metric spaces.

Given $0<p\le 1$, a subset $\mathcal{C}$ of a vector space V is said to be *absolutely p-convex* if for any x and $y \in \mathcal{C}$ and any scalars λ and μ with $|\lambda|^p + |\mu|^p \le 1$ we have $\lambda\, x + \mu\, y \in \mathcal{C}$. The Minkowski functional $\|\cdot\|_{\mathcal{C}}$ of an absolutely p-convex set $\mathcal{C}$, given by

$$\|x\|_{\mathcal{C}} = \inf\left\{\lambda > 0\colon \lambda^{-1}x \in \mathcal{C}\right\},$$

defines a p-seminorm on $\operatorname{span}(\mathcal{C})$.

Given a nonempty subset Z of a vector space V there is a method for building a p-Banach space from it. Let $\operatorname{co}_p(Z)$ denote the *p-convex hull* of Z, i.e., the smallest absolutely p-convex set containing Z. If $N = \{x \in \operatorname{span}(Z) \colon \|x\|_{\operatorname{co}_p(Z)} = 0\}$, then the quotient space $\operatorname{span}(Z)/N$ equipped with $\|\cdot\|_{\operatorname{co}_p(Z)}$ is a p-normed linear space. In the case when $\operatorname{span}(Z)^*$ separates the points of $\operatorname{span}(Z)$ then $\|\cdot\|_{\operatorname{co}_p(Z)}$ is a p-norm.

Definition 2.2. The completion of $(\operatorname{span}(Z)/N, \|\cdot\|_{\operatorname{co}_p(Z)})$ will be called the *p-Banach space constructed from Z by the p-convexification method* and will be denoted by $(X_{p,Z}, \|\cdot\|_{p,Z})$.

Notice that it is possible to give an explicit expression for $\|\cdot\|_{p,Z}$. As a matter of fact, for $x \in X_{p,Z}$ we have

$$\|x\|_{p,Z} = \inf \left\{ \left(\sum_{j=1}^\infty |a_j|^p \right)^{1/p} : x = \sum_{j=1}^\infty a_i \, x_i, \quad x_i \in Z \right\}. \tag{2.4}$$

When dealing with a quasi-Banach space X it is often convenient to know which is the “smallest” Banach space containing X or, more generally, given $0 < q \le 1$, the smallest q-Banach space containing X. To be precise:

Definition 2.3. Given a quasi-Banach space X and $0 < q \le 1$, the *q-Banach envelope* of X (resp. *Banach envelope* for $q = 1$), denoted $(\widehat{X^q}, \|\cdot\|_{c,q})$ (resp. $(\widehat{X}, \|\cdot\|_c)$ for $q = 1$) is the q-Banach space obtained by applying to the unit ball B_X of X the q-convexification method.

Obviously $\|x\|_{c,q} \le \|x\|$ for all $x \in X$, so that the identity map on X induces a (not necessarily one-to-one) bounded linear map $i_{X,q} \colon X \to \widehat{X^q}$ whose range is dense in $\widehat{X^q}$. This map possesses the following universal property: if $T \colon X \to Y$ is a bounded linear map and Y is an arbitrary q-Banach space then T factors through $i_{X,q}$,

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "i_{X,q}"'] & & Y \\
& \widehat{X^q} \arrow[ur, "\widehat{T}"'] &
\end{tikzcd}$$

and the unique “extension” $\widehat{T} \colon \widehat{X^q} \to Y$ has the same norm as T. In particular, X and $\widehat{X^q}$ have the same dual space.

For instance, the q-Banach envelope of ℓ_p for $0 < p < q \le 1$ is ℓ_q.

The following formula for the q-Banach envelope quasi-norm will be very useful. The case $q = 1$ was shown by Peetre in [21].

Lemma 2.4. *Let X be a quasi-Banach space and $0 < q \le 1$. Then for $x \in X$,*

$$\|x\|_{c,q} = \inf \left\{ \left(\sum_{i=1}^n \|x_i\|^q \right)^{1/q} : \sum_{i=1}^n x_i = x,\, x_i \in X,\, n \in \mathbb{N} \right\}. \tag{2.5}$$

Proof. Let $\|\cdot\|_0$ be the q-seminorm on X defined by the expression in (2.5) and X_0 be the q-Banach space obtained from $(X, \|\cdot\|_0)$ by the completion method. If $T\colon X \to Y$ is a bounded linear map and Y is q-Banach, then $\|T(\mathbf{x})\| \le \|T\|\|\mathbf{x}\|_0$. Consequently, X_0 has the same universal property as $\widehat{X^q}$, thus X_0 and $\widehat{X^q}$ are isometric. □

2.3. p-norming sets in quasi-Banach spaces.

Definition 2.5. Given a quasi-Banach space X and $0 < p \le 1$, we say that a subset Z of X is a *p-norming set with constants C and D* if

$$\frac{1}{C}\overline{\operatorname{co}_p(Z)} \subseteq B_X \subseteq D\overline{\operatorname{co}_p(Z)}.$$

In the case when $C = D = 1$ we say that Z is *isometrically p-norming.*

Note that Z is a p-norming set of X if and only if $\|\cdot\|_{p,Z}$ defines an equivalent quasi-norm on X. Consequently, if X admits a p-norming set then X is isomorphic to a p-Banach space. Conversely, if X is a p-Banach space, then a set $Z \subseteq X$ is p-norming with constants C and D if and only if

$$\frac{1}{C} Z \subseteq B_X \subseteq D\overline{\operatorname{co}_p(Z)}. \tag{2.6}$$

Adopting the terminology from harmonic analysis it can be said that a set Z is p-norming in X if and only if (Z, ℓ_p) is an atomic decomposition of X. Recall that a pair $(\mathcal{A}, \mathbb{S})$, where $\mathcal{A}$ is a subset of X and $\mathbb{S}$ is a symmetric sequence space, is said to be an *atomic decomposition* of X if there are constants $0 < C, D < \infty$ such that

(i) Given $f = (a_n)_{n=1}^\infty \in \mathbb{S}$ and $(\alpha_n)_{n=1}^\infty \subset \mathcal{A}$ then $\sum_{n=1}^\infty a_n\, \alpha_n$ converges in X to a vector x verifying $\|x\| \le C\|f\|_{\mathbb{S}}$, and
(ii) for any $x \in X$ there are $f = (a_n)_{n=1}^\infty \in \mathbb{S}$ and $(\alpha_n)_{n=1}^\infty \subset \mathcal{A}$ such that $x = \sum_{n=1}^\infty a_n\, \alpha_n$ and $\|f\|_{\mathbb{S}} \le D\|x\|$.

We conclude this preliminary section enunciating for future reference a few straightforward auxiliary results on p-norming sets.

Lemma 2.6. *Suppose Z_1 and Z_2 are subsets of a quasi-Banach X such that $Z_1 \subseteq Z_2$, Z_1 is dense in Z_2, and Z_2 is p-norming in X. Then Z_1 is a p-norming set in X with the same constants as Z_2.*

Lemma 2.7. *Suppose that Z_1 and Z_2 are p-norming sets for quasi-Banach spaces X_1 and X_2, respectively. Let T be a one-to-one linear map from* $\operatorname{span}(Z_1)$ *into X_2 such that $T(Z_1) = Z_2$. Then T extends to an onto isomorphism $\widetilde{T}\colon X_1 \to X_2$. Moreover, in the case when Z_1 and Z_2 are both isometrically p-norming sets, $\widetilde{T}$ is an isometry.*

Lemma 2.8. *Suppose that Z is a p-norming set for a quasi-Banach space X with constants C_1 and C_2 and that $Z_0 \subseteq Z$. If there is a constant C such that every $x \in Z$ can be written as $x = \sum_{n=1}^\infty a_n\, x_n$ for some $f = (a_n)_{n=1}^\infty \in \ell_p$ with $\|f\|_p \le C$ and $(x_n)_{n=1}^\infty$ in Z_0, then Z_0 is a p-norming set for X with constants C_1 and CC_2.*

Proof. By hypothesis $Z \subseteq C\,\overline{\operatorname{co}_p(Z_0)}$. Therefore $\overline{\operatorname{co}_p(Z)} \subseteq C\,\overline{\operatorname{co}_p(Z_0)}$, and so

$$\frac{1}{C_1}\overline{\operatorname{co}_p(Z_0)} \subseteq \frac{1}{C_1}\overline{\operatorname{co}_p(Z)} \subseteq B_X \subseteq C_2\,\overline{\operatorname{co}_p(Z)} \subseteq CC_2\,\overline{\operatorname{co}_p(Z_0)},$$

as desired. □

3. The metric envelope of a quasimetric space

Suppose $(\mathcal{M}, \rho)$ is a pointed quasimetric space. By analogy with the universal extension property of the Banach envelope of a quasi-Banach space, we are interested in the question on how to construct a metric space $(\widetilde{\mathcal{M}}, \widetilde{\rho})$, and a map $Q\colon \mathcal{M} \to \widetilde{\mathcal{M}}$ with $\operatorname{Lip}(Q) \le 1$ such that whenever (M, d) is a metric space and $f\colon \mathcal{M} \to M$ verifies the Lipschitz condition

$$d(f(x), f(y)) \le C\rho(x, y), \qquad x, y \in \mathcal{M}, \tag{3.7}$$

then f induces a Lipschitz map $\widetilde{f}\colon \widetilde{\mathcal{M}} \to M$ without increasing the Lipschitz norm, i.e., $d(\widetilde{f}(x), \widetilde{f}(y)) \le C\widetilde{\rho}(x, y)$ for all $x, y \in \widetilde{\mathcal{M}}$.

Note that if f verifies (3.7), then we will have

$$d(f(Q(x)), f(Q(y))) \le C\sum_{i=0}^n \rho(x_i, x_{i+1}),$$

for any finite sequence $x = x_0, x_1, \dots, x_{n+1} = y$ of (possibly repeated) points in $\mathcal{M}$. Therefore, in all fairness we define, for $x, y \in \mathcal{M}$,

$$\widetilde{\rho}(x, y) = \inf \sum_{i=0}^n \rho(x_i, x_{i+1}), \tag{3.8}$$

where the infimum is taken over all sequences $x = x_0, x_1, \dots, x_{n+1} = y$ of finitely-many points in $\mathcal{M}$. Clearly, $\widetilde{\rho}$ is symmetric, satisfies the triangle inequality, and does not exceed ρ. Before going on, let us point out that $\widetilde{\rho}(x, y)$ cand be zero for different points x, y in $\mathcal{M}$.

Example 3.1. A metric space $(\mathcal{M}, d)$ is metrically convex (see [6]) if for every $x, y \in \mathcal{M}$ and any $0 < \lambda < 1$ there exists $z_\lambda \in \mathcal{M}$ with

$$d(x, z_\lambda) = \lambda d(x, y) \quad \text{and} \quad d(y, z_\lambda) = (1-\lambda) d(x, y).$$

Let $(\mathcal{M}, d)$ be a metrically convex space and, for $0 < p < 1$, consider the p-metric $\rho = d^{1/p}$ on $\mathcal{M}$. Then $\widetilde{\rho}(x, y) = 0$ for any $x, y \in \mathcal{M}$. Indeed, given $x \neq y$ in $\mathcal{M}$, by the metric convexity of $\mathcal{M}$ for every $n \in \mathbb{N}$ we can find a chain of points $\{x_0, x_1, \dots, x_n\}$ where $x_0 = x$, $x_n = y$, and $d(x_{j-1}, x_j) = d(x, y)/n$ for each $j = 1, 2, \dots, n$. By the definition we then have

$$\widetilde{\rho}(x, y) \leq \left(\frac{d(x, y)}{n} \right)^{1/p} n = \frac{d(x, y)}{n^{1/p-1}} \to 0.$$

Thus, $\widetilde{\rho}(x, y) = 0$.

In view of that, we shall identify points in $\mathcal{M}$ that are at a zero $\widetilde{\rho}$-distance, which leads to the following definition.

Definition 3.2. Let $(\mathcal{M}, \rho)$ be a quasimetric space and $\tilde{\rho}$ as in (3.8). We consider the equivalence relation

$$x \sim y \Longleftrightarrow \widetilde{\rho}(x, y) = 0,$$

and define $\widetilde{\mathcal{M}}$ to be the quotient space $\mathcal{M}/\sim$. If $\widetilde{x}$ and $\widetilde{y}$ denote the respective equivalence classes of x and y, we put $\tilde{\rho}(\widetilde{x}, \widetilde{y}) = \widetilde{\rho}(x, y)$. The metric space $(\widetilde{\mathcal{M}}, \widetilde{\rho})$, together with the quotient map $Q \colon \mathcal{M} \to \widetilde{\mathcal{M}}$ will be called the *metric envelope* of $(\mathcal{M}, \rho)$.

Our discussion yields that the metric envelope of a quasimetric space is characterized by the following universal property.

Theorem 3.3. *Suppose* $(\widetilde{\mathcal{M}}, \widetilde{\rho}, Q)$ *is the metric envelope of a quasimetric space* $(\mathcal{M}, \rho)$*. Then:*

(i) $\operatorname{Lip}(Q) = 1$*, and*

(ii) *whenever* (M, d) *is a metric space and* $f \colon (\mathcal{M}, \rho) \to (M, d)$ *is* C*-Lipschitz, there is a unique map* $\widetilde{f} \colon (\widetilde{\mathcal{M}}, \widetilde{\rho}) \to (M, d)$ *such that* $f = \widetilde{f} \circ Q$ *is* C*-Lipschitz. Pictorially,*

$$\begin{tikzcd}
(\mathcal{M}, \rho) \arrow[rr, "f"] \arrow[dr, "Q"'] & & (M, d) \\
& (\widetilde{\mathcal{M}}, \widetilde{\rho}) \arrow[ur, "\widetilde{f}"'] &
\end{tikzcd}$$

Remark 3.4. Theorem 3.3 can be rephrased as saying that for every metric space (M, d) the mapping $g \mapsto Q \circ g$ defines an isometry from

$\operatorname{Lip}_0(\widetilde{\mathcal{M}}, M)$ onto $\operatorname{Lip}_0(\mathcal{M}, M)$, and so these two spaces can be naturally identified.

Note that, in this language, Example 3.1 yields that for $0 < p < 1$, the metric envelope of $\mathbb{R}$ equipped with the p-metric $\rho(x, y) = |x-y|^{1/p}$ is trivial. On the other hand, we saw in Example 2.1 that every real Lipschitz map on $(\mathbb{R}, \rho)$ that carries 0 to 0 is the zero function. Next we see that this is not a coincidence.

Proposition 3.5. *Given a quasimetric space* $(\mathcal{M}, \rho)$ *the following are equivalent.*

- $(\widetilde{\mathcal{M}}, \tilde{\rho})$ *is trivial.*
- $\operatorname{Lip}_0(\widetilde{\mathcal{M}}, M) = \{0\}$ *for any metric space* (M, d)*.*
- $\operatorname{Lip}_0(\mathcal{M}) = \{0\}$*.*

Proof. If $(\widetilde{\mathcal{M}}, \tilde{\rho})$ is trivial it is clear that $\operatorname{Lip}_0(\widetilde{\mathcal{M}}, M) = \{0\}$ for any metric space M. Using Remark 3.4 we get $\operatorname{Lip}_0(\mathcal{M}, M) = \{0\}$.

If $\operatorname{Lip}_0(\widetilde{\mathcal{M}}, M) = \{0\}$ for any metric space M in particular it holds for $M = \mathbb{R}$, i.e., $\operatorname{Lip}_0(\mathcal{M}) = \{0\}$.

Finally, if $(\widetilde{\mathcal{M}}, \tilde{\rho})$ is non-trivial then clearly $\operatorname{Lip}_0(\widetilde{\mathcal{M}})$ is non-trivial and so by Remark 3.4 we get $\operatorname{Lip}_0(\mathcal{M}) \neq \{0\}$. □

Example 3.6. Let $0 < p < 1$. We know that the p-metric space $L_p[0, 1]$ equipped with the usual p-metric induced by the p-norm, given by

$$\rho(f, g) = \|f - g\|_p^p, \quad f, g \in L_p[0, 1]$$

has $\operatorname{Lip}_0(L_p[0, 1]) = \{0\}$ (see [1, Proposition 2.8]). Then, by Proposition 3.5 we infer that its metric envelope is trivial.

Let us next show that the fact $\widetilde{L_p[0, 1]} = \{0\}$ is related to the well-known property that the Banach envelope of the p-Banach space $L_p[0, 1]$ for $0 < p < 1$ is trivial. In fact, metric and Banach envelopes are related by the following result.

Proposition 3.7. *Let* $(X, \|\cdot\|)$ *be a p-normed space. Consider on* X *the p-metric* ρ *given by* $\rho(x, y) = \|x - y\|$ *and let* 0 *be the distinguished point of* X*. Then* $\widetilde{\rho}(x, y) = \|x - y\|_c$ *for all* $x, y \in X$*.*

Proof. The set of all tuples $(y_j)_{j=0}^n$ with $y_0 = x$ and $y_n = y$ coincides with the set of all tuples of the form $(x + \sum_{k=0}^j x_k)_{j=0}^n$, where $x_0 = 0$ and $\sum_{j=1}^n x_j = y - x$. Hence,

$$\tilde{\rho}(x, y) = \inf \left\{ \sum_{j=1}^n \rho\left(x + \sum_{k=1}^j x_k, x + \sum_{k=1}^{j-1} x_k\right) : \sum_{j=1}^n x_j = y - x \right\}$$

$$\begin{aligned} &= \inf\left\{ \sum_{j=1}^n \|x_j\| \colon \sum_{j=1}^n x_j = y - x \right\} \\ &= \|y-x\|_c. \end{aligned}$$

□

Remark 3.8. Note that it is possible to extend Definition 3.2 and Theorem 3.3 to the case when $0 < q < 1$. Indeed, given a pointed quasimetric space $(\mathcal{M}, \rho)$ we define its *q-metric envelope* $(\widetilde{\mathcal{M}^q}, \widetilde{\rho_q})$ following the same steps as in the construction of its metric envelope. The q-metric $\widetilde{\rho_q}$ is given by

$$\widetilde{\rho_q}(x,y) = \inf \left(\sum_{i=0}^n \rho^q(x_i, x_{i+1}) \right)^{1/q}, \qquad x, y \in \mathcal{M},$$

the infimum being taken over all finite sequences $x = x_0, x_1, \dots, x_{n+1} = y$ of points in $\mathcal{M}$, and $\widetilde{\mathcal{M}^q}$ is the quotient space $\mathcal{M}/\sim_q$. The equivalence relation here is the expected one, i.e.,

$$x \sim_q y \Longleftrightarrow \widetilde{\rho_q}(x,y) = 0.$$

Thus $(\widetilde{\mathcal{M}^q}, \widetilde{\rho_q})$ is the pointed q-metric space (having as a distinguished point the equivalence class of 0) characterized by the following universal property: whenever (M, d) is a q-metric space and $f\colon (\mathcal{M}, \rho) \to (M, d)$ is C-Lipschitz then the map $\widetilde{f}\colon (\widetilde{\mathcal{M}^q}, \widetilde{\rho_q}) \to (M, d)$ such that $f = \widetilde{f} \circ Q$ is C-Lipschitz, where $Q\colon \mathcal{M} \to \widetilde{\mathcal{M}^q}$ denotes the canonical quotient map:

$$\begin{tikzcd} (\mathcal{M}, \rho) \arrow[rr, "f"] \arrow[dr, "Q"'] & & (M, d) \\ & (\widetilde{\mathcal{M}^q}, \widetilde{\rho_q}) \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

Notice also that if we regard a p-Banach space $(X, \|\cdot\|_X)$ as a pointed p-metric space in the obvious way (i.e., by taking 0 as the origin of the vector space X equipped with the p-metric $\rho(x,y) = \|x-y\|_X$), Proposition 3.7 can be generalized as well, and we can come to the conclusion that the q-Banach envelope of X is the completion of its q-metric envelope. We leave out for the reader to check the straightforward details.

4. Lipschitz free p-spaces over quasimetric spaces

Every metric space embeds isometrically into a Banach space. Similarly, the natural environment to isometrically embed quasimetric spaces will be p-Banach spaces. To get started, notice that for $0 < p < 1$, every pointed p-metric space $\mathcal{M}$ embeds isometrically into a "huge" p-Banach space, namely the space $Y = \ell_\infty(\mathcal{M}; L_p(0,\infty))$ of bounded functions from $\mathcal{M}$ into the real space $(L_p(0,\infty), \|\cdot\|_p)$ endowed with the p-norm

$$\|f\|_Y = \sup_{x\in\mathcal{M}} \|f(x)\|_p.$$

Indeed, with the convention that $\chi_{(a,b]} = -\chi_{(b,a]}$ if $a > b$, the map $\Psi\colon \mathcal{M} \to \ell_\infty(\mathcal{M}; L_p(0,\infty))$ given by

$$\Psi(x) = \left(\chi_{(\rho^p(0,y),\rho^p(x,y)]}\right)_{y\in\mathcal{M}}$$

does the job (see [3, Proposition 3.3]). Of course, depending on the p-metric space we can find simpler embeddings. For instance, the mapping

$$\Phi\colon (\mathbb{R}, |\cdot|^{1/p}) \to L_p(\mathbb{R}), \quad \Phi(x) = \chi_{(0,x]} \tag{4.9}$$

is an isometric embedding.

Once we have accomplished the task to embed a p-metric space $\mathcal{M}$ into a p-Banach space, it seems natural to look for an "optimal" way to do it, in the sense that every Lipschitz map from $\mathcal{M}$ into a p-Banach space factors through it. The following construction from [3] attains this goal.

Let $\mathbb{R}_0^{\mathcal{M}}$ be the space of all (not necessarily continuous) maps $f\colon \mathcal{M} \to \mathbb{R}$ so that $f(0) = 0$ and let $\mathcal{P}(\mathcal{M})$ be the linear span in the linear dual $(\mathbb{R}_0^{\mathcal{M}})^\#$ of the evaluations $\delta(x)$, where x runs through $\mathcal{M}$, defined by

$$\langle \delta(x), f\rangle = f(x), \qquad f \in \mathbb{R}^{\mathcal{M}}. \tag{4.10}$$

Note that $\delta(0) = 0$.

If $\mu = \sum_{j=1}^N a_j\delta(x_j) \in \mathcal{P}(\mathcal{M})$, put

$$\|\mu\|_{\mathcal{F}_p(\mathcal{M})} = \sup \left\| \sum_{j=1}^N a_j f(x_j) \right\|_Y, \tag{4.11}$$

the supremum being taken over all p-normed spaces $(Y, \|\cdot\|_Y)$ and all 1-Lipschitz maps $f\colon \mathcal{M} \to Y$ with $f(0) = 0$. It is straightforward to check that formula (4.11) defines a p-seminorm on $\mathcal{P}(\mathcal{M})$. In fact, the following proposition shows that $\|\cdot\|_{\mathcal{F}_p(\mathcal{M})}$ is a p-norm, thus settling a question posed in [3].

Proposition 4.1. *Let $(\mathcal{M},\rho)$ be a pointed p-metric space, $0 < p \le 1$. Then $(\mathcal{P}(\mathcal{M}), \|\cdot\|_{\mathcal{F}_p(\mathcal{M})})$ is a p-normed space.*

Proof. Suppose that $\|\sum_{j=1}^N a_j\delta(x_j)\|_{\mathcal{F}_p(\mathcal{M})} = 0$ for some $(a_j)_{j=1}^N$ scalars and some $(x_j)_{j=1}^N$ in $\mathcal{M}\setminus\{0\}$. Then $\sum_{j=1}^N a_j f(x_j) = 0$ for every p-Banach space X and every Lipschitz map $f\colon \mathcal{M} \to X$ with $f(0) = 0$.

Pick $i \in \{1, \dots, N\}$ and for the sake of convenience denote the distinguished point of $\mathcal{M}$ by x_0. Since the set $\mathcal{N} = \{x_j \colon 0 \le j \le N\}$ is finite, the map from the metric space $(\mathcal{N}, \rho^p)$ into $(\mathbb{R}, |\cdot|)$ given by $x_i \mapsto 1$ and $x_j \mapsto 0$ for $j \neq i$ is Lipschitz. By McShane's theorem, it extends to a Lipschitz map g from $(\mathcal{M}, \rho^p)$ into $(\mathbb{R}, |\cdot|)$. In other words, the map g is Lipschitz from $(\mathcal{M}, \rho)$ into $(\mathbb{R}, |\cdot|^{1/p})$. If Φ is as in (4.9), then $f := \Phi \circ g \colon \mathcal{M} \to L_p(\mathbb{R})$ is Lipschitz as well. Since $f(0) = 0$, we infer that $a_i \chi_{(0,1]} = \sum_{j=1}^N a_j f(x_j) = 0$. Hence, $a_i = 0$. $\square$

Definition 4.2 (cf. [3]). Given a p-metric space $\mathcal{M}$, the *Lipschitz free p-space over $\mathcal{M}$*, denoted by $\mathcal{F}_p(\mathcal{M})$, is the p-Banach space resulting from the completion of the p-normed space $(\mathcal{P}(\mathcal{M}), \|\cdot\|_{\mathcal{F}_p(\mathcal{M})})$. We will refer to the map $\delta_{\mathcal{M}} \colon \mathcal{M} \to \mathcal{F}_p(\mathcal{M})$ given by $\delta_{\mathcal{M}}(x) = \delta(x)$ as the natural embedding of $\mathcal{M}$ into $\mathcal{F}_p(\mathcal{M})$.

Remark 4.3. Note that the choice of a base point in $\mathcal{M}$ is not relevant in the definition of $\mathcal{F}_p(\mathcal{M})$. Indeed, if we change the origin in $\mathcal{M}$ and apply the construction, we have a natural linear isometry between the resulting Lipschitz free p-spaces.

For expositional ease and further reference, let us point out the following easy consequence of Proposition 4.1.

Lemma 4.4. *Let $(\mathcal{M}, \rho)$ be an infinite p-metric space, $0 < p \le 1$. Then $\mathcal{F}_p(\mathcal{M})$ is infinite dimensional.*

Similarly to Lipschitz free Banach spaces over metric spaces, the spaces $\mathcal{F}_p(\mathcal{M})$ for $0 < p < 1$ are uniquely characterised by the universal property included in the following result from [3].

Theorem 4.5. *Let $(\mathcal{M}, \rho)$ be a pointed p-metric space, $0 < p \le 1$. Then:*

(a) *$\delta_{\mathcal{M}}$ is an isometric embedding.*
(b) *The linear span of $\{\delta_{\mathcal{M}}(x) \colon x \in \mathcal{M}\}$ is dense in $\mathcal{F}_p(\mathcal{M})$.*
(c) *$\mathcal{F}_p(\mathcal{M})$ is the unique (up to isometric isomorphism) p-Banach space such that for every p-Banach space X and every Lipschitz map $f \colon \mathcal{M} \to X$ with $f(0) = 0$ there exists a unique linear map $T_f \colon \mathcal{F}_p(\mathcal{M}) \to X$ with $T_f \circ \delta_{\mathcal{M}} = f$. Moreover $\|T_f\| = \operatorname{Lip}(f)$. Pictorially,*

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "f"] \arrow[dr, "\delta_{\mathcal{M}}"'] & & X \\
& \mathcal{F}_p(\mathcal{M}) \arrow[ur, "T_f"'] &
\end{tikzcd}$$

Corollary 4.6. *The space $\mathcal{F}_p(\mathcal{M})$ is separable whenever $\mathcal{M}$ is.*

Proof. Note that the map $\delta\colon \mathcal{M} \to \mathcal{F}_p(\mathcal{M})$ is an isometric embedding and that $\mathcal{F}_p(\mathcal{M})$ is the closed linear span of $\delta(\mathcal{M})$. □

Remark 4.7. If $p = 1$ (so that ρ is a metric) then it follows from the Hahn-Banach theorem that $\mathcal{F}_1(\mathcal{M})$ is the space denoted by $\mathcal{F}(\mathcal{M})$ in [14, 17] and that the norm of $\mu = \sum_{j=1}^N a_j x_j \in \mathcal{P}(\mathcal{M})$ can be computed as

$$\|\mu\|_{\mathcal{F}_1(\mathcal{M})} = \sup \left| \sum_{j=1}^N a_j f(x_j) \right|,$$

the supremum being taken over all 1-Lipschitz maps $f\colon \mathcal{M} \to \mathbb{R}$ with $f(0) = 0$. Moreover, it is known (see, e.g., [23]) that $\mathcal{F}_1(\mathcal{M})^* = \operatorname{Lip}_0(\mathcal{M})$. We advance that the corresponding result also holds for $p < 1$, i.e., $\mathcal{F}_p(\mathcal{M})^* = \operatorname{Lip}_0(\mathcal{M})$. We will prove this in Corollary 4.23.

Lipschitz free p-spaces provide a canonical linearization process of Lipschitz maps between p-metric spaces: if we identify (through the map $\delta_{\mathcal{M}}$) a p-metric space $\mathcal{M}$ with a subset of $\mathcal{F}_p(\mathcal{M})$, then any Lipschitz map from $\mathcal{M}_1$ to a metric space $\mathcal{M}_2$ which maps 0 to 0 extends to a continuous linear map from $\mathcal{F}_p(\mathcal{M}_1)$ to $\mathcal{F}_p(\mathcal{M}_2)$. That is:

Lemma 4.8 (cf. [14, Lemma 2.2]). *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be pointed p-metric spaces $(0 < p \leq 1)$ and suppose $f\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a Lipschitz map such that $f(0) = 0$. Then there exists a unique linear operator $L_f\colon \mathcal{F}_p(\mathcal{M}_1) \to \mathcal{F}_p(\mathcal{M}_2)$ such that $L_f \delta_{\mathcal{M}_1} = \delta_{\mathcal{M}_2} f$, i.e., the following diagram commutes*

$$\begin{tikzcd}
\mathcal{M}_1 \arrow[r, "f"] \arrow[d, "\delta_{\mathcal{M}_1}"'] & \mathcal{M}_2 \arrow[d, "\delta_{\mathcal{M}_2}"] \\
\mathcal{F}_p(\mathcal{M}_1) \arrow[r, "L_f"] & \mathcal{F}_p(\mathcal{M}_2)
\end{tikzcd}$$

and $\|L_f\| = \operatorname{Lip}(f)$. In particular, if f is bi-Lipschitz bijection then L_f is an isomorphism.

Proof. Since δ_{M_2} is an isometric embedding, the map $g := \delta_{M_2} \circ f$ is Lipschitz with $g(0) = 0$ and $\operatorname{Lip}(g) = \operatorname{Lip}(f)$. Now the result follows from Theorem 4.5. □

4.1. **Molecules and atomic decompositions.** Given a set $\mathcal{M}$ and $x \in \mathcal{M}$, let χ_x denote the indicator function of the singleton set $\{x\}$. Now, for x and $y \in \mathcal{M}$ we put

$$m_{x,y} := \chi_x - \chi_y.$$

Let $(\mathcal{M}, \rho)$ be a p-metric space for some $0 < p \le 1$. A *molecule* of $\mathcal{M}$ is a function $m\colon \mathcal{M} \to \mathbb{R}$ that is supported on a finite subset of $\mathcal{M}$ and that satisfies $\sum_{x \in \mathcal{M}} m(x) = 0$. The vector space of all molecules of a metric space $\mathcal{M}$ will be denoted by $\operatorname{Mol}(\mathcal{M})$.

A simple induction argument shows that every molecule has at least one expression as a linear combination of molecules of the form $m_{x,y}$, so that $\operatorname{Mol}(\mathcal{M})$ coincides with the linear span of the family of molecules

$$\mathcal{A}'(\mathcal{M}) = \left\{ \frac{m_{x,y}}{\rho(x,y)} \colon x, y \in \mathcal{M}, x \neq y \right\} \subseteq \mathbb{R}^{\mathcal{M}}.$$

Definition 4.9. We define the *Arens-Eells p-space over $\mathcal{M}$*, denoted $\text{Æ}_p(\mathcal{M})$, as the p-Banach space constructed from the set $\mathcal{A}'(\mathcal{M})$ using the p-convexification method (see Definition 2.2).

This way, if we give $\operatorname{Mol}(\mathcal{M})$ the p-seminorm

$$\|m\|_{\text{Æ}_p} = \inf \left\{ \left(\sum_{i=1}^N |a_i|^p \right)^{1/p} \colon m = \sum_{i=1}^N a_i \frac{m_{x_i,y_i}}{\rho(x_i,y_i)},\ N \in \mathbb{N} \right\}, \quad (4.12)$$

we have that $\text{Æ}_p(\mathcal{M})$ is the completion of $\operatorname{Mol}(\mathcal{M})$ (a priori, modulo the set of molecules with zero p-seminorm) with respect to $\|\cdot\|_{\text{Æ}_p}$. However, as we will see below, formula (4.12) defines in fact a p-norm on $\operatorname{Mol}(\mathcal{M})$.

The following result establishes that the Arens-Eells p-space over $\mathcal{M}$ can be identified with the Lipschitz free p-space over $\mathcal{M}$.

Theorem 4.10. *Let $0 < p \le 1$ and $(\mathcal{M}, \rho)$ be a pointed p-metric space. Then $\mathcal{F}_p(\mathcal{M})$ and $\text{Æ}_p(\mathcal{M})$ are isometrically isomorphic. In fact, there is a linear onto isometry $T\colon \mathcal{F}_p(\mathcal{M}) \to \text{Æ}_p(\mathcal{M})$ such that $T(\delta(x)) = \chi_x - \chi_0 = m_{x,0}$ for all $x \in \mathcal{M}$.*

Proof. Consider the map $f\colon \mathcal{M} \to \text{Æ}_p(\mathcal{M})$ given by $f(x) = m_{x,0}$ for $x \in \mathcal{M}$. Clearly, $f(0) = 0$ and $f(x) - f(y) = m_{x,y}$ for all x, $y \in \mathcal{M}$. Since f is 1-Lipschitz, Theorem 4.5 yields a norm-one linear map $T_f\colon \mathcal{F}_p(\mathcal{M}) \to \text{Æ}_p(\mathcal{M})$ such that $T_f(\delta(x)) = m_{x,0}$.

Since $(\chi_x)_{x \in \mathcal{M}}$ is a linearly independent family in $\mathbb{R}^{\mathcal{M}}$, there is a linear map from $\operatorname{span}\{\chi_x \colon x \in \mathcal{M}\}$ into $\mathcal{P}(\mathcal{M})$ that takes χ_x to $\delta(x)$ for every $x \in \mathcal{M}$. Let S_1 be its restriction to $\operatorname{Mol}(\mathcal{M})$. For $x, y \in \mathcal{M}$ with $x \neq y$ we have

$$S_1\left(\frac{m_{x,y}}{\rho(x,y)} \right) = \frac{\delta(x) - \delta(y)}{\rho(x,y)},$$

and

$$\left\| \frac{\delta(x)-\delta(y)}{\rho(x,y)} \right\|_{\mathcal{F}_p(\mathcal{M})} \le 1,$$

so by density S_1 extends to a norm-one operator S from $\text{Æ}_p(\mathcal{M})$ into $\mathcal{F}_p(\mathcal{M})$. Since $T(S(m)) = m$ for every molecule m, and $S(T(\mu)) = \mu$ for every $\mu \in \mathcal{P}(\mathcal{M})$, by continuity and density it follows that $T \circ S = \mathrm{Id}_{\text{Æ}_p(\mathcal{M})}$ and $S \circ T = \mathrm{Id}_{\mathcal{F}_p(\mathcal{M})}$. □

The following two results are re-formulations of Theorem 4.10. While the expression of the norm in (4.11) relies on extraneous ingredients, Corollary 4.12 provides an intrinsic formula for the p-norm on $\mathcal{F}_p(\mathcal{M})$, i.e., an expression that relies only on the quasimetric on the space $\mathcal{M}$.

Corollary 4.11. *Let $(\mathcal{M}, \rho)$ be a pointed p-metric space, $0 < p \le 1$. The subset of $\mathcal{P}(\mathcal{M})$ given by*

$$\mathcal{A}(\mathcal{M}) = \left\{ \frac{\delta(y)-\delta(x)}{\rho(x,y)} : x, y \in \mathcal{M},\, x \neq y \right\}$$

is isometrically p-norming for $\mathcal{F}_p(\mathcal{M})$.

Corollary 4.12. *Let $(\mathcal{M}, \rho)$ be a pointed p-metric space, $0 < p \le 1$. For $\mu \in \mathcal{F}_p(\mathcal{M})$ we have*

$$\|\mu\|_{\mathcal{F}_p(\mathcal{M})} = \inf \left\{ \left(\sum_{k=1}^{\infty} |a_k|^p \right)^{1/p} : \mu = \sum_{k=1}^{\infty} a_k \frac{\delta(x_k)-\delta(y_k)}{\rho(x_k, y_k)} \right\}.$$

4.2. **Applications: early examples and results.** Next we use Corollary 4.11 to identify the first examples of Lipschitz-free p-spaces over quasimetric spaces for $0 < p < 1$.

Theorem 4.13. *Let $0 < p \le 1$. Let I be an interval of the real line equipped with the p-metric $\rho(x,y) = |x-y|^{1/p}$ for $x, y \in I$. Then*

$$\mathcal{F}_p(I) \approx L_p(I)$$

isometrically. To be precise, if a is the base point of I, the mapping

$$\mathcal{F}_p(I) \to L_p(I), \quad \delta_I(x) \mapsto \chi_{(a,x]}$$

extends to a linear isometry.

Proof. Choose an arbitrary $a \in I$ as the base point of $(I, |\cdot|^{1/p})$. Let

$$\mathcal{A}_{p,I} = \left\{ \frac{\chi_{(x,y]}}{|y-x|^{1/p}} : x, y \in I,\, x < y \right\},$$

and let $T \colon \mathcal{P}(I) \to \mathbb{R}^I$ be the linear map determined by

$$\delta(x) \mapsto \chi_{(a,x]}, \quad x \in I \setminus \{a\}.$$

Using the notation of Corollary 4.11, we put

$$\mathcal{A}(I)=\left\{\frac{\delta(x)-\delta(y)}{|x-y|^{1/p}}: x,y\in I, x\neq y\right\}.$$

Since $T(\mathcal{A}(I))=\{\pm f\colon f\in\mathcal{A}_{p,I}\}$, taking into account Corollary 4.11 and Lemma 2.7, it suffices to prove that $\mathcal{A}_{p,I}$ is an isometric p-norming set for $L_p(I)$. To that end we need to verify that $f\in\overline{\operatorname{co}_p}(\mathcal{A}_{p,I})$ for every $f\in L_p(I)$ with $\|f\|_p\le 1$. By density it is sufficient to prove it for step functions. Let $f\colon I\to\mathbb{R}$ be a step function, i.e.,

$$f=\sum_{j=1}^{N} a_j\chi_{(x_{j-1},x_j]},$$

for some $x_0<x_1<\cdots<x_{j-1}<x_j<\cdots<x_N$ in I and some scalars $(a_j)_{j=1}^N$. Then, if $b_j=(x_j-x_{j-1})^{1/p}a_j$, we have $f=\sum_{j=1}^N b_j\, f_j$ with $f_j\in\mathcal{A}_{p,I}$ and

$$\sum_{n=1}^{N}|b_j|^p=\sum_{j=1}^{n}|a_j|^p(x_j-x_{j-1})=\|f\|_p^p. \qquad \square$$

Recall that a quasimetric space $\mathcal{M}$ is *uniformly separated* if

$$\inf\{\rho(x,y)\colon x,y\in\mathcal{M}, x\neq y\}>0.$$

Let us note that each bounded and uniformly separated quasimetric space is Lipschitz isomorphic to the $\{0,1\}$-*metric space*, i.e., the metric space whose distance attains only the values 0 and 1.

Theorem 4.14. *Let $\mathcal{M}$ be a bounded and uniformly separated quasimetric space. For $0<p\le 1$ we have*

$$\mathcal{F}_p(\mathcal{M})\approx\ell_p(\mathcal{M}\setminus\{0\}).$$

To be precise, the map

$$\mathcal{F}_p(\mathcal{M})\to\ell_p(\mathcal{M}\setminus\{0\}),\quad \delta_{\mathcal{M}}(x)\mapsto\mathbf{e}_x$$

extends to a linear isomorphism.

Proof. Without loss of generality we can assume that $(\mathcal{M},\rho)$ is the $\{0,1\}$-metric space. If x and y are two different points in $\mathcal{M}\setminus\{0\}$ we can write

$$\frac{\delta(y)-\delta(x)}{\rho(x,y)}=a_{xy}\frac{\delta(x)}{\rho(0,x)}+b_{x,y}\frac{\delta(y)}{\rho(0,y)},$$

where $a_{x,y}=-1$ and $b_{x,y}=1$. Since $|a_{x,y}|^p+|b_{x,y}|^p=2$, by Corollary 4.11 and Lemma 2.8, the set

$$\mathcal{A}=\left\{\frac{\delta(x)}{\rho(0,x)}\colon x\in\mathcal{M}\setminus\{0\}\right\}=\{\delta(x)\colon x\in\mathcal{M}\setminus\{0\}\},$$

is p-norming for $\mathcal{F}_p(\mathcal{M})$ with constants 1 and $2^{1/p}$. Consider the linear map $T\colon \mathcal{P}(\mathcal{M}) \to \mathbb{R}^{\mathcal{M}\setminus\{0\}}$ given by

$$\delta(x) \mapsto \mathbf{e}_x, \quad x \in \mathcal{M} \setminus \{0\},$$

where $\mathbf{e}_x$ denotes the indicator function of the singleton $\{x\}$. We have that $T(\mathcal{A}) = \mathcal{A}(\mathcal{M} \setminus \{0\}) := \{\mathbf{e}_x \colon x \in \mathcal{M} \setminus \{0\}\}$. Since $\mathcal{A}(\mathcal{M} \setminus \{0\})$ is an isometrically p-norming set for $\ell_p(\mathcal{M}\setminus\{0\})$, an appeal to Lemma 2.7 finishes the proof. □

Notice that, quantitatively, the proof of Theorem 4.14 gives that if $\mathcal{M}$ is equipped with the $\{0,1\}$-metric, then

$$2^{-1/p}\left(\sum_{x\in\mathcal{M}\setminus\{0\}} |a_x|^p\right)^{1/p} \le \left\| \sum_{x\in\mathcal{M}\setminus\{0\}} a_x\,\delta(x)\right\|_{\mathcal{F}_p(\mathcal{M})} \le \left(\sum_{x\in\mathcal{M}\setminus\{0\}} |a_x|^p\right)^{1/p}$$

for all scalars $(a_x)_{x\in\mathcal{M}\setminus\{0\}}$ eventually null. Going further we are going to be able to compute the quasi-norms $\left\|\sum_{x\in\mathcal{M}\setminus\{0\}} a_x\,\delta(x)\right\|_{\mathcal{F}_p(\mathcal{M})}$ in the case when $a_x \ge 0$. Our argument relies on the construction of a suitable d-dimensional absolutely p-convex body for every $d \in \mathbb{N}$.

Proposition 4.15. *For every $d \in \mathbb{N}$ and every $0 < p \le 1$, there is a p-norm $\|\cdot\|_{(p)}$ on $\mathbb{R}^d$ such that:*

- (a) $\|(x_j)_{j=1}^n\|_{(p)} = (\sum_{j=1}^d x_j^p)^{1/p}$ *if $x_j \ge 0$ for $j \in \{1,\dots,d\}$, and*
- (b) $\|\mathbf{e}_i - \mathbf{e}_j\|_{(p)} \le 1$ *for all i, $j \in \{1,\dots,d\}$.*

Proof. Given a vector space V and $Z \subseteq V$, set

$$\operatorname{co}_p^+(Z) = \left\{ \sum_{j=1}^k \lambda_j\, v_j \colon k \in \mathbb{N},\, \lambda_j \ge 0,\, \sum_{j=1}^k \lambda_j^p \le 1, v_j \in Z \right\}.$$

For $d \in \mathbb{N}$, put $\mathbb{N}[d] = \{1,\dots,d\}$. Given $\mathbf{s} = (s_j)_{j=1}^d \in \mathbb{R}^d$, we let $M_{\mathbf{s}}$ be the endomorphism of $\mathbb{R}^d$ given by

$$M_{\mathbf{s}}((x_j)_{j=1}^d) = (s_j\, x_j)_{j=1}^d.$$

Given $A \subseteq \mathbb{N}[d]$, we put $M_A = M_{\mathbf{s}}$ where $\mathbf{s} = (s_j)_{j=1}^n$ is defined by $s_j = 1$ for $j \in \mathbb{N}[d] \setminus A$, and $s_j = -1$ for $j \in A$; that is, M_A is the symmetry with respect to the subspace $\{(x_j)_{j=1}^d \in \mathbb{R}^d \colon x_j = 0 \text{ for all } j \in A\}$. Denote

$$\mathbb{R}^d_+ = \left\{(x_j)_{j=1}^d \in \mathbb{R}^d \colon x_j \ge 0 \text{ for all } j \in \mathbb{N}[d]\right\},$$

$$B_p = \left\{ (x_j)_{j=1}^d \in \mathbb{R}^d \colon \sum_{j=1}^d |x_j|^p \le 1 \right\},$$

$$B_\infty = \left\{ (x_j)_{j=1}^d \in \mathbb{R}^d \colon |x_j| \le 1 \text{ for all } j \in \mathbb{N}[d] \right\}.$$

Given $i, j \in \mathbb{N}[d]$ with $i \neq j$, we define $Z_{i,j} \subseteq \mathbb{R}^d_+$ by

$$Z_{i,j} = \{a\,\mathbf{e}_i + b\,\mathbf{e}_j \colon 0 \le a, b \le 1\}.$$

Given disjoint sets A, $B \subseteq \mathbb{N}[d]$ we define $\mathcal{T}_{A,B} \subseteq \mathbb{R}^d_+$ by

$$\mathcal{T}_{A,B} = \begin{cases} \{0\} & \text{if } A = B = \emptyset \\ \operatorname{co}_p^+(\{\mathbf{e}_i \colon i \in A\})) & \text{if } A \neq \emptyset \text{ and } B = \emptyset, \\ \operatorname{co}_p^+(\{\mathbf{e}_j \colon j \in B\}) & \text{if } A = \emptyset \text{ and } B \neq \emptyset, \\ \operatorname{co}_p^+(\cup_{(i,j)\in A\times B} Z_{i,j}) & \text{otherwise.} \end{cases}$$

It is routine to prove that the the family of bodies $(\mathcal{T}_{A,B})$ enjoys the following properties.

(a1) If λ, $\mu \ge 0$ are such that $\lambda^p + \mu^p \le 1$, then $\lambda\,\mathcal{T}_{A,B} + \mu\,\mathcal{T}_{A,B} \subseteq \mathcal{T}_{A,B}$ for all disjoint $A, B \subseteq \mathbb{N}[d]$.
(a2) $B_p \cap \mathbb{R}^d_+ \subseteq \mathcal{T}_{\mathbb{N}[d]\setminus A, A} \subseteq B_\infty$, for all $A \subseteq \mathbb{N}[d]$.
(a3) $B_p \cap \mathbb{R}^d_+ = \mathcal{T}_{\mathbb{N}[d],\emptyset}$.
(a4) If $A \subseteq A_1$ and $B \subseteq B_1$, then $\mathcal{T}_{A,B} \subseteq \mathcal{T}_{A_1,B_1}$.
(a5) If $\mathbf{x} = (x_j)_{j=1}^d \in \mathcal{T}_{A,B}$ and $x_j = 0$ for every $j \notin D$, where $D \subseteq \mathbb{N}[d]$, then $\mathbf{x} \in \mathcal{T}_{A\cap D, B\cap D}$.
(a6) If $0 \le x_j \le y_j$ for every $j \in \mathbb{N}[d]$ and $(y_j)_{j=1}^d \in \mathcal{T}_{A,B}$, for $A, B \subseteq \mathbb{N}[d]$ disjoint, then $(x_j)_{j=1}^d \in \mathcal{T}_{A,B}$.

Given $A \subseteq \mathbb{N}[d]$, put $\mathcal{C}_A = M_A(\mathcal{T}_{\mathbb{N}[d]\setminus A, A})$. By definition,

(b1) $-\mathcal{C}_A = \mathcal{C}_{\mathbb{N}[d]\setminus A}$.

We infer from (a1), (a2), (a3) and (a6), respectively, that

(b2) if λ, $\mu \ge 0$ are such that $\lambda^p + \mu^p \le 1$, then $\lambda\,\mathcal{C}_A + \mu\,\mathcal{C}_A \subseteq \mathcal{C}_A$,
(b3) $\{(x_j)_{j=1}^d \in B_p \colon \{j \colon x_j < 0\} = A\} \subseteq \mathcal{C}_A \subseteq B_\infty$,
(b4) $\mathcal{C}_\emptyset = B_p \cap \mathbb{R}^d_+$, and
(b5) if $\mathbf{s} \in [0,1]^d$ and $\mathbf{x} \in \mathcal{C}_A$ then $M_{\mathbf{s}}(\mathbf{x}) \in \mathcal{C}_A$.

Consider the d-dimensional body $\mathcal{C}_{(p)} = \cup_{A\subseteq\mathbb{N}[d]} \mathcal{C}_A$. Properties (b2), (b3) and (b5) give, respectively,

(c1) $-\mathcal{C}_{(p)} = \mathcal{C}_{(p)}$,
(c2) $B_p \subseteq \mathcal{C}_{(p)} \subseteq B_\infty$, and that
(c3) if $\mathbf{x} \in \mathcal{C}_{(p)}$ and $\mathbf{s} \in [0,1]^d$, then $M_{\mathbf{s}}(\mathbf{x}) \in \mathcal{C}_{(p)}$.

We infer from (a4) and (a5) that

(c4) if $\mathbf{x} = (x_j)_{j=1}^d \in \mathcal{C}_{(p)}$ and
$$\{j \in \mathbb{N}[d] \colon x_j < 0\} \subseteq B \subseteq \{j \in \mathbb{N}[d] \colon x_j \le 0\},$$
then $\mathbf{x} \in \mathcal{C}_B$.

Combining (c4) with (b4) we obtain

(c5) $\mathcal{C}_{(p)} \cap \mathbb{R}^d_+ = B_p \cap \mathbb{R}^d_+$.

Let us prove that $\mathcal{C}_{(p)}$ is absolutely p-convex. Let $\mathbf{x} = (x_j)_{j=1}^d$, $\mathbf{y} = (y_j)_{j=1}^d \in \mathcal{C}_{(p)}$ and $\lambda, \mu \in \mathbb{R}$ with $|\lambda|^p + |\mu|^p \le 1$. By (c1) we can assume that $\lambda, \mu \ge 0$. Let

$$\begin{aligned} A &= \{j \in \mathbb{N}[d] \colon \operatorname{sgn}(x_j)\operatorname{sgn}(y_j) \neq -1\}, \\ D &= \{j \in \mathbb{N}[d] \setminus A \colon \operatorname{sgn}(\lambda x_j + \mu y_j) = 0\}, \\ E &= \{j \in \mathbb{N}[d] \setminus A \colon \operatorname{sgn}(\lambda x_j + \mu y_j) = \operatorname{sgn}(x_j)\}, \\ F &= \{j \in \mathbb{N}[d] \setminus A \colon \operatorname{sgn}(\lambda x_j + \mu y_j) = \operatorname{sgn}(y_j)\}. \end{aligned}$$

By construction (A, D, E, F) is a partition of $\mathbb{N}[d]$. Note that $\lambda > 0$ if $E \neq \emptyset$ and $\mu > 0$ if $F \neq \emptyset$. We define $\tilde{\mathbf{x}} = (x_j)_{j=1}^d$, $\tilde{\mathbf{y}} = (y_j)_{j=1}^d$ and $\mathbf{s} = (s_j)_{j=1}^d$ by

$$(\tilde{x}_j, \tilde{y}_j, s_j) = \begin{cases} (x_j, y_j, 1) & \text{if } j \in A, \\ (0,0,0) & \text{if } j \in D, \\ (x_j, 0, (\lambda x_j)^{-1}(\lambda x_j + \mu y_j)) & \text{if } j \in E, \\ (0, y_j, (\mu y_j)^{-1}(\lambda x_j + \mu y_j)) & \text{if } j \in F. \end{cases}$$

By construction, $\mathbf{s} \in [0,1]^d$ and $\lambda \mathbf{x} + \mu \mathbf{y} = M_{\mathbf{s}}(\lambda \tilde{\mathbf{x}} + \mu \tilde{\mathbf{y}})$. Hence, taking into account (c3), it suffices to prove that $\lambda \tilde{\mathbf{x}} + \mu \tilde{\mathbf{y}} \in \mathcal{C}_{(p)}$. Note that, by construction, $\operatorname{sgn}(\tilde{x}_j)\operatorname{sgn}(\tilde{y}_j) \neq -1$ for every $j \in \mathbb{N}[d]$. Therefore, the set $\{j \in \mathbb{N}[d] \colon \tilde{x}_j < 0\} \cup \{j \in \mathbb{N}[d] \colon \tilde{y}_j < 0\}$ is contained in

$$B := \{j \in \mathbb{N}[d] \colon \tilde{x}_j \le 0\} \cap \{j \in \mathbb{N}[d] \colon \tilde{y}_j \le 0\}.$$

Since, by (c3), $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}} \in \mathcal{C}_{(p)}$, we infer from (c4) that $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}} \in \mathcal{C}_B$. Then, by (b2), $\lambda \tilde{\mathbf{x}} + \mu \tilde{\mathbf{y}} \in \mathcal{C}_B \subseteq \mathcal{C}_{(p)}$.

Let $\|\cdot\|_{(p)}$ be the Minkowski functional associated to $\mathcal{C}_{(p)}$. Taking into account (c2) we infer that $\|\cdot\|_{(p)}$ is a p-norm on $\mathbb{R}^d$. By (c5), $\|\mathbf{x}\|_{(p)} = \|\mathbf{x}\|_p$ for every $\mathbf{x} \in \mathbb{R}^d_+$. $\square$

Proposition 4.16. *Let $\mathcal{M}$ be the $\{0,1\}$-metric space and $(a_x)_{x \in \mathcal{M} \setminus \{0\}}$ be an eventually null family of scalars. Then*

$$\left\| \sum_{x \in \mathcal{M} \setminus \{0\}} a_x \, \delta(x) \right\|_{\mathcal{F}_p(\mathcal{M})} \ge \left(\sum_{\substack{x \in \mathcal{M} \setminus \{0\} \\ a_x \ge 0}} a_x^p \right)^{1/p}.$$

In particular, if $a_x \ge 0$ for all $x \in \mathcal{M} \setminus \{0\}$,

$$\left\| \sum_{x \in \mathcal{M} \setminus \{0\}} a_x \, \delta(x) \right\|_{\mathcal{F}_p(\mathcal{M})} = \left(\sum_{x \in \mathcal{M} \setminus \{0\}} a_x^p \right)^{1/p}.$$

Proof. Let $(a_x)_{x\in\mathcal{M}\setminus\{0\}} \in [0,\infty)^{\mathcal{M}\setminus\{0\}}$ be eventually null. Pick $d\in\mathbb{N}$ and a one-to-one map $\phi\colon \mathbb{N}[d]\to\mathcal{M}\setminus\{0\}$ such that

$$\{x\in\mathcal{M}\setminus\{0\}\colon a_x>0\}\subseteq\phi(\mathbb{N}[d])\subseteq\{x\in\mathcal{M}\setminus\{0\}\colon a_x\ge 0\}.$$

Let $\|\cdot\|_{(p)}$ be as in Proposition 4.15 and consider the mapping $f\colon\mathcal{M}\to(\mathbb{R}^d,\|\cdot\|_{(p)})$ given by $\phi(k)\mapsto\mathbf{e}_k$ for all $k\in\mathbb{N}[d]$ and $x\mapsto 0$ if $x\notin\phi(\mathbb{N}[d])$. Since $\|\mathbf{e}_i\|_{(p)}$, $\|\mathbf{e}_i-\mathbf{e}_j\|_{(p)}\le 1$ for every $i,j\in\mathbb{N}[d]$, f is 1-Lipschitz. Therefore

$$\begin{aligned}\left\|\sum_{x\in\mathcal{M}\setminus\{0\}} a_x\,\delta(x)\right\|_{\mathcal{F}_p(\mathcal{M})} &\ge \left\|\sum_{x\in\mathcal{M}\setminus\{0\}} a_x\,f(x)\right\|_{(p)}\\ &=\left\|\sum_{k=1}^d a_{\phi(k)}\mathbf{e}_k\right\|_{(p)}\\ &=\left(\sum_{k=1}^d a_{\phi(k)}^p\right)^{1/p}\\ &=\left(\sum_{\substack{x\in\mathcal{M}\setminus\{0\}\\ a_x\ge 0}} a_x^p\right)^{1/p}.\end{aligned}$$

From this inequality, the later identity is clear. □

On occasion it will be convenient to know that the Lipschitz free p-space over a quasimetric space and the Lipschitz free p-space over its completion are the same. Let us state this basic fact for reference and provide a proof using the tools that we introduced before.

Proposition 4.17. *Let $\mathcal{M}$ be a p-metric space for some $0<p\le 1$ and let $\mathcal{N}$ be a dense subset of $\mathcal{M}$ equipped with the same quasimetric. Then $\mathcal{F}_p(\mathcal{N})\approx\mathcal{F}_p(\mathcal{M})$ isometrically. In fact, the canonical linear map is an isometry.*

Proof. The canonical linear map $L_\jmath\colon\mathcal{P}(\mathcal{N})\to\mathcal{P}(\mathcal{M})$ induced by the inclusion $\jmath$ from $\mathcal{N}$ into $\mathcal{M}$ is one-to-one on $\mathcal{P}(\mathcal{N})$. By density, the set of molecules of $\mathcal{M}$ of the form

$$L_\jmath(\mathcal{A}(\mathcal{N}))=\left\{\frac{\delta_{\mathcal{M}}(y)-\delta_{\mathcal{M}}(x)}{\rho(x,y)}\colon x,y\in\mathcal{N},\,x\ne y\right\}$$

is an isometrically p-norming set in $\mathcal{F}_p(\mathcal{M})$. Now Corollary 4.11 and Lemma 2.7 yield that $L_\jmath$ extends to a linear isometry from $\mathcal{F}_p(\mathcal{N})$ onto $\mathcal{F}_p(\mathcal{M})$. □

We will study in detail some properties of the canonical map $L_\jmath$ in Section 6. For the time being, to finish this section we provide a sufficient condition for $L_\jmath$ to be an isomorphic embedding.

Definition 4.18. Let $\mathcal{M}$ be a p-metric space, $0 < p \le 1$, and let $\mathcal{N}$ be a subset of $\mathcal{M}$. A Lipschitz map $r\colon \mathcal{M} \to \mathcal{N}$ is called a *Lipschitz retraction* if it is the identity on $\mathcal{N}$. When such a Lipschitz retraction exists we say that $\mathcal{N}$ is a *Lipschitz retract* of $\mathcal{M}$.

Lemma 4.19 (cf. [14, Lemma 2.2])**.** *Let $\mathcal{M}$ be a pointed p-metric space $(0 < p \le 1)$ and $\mathcal{N}$ be a Lipschitz retract of $\mathcal{M}$. Then the inclusion map $\jmath\colon \mathcal{N} \to \mathcal{M}$ induces an isomorphic embedding $L_\jmath\colon \mathcal{F}_p(\mathcal{N}) \to \mathcal{F}_p(\mathcal{M})$ onto a complemented subspace.*

Proof. Without loss of generality we may and do assume that $0 \in \mathcal{N}$. Let $\jmath\colon \mathcal{N} \to \mathcal{M}$ be the inclusion map and let $r\colon \mathcal{M} \to \mathcal{N}$ be a Lipschitz retraction. Lemma 4.8 yields $L_r \circ L_\jmath = \mathrm{Id}_{\mathcal{F}_p(\mathcal{N})}$, i.e., $L_\jmath \circ L_r$ is a linear projection from $\mathcal{F}_p(\mathcal{M})$ onto the linear subspace $L_\jmath(\mathcal{F}_p(\mathcal{N}))$ of $\mathcal{F}_p(\mathcal{M})$ and $L_\jmath$ is an isomorphism. $\square$

4.3. **Lipschitz free p-spaces, q-metric envelopes, and duality.**

Proposition 4.20. *Let $0 < p < q \le 1$. Suppose $\mathcal{M}$ is a pointed p-metric space with q-metric envelope $\widetilde{\mathcal{M}^q}$. Then:*

(a) *$\mathcal{F}_q(\widetilde{\mathcal{M}^q})$ is the q-Banach envelope of the p-Banach space $\mathcal{F}_p(\mathcal{M})$.*
(b) *In the particular case that $\mathcal{M}$ is a p-Banach space X with q-Banach envelope $\widehat{X^q}$ then the q-Banach envelope of the p-Banach space $\mathcal{F}_p(X)$ is the q-Banach space $\mathcal{F}_q(\widehat{X^q})$.*

Proof. The universal properties of q-metric envelopes and q-Banach envelopes yield the commutative diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "Q"] \arrow[d, "\delta"'] & \widetilde{\mathcal{M}^q} \arrow[d, "\delta"] \\
\mathcal{F}_p(\mathcal{M}) \arrow[r, "L_Q"] \arrow[d, "i"'] & \mathcal{F}_q(\widetilde{\mathcal{M}^q}) \\
\widehat{\mathcal{F}_p(\mathcal{M})^q} \arrow[ur, "\widetilde{L_Q}"'] &
\end{tikzcd}$$

Since $\mathrm{co}_p(\mathcal{A}(\mathcal{M}))$ is dense in the unit ball B of $\mathcal{F}_p(\mathcal{M})$ and $i(\mathrm{co}_q(B))$ is dense in the unit ball of $X := \widehat{\mathcal{F}_p(\mathcal{M})^q}$, we infer that $\mathrm{co}_q(i(\mathcal{A}(\mathcal{M})))$ is dense in the unit ball of X. Therefore, by Lemma 2.6, $A := i(\mathcal{A}(\mathcal{M}))$ is an isometrically q-norming set for X. Moreover, $\widetilde{L_Q}$ is a bijection from

$i(\mathcal{P}(\mathcal{M}))$ onto $\mathcal{P}(\widetilde{\mathcal{M}^q}) = \mathcal{P}(Q(\mathcal{M}))$ and $\widetilde{L_Q}(A) = \mathcal{A}(\widetilde{\mathcal{M}^q}) = \mathcal{A}(Q(\mathcal{M}))$. We deduce from Lemma 2.7 that $\widetilde{L_Q}$ is an isometric isomorphism. □

Remark 4.21. The previous proposition implies, for example, that $\mathcal{F}_p(\mathbb{R})$ is not isomorphic to L_p for $p < 1$. Indeed, its Banach envelope is L_1, while the Banach envelope of L_p is trivial, and if two p-spaces are isomorphic, their envelopes are also isomorphic.

Remark 4.22. Roughly speaking, it could be argued that given a r-metric space $\mathcal{M}$, $0 < r \le 1$, the family $(\mathcal{F}_p(\widetilde{\mathcal{M}^p}))_{0<p\le 1}$, where $\mathcal{F}_p(\widetilde{\mathcal{M}^p}) = \mathcal{F}_p(\mathcal{M})$ if $p \le r$, forms a scale of quasi-Banach spaces in the same way as the family $(\ell_p)_{0<p\le 1}$ does. Indeed, given $p \le q < 1$, Proposition 4.20 provides a canonical range-dense linear map $L_{q,p}\colon \mathcal{F}_p(\widetilde{\mathcal{M}^p}) \to \mathcal{F}_q(\widetilde{\mathcal{M}^q})$ with $\|L_{q,p}\| \le 1$ and, if $q < s \le 1$, we have $L_{s,p} = L_{s,q} \circ L_{q,p}$.

Let us restrict our attention to the case when $p < r$. Then

$$L_{r,p}\colon \mathcal{F}_p(\mathcal{M}) \to \mathcal{F}_r(\mathcal{M})$$

is the identity map on $\mathcal{P}(\mathcal{M})$ and, hence, it is one-to-one on a dense subspace. However we do not know if this map is always injective. In the case when $r = 1$ we would like to point out that the map $L_{1,p}\colon \mathcal{F}_p(\mathcal{M}) \to \mathcal{F}(\mathcal{M})$ is one-to-one if and only if $\mathcal{F}_p(\mathcal{M})^*$ separates the points of $\mathcal{F}_p(\mathcal{M})$.

Corollary 4.23. *Let $\mathcal{M}$ be a pointed p-metric space, $0 < p \le 1$. Then $\mathcal{F}_p(\mathcal{M})^* = \operatorname{Lip}_0(\mathcal{M})$, in the sense that given $\phi \in \mathcal{F}_p(\mathcal{M})^*$ there exists a unique $f \in \operatorname{Lip}_0(\mathcal{M})$ such that $\phi\big(\sum a_i\delta(x_i)\big) = \sum a_i f(x_i)$ for every $\sum a_i\delta(x_i) \in \mathcal{F}_p(\mathcal{M})$, and the map $\phi \mapsto f$ is a linear isometry of $\mathcal{F}_p(\mathcal{M})^*$ onto $\operatorname{Lip}_0(\mathcal{M})$. In particular, $\mathcal{F}_p(\mathcal{M})^* = \{0\}$ if $\operatorname{Lip}_0(\mathcal{M}) = \{0\}$.*

Proof. By identifying $\mathcal{M}$ with $\delta(\mathcal{M}) \subseteq \mathcal{F}_p(\mathcal{M})$ we get that the restriction of any $\phi \in \mathcal{F}_p(\mathcal{M})^*$ to $\mathcal{M}$ belongs to $\operatorname{Lip}_0(\mathcal{M})$. And conversely, any $f \in \operatorname{Lip}_0(\mathcal{M})$ uniquely extends by the universal property to an element of $\mathcal{F}_p(\mathcal{M})^*$. This correspondence is a linear isometry. □

Corollary 4.24. *Let $\mathcal{M}$ and $\mathcal{N}$ be pointed metric spaces and suppose $0 < p < 1$. If $\mathcal{F}_p(\mathcal{M}) \approx \mathcal{F}_p(\mathcal{N})$ then $\mathcal{F}(\mathcal{M}) \approx \mathcal{F}(\mathcal{N})$.*

The last theorem of this section extends to the case when $0 < p < 1$ a result of Naor and Schechtman [20].

Theorem 4.25. *For any $0 < p \le 1$, the p-Banach spaces $\mathcal{F}_p(\mathbb{R})$ and $\mathcal{F}_p(\mathbb{R}^2)$ are not isomorphic.*

Proof. The case $p = 1$ was proved in [20]. The case $0 < p < 1$ is taken care of in Corollary 4.24. □

5. Lipschitz free p-spaces over ultrametric spaces

The spaces $\mathcal{F}_p(\mathcal{M})$ over quasimetric (or even metric) spaces provide a new class of quasi-Banach spaces that in general are difficult to identify. The point of this section is to see that by imposing a stronger condition on $\mathcal{M}$, namely being ultrametric, we can recognize the Lipschitz free p-space over $\mathcal{M}$.

Recall that a distance d on a set $\mathcal{M}$ is called an *ultrametric* provided that in place of the triangle inequality, d satisfies the stronger condition

$$d(x,z) \le \max\{d(x,y), d(y,z)\}, \quad x,y,z \in \mathcal{M}.$$

Note that ultrametrics can be characterized as metrics d such that d^p is a metric for every $p > 0$. Indeed, if $(\mathcal{M}, d^p)$ is a metric space for $p \in A$, and the set $A \subseteq \mathbb{R}$ is unbounded, then

$$d(x,z) \le \left(d^p(x,y) + d^p(y,z)\right)^{1/p}, \quad x,y,z \in \mathcal{M},\, p \in A.$$

Letting p tend to infinity we get $d(x,z) \le \max\{d(x,y), d(y,z)\}$. The converse implication is clear.

Before proceeding, let us digress a bit with the help of an example. Let $(\mathcal{M}, \le)$ be a totally ordered set and $\lambda = (\lambda_x)_{x\in\mathcal{M}}$ a non-decreasing family of positive numbers (it could also be non-increasing, in which case we would consider the reverse order on $\mathcal{M}$). Equipped with $d_\lambda \colon \mathcal{M}\times\mathcal{M} \to [0,\infty)$ defined by

$$d_\lambda(x,y) = \begin{cases} \lambda_{\max\{x,y\}} & \text{if } x \ne y \\ 0 & \text{if } x = y, \end{cases}$$

$\mathcal{M}$ is an ultrametric space. Since every set can be equipped with a total order, if we put $\lambda_x = 1$ for all $x \in \mathcal{M}$, we infer that the $\{0,1\}$-metric on $\mathcal{M}$ and the ultrametric d_λ coincide.

Thus the following theorem, which extends to the case $p < 1$ a result from [7], also extends Theorem 4.14 in the separable case.

Theorem 5.1. *Let $(\mathcal{M}, d)$ be an infinite separable pointed ultrametric space. Then $\mathcal{F}_p(\mathcal{M}, d) \approx \ell_p$ for every $0 < p \le 1$.*

The techniques we use to prove this theorem rely on the concepts of $\mathbb{R}$-tree and length measure. For the convenience of the reader we shall include these definitions, which we borrow from [13, Sect. 2]. For more details concerning $\mathbb{R}$-trees one may consult for instance [12, Chapter 3].

Definition 5.2. An $\mathbb{R}$-*tree* is a metric space $(\mathcal{T}, d)$ satisfying the following two conditions:

(i) For any points a and b in $\mathcal{T}$, there exists a unique isometry ϕ from the closed interval $[0, d(a,b)]$ into $\mathcal{T}$ such that $\phi(0) = a$ and $\phi(d(a,b)) = b$.
(ii) Any one-to-one continuous mapping $\varphi\colon [0,1] \to \mathcal{T}$ has the same range as the isometry ϕ associated to the points $a = \varphi(0)$ and $b = \varphi(1)$.

If $\mathcal{T}$ is an $\mathbb{R}$-tree, given any x and y in $\mathcal{T}$ we denote by ϕ_{xy} the unique isometry associated to x and y as in Definition 5.2, and write $[x,y]$ for the range of $\phi_{x,y}$. Such subsets of $\mathcal{T}$ are called *segments*. Moreover, we say that $v \in \mathcal{T}$ is a *branching point* of $\mathcal{T}$ if there are three points $x_1, x_2, x_3 \in \mathcal{T} \setminus \{v\}$ such that $[x_i, v] \cap [x_j, v] = \{v\}$ whenever $i, j \in \{1,2,3\}$, $i \neq j$. We say that a subset A of $\mathcal{T}$ is measurable whenever $\phi_{xy}^{-1}(A)$ is Lebesgue-measurable for any x and y in $\mathcal{T}$. If A is measurable and S is a segment $[x,y]$, we write $\lambda_S(A)$ for $\lambda(\phi_{xy}^{-1}(A))$, where λ is the Lebesgue measure on $\mathbb{R}$. We denote by $\mathcal{R}$ the set of all subsets of $\mathcal{T}$ that can be written as a finite union of disjoint segments. For $R = \cup_{k=1}^n S_k$ (with disjoint S_k) in $\mathcal{R}$, we put

$$\lambda_R(A) = \sum_{k=1}^n \lambda_{S_k}(A).$$

Now,

$$\lambda_{\mathcal{T}}(A) = \sup_{R \in \mathcal{R}} \lambda_R(A)$$

defines a measure on the σ-algebra of $\mathcal{T}$-measurable sets called the *length measure*. Note that this is nothing but the 1-dimensional Hausdorff measure (multiplied by the constant 2).

Suppose $(\mathcal{S}, d)$ is a closed subset of an $\mathbb{R}$-tree $\mathcal{T}$ with a base point $0 \in \mathcal{S}$. For $s \in \mathcal{S}$ we put

$$L_{\mathcal{S}}(s) := \inf_{x \in [0,s) \cap \mathcal{S}} d(s,x).$$

If $L_{\mathcal{S}}(s) > 0$, we denote by $\sigma_{\mathcal{S}}(s)$ the unique point from $[0,s) \cap \mathcal{S}$ with $d(s, \sigma_{\mathcal{S}}(s)) = L(s)$. Finally, we put

$$\mathcal{S}_+ := \{s \in \mathcal{S} \colon L_{\mathcal{S}}(s) > 0\}.$$

Lemma 5.3. *Let $(\mathcal{S}, d)$ be a closed subset of an $\mathbb{R}$-tree $\mathcal{T}$ with a point $0 \in \mathcal{S}$. Let $\mathcal{S}$ have length measure zero. Then for all $y \in \mathcal{S}$ and all $x \in [0,y] \cap \mathcal{S}$,*

$$d(x,y) = \sum_{z \in (x,y] \cap \mathcal{S}_+} L_{\mathcal{S}}(z).$$

Proof. Using the transformation $\phi_{0,y}$, we can assume without loss of generality that $x, y \in \mathbb{R}$, $0 \le x \le y$ and $\mathcal{S} \subseteq [0, y]$. Then, the subset $(x, y) \setminus \mathcal{S}$ of $\mathbb{R}$ is open and, so, it can be expressed as $\cup_{i\in I}(a_i, b_i)$, where the intervals are disjoint. Since $\mathcal{S}$ has measure zero, we have $y - x = \sum_{i\in I}(b_i - a_i)$. It is clear that $b_i \in \mathcal{S}_+$ and $\sigma_{\mathcal{S}}(b_i) = a_i$ for every $i \in I$. Thus, it suffices to see that the map $b \colon I \to \mathcal{S}_+$, $i \mapsto b_i$ is onto. Given $s \in \mathcal{S}_+$, since $(\sigma_{\mathcal{S}}(s), s) \cap \mathcal{S} = \emptyset$, there exists $i \in I$ such that $(\sigma_{\mathcal{S}}(s), s) \subseteq (a_i, b_i)$. Taking into account that neither $\sigma_{\mathcal{S}}(s)$ nor s belong to (a_i, b_i), we infer that $a_i = \sigma_{\mathcal{S}}(s)$ and $b_i = s$. □

In the case when $p = 1$ the following result was proved by Godard (see [13, Proposition 2.3]). Below we give an alternative proof which works for every $0 < p \le 1$ and for not necessarily separable $\mathbb{R}$-trees $\mathcal{T}$.

Proposition 5.4. *Let $(\mathcal{S}, d)$ be a closed subset of an $\mathbb{R}$-tree $\mathcal{T}$ such that $\mathcal{S}$ contains all the branching points of $\mathcal{T}$ and has length measure zero. Then $\mathcal{F}_p(\mathcal{S}, d^{1/p}) \approx \ell_p(\mathcal{S}_+)$ isometrically. To be precise, the map*

$$T(\delta(s)) := \sum_{x\in(0,s]\cap\mathcal{S}_+} (L_{\mathcal{S}}(x))^{1/p}\mathbf{e}_x, \qquad s \in \mathcal{S},$$

extends to a linear isometry between $\mathcal{F}_p(\mathcal{S}, d^{1/p})$ and $\ell_p(\mathcal{S}_+)$.

Proof. Without loss of generality we assume that $0 \in \mathcal{S}$, and for simplicity, for each $s \in \mathcal{S}_+$ we denote $\sigma_{\mathcal{S}}(s)$ by $s^{\#}$.

For every $y \in \mathcal{S}$ and $x \in [0, y] \cap \mathcal{S}$ we have

$$\delta(y) - \delta(x) = \sum_{z\in(x,y]\cap\mathcal{S}_+} \big(\delta(z) - \delta(z^{\#})\big). \tag{5.13}$$

To see this, if we consider in $[x, y]$ the total order induced by the isometry $\phi_{x,y}$, by Lemma 5.3 for any $\varepsilon > 0$ we can get $z_1 < z_2 < \cdots < z_n \in (x, y] \cap \mathcal{S}_+$ with $D := \left|\sum_{i=1}^n d(z_i, z_i^{\#}) - d(x, y)\right| \le \epsilon$. With the convention that $z_0 = x$,

$$\begin{aligned}
&\left\|\delta(y) - \delta(x) - \sum_{i=1}^n (\delta(z_i) - \delta(z_i^{\#}))\right\|^p \\
&\qquad\qquad \le \|\delta(y) - \delta(z_n)\|^p + \sum_{i=1}^n \|\delta(z_i^{\#}) - \delta(z_{i-1})\|^p \\
&\qquad\qquad \le d(y, z_n) + \sum_{i=1}^n d(z_i^{\#}, z_{i-1}) \\
&\qquad\qquad = D \le \varepsilon,
\end{aligned}$$

where, in the last equality we used the fact that $z_{i-1} \le z_i^{\#}$ for every $i = 1, \dots, n$.

For $s \in \mathcal{S}_+$, put

$$\mathbf{a}_s := \frac{\delta(s) - \delta(s^{\#})}{d^{1/p}(s, s^{\#})},$$

and set

$$\mathcal{A} := \overline{\operatorname{co}_p}\left\{\mathbf{a}_s \colon s \in \mathcal{S}_+\right\}.$$

Note that $T(\mathbf{a}_s) = \mathbf{e}_s$ since $(s^{\#}, s) \cap \mathcal{S} = \emptyset$. By Lemma 2.7, in order to show that T extends to a surjective isometry it suffices to show that

$$a_{x,y} := \frac{\delta(x) - \delta(y)}{d^{1/p}(x, y)} \in \mathcal{A}, \tag{5.14}$$

for each $x, y \in \mathcal{S}$. To that end, given $x, y \in \mathcal{S}$ there are three cases to take into account:

• Case 1: If $x \in [0, y]$, then by the identity (5.13),

$$a_{x,y} = \sum_{z \in (x,y] \cap \mathcal{S}_+} \frac{(L_{\mathcal{S}}(z))^{1/p}}{d^{1/p}(x, y)} \mathbf{a}_z,$$

which, by Lemma 5.3, is a p-convex combination of $\mathbf{a}_z$'s, and so $a_{x,y} \in \mathcal{A}$.

• Case 2: If $y \in [0, x]$, switching the roles of x and y in the previous case we easily get $a_{x,y} \in \mathcal{A}$.

• Case 3: If neither Case 1 nor Case 2 occurs, there exists a branching point $c \in [x, y]$ with $[0, x] \cap [0, y] = [0, c]$. Then we can write

$$a_{x,y} = \lambda a_{y,c} + \mu a_{c,x}, \quad \lambda = \frac{d^{1/p}(y, c)}{d^{1/p}(x, y)}, \quad \mu = \frac{d^{1/p}(c, x)}{d^{1/p}(x, y)}.$$

From the two previous cases we have $a_{c,y}$, $a_{c,x} \in \mathcal{A}$, and since

$$\lambda^p + \mu^p = \frac{d(x, c) + d(c, y)}{d(x, y)} = 1,$$

we conclude that $a_{x,y} \in \mathcal{A}$. Hence, (5.14) is fulfilled. □

Since the real line is a trivial example of an $\mathbb{R}$-tree we obtain:

Corollary 5.5. *Let $\mathcal{M}$ be an infinite subset of $\mathbb{R}$ and $0 < p \le 1$. If the closure of $\mathcal{M}$ has measure zero then*

$$\mathcal{F}_p(\mathcal{M}, |\cdot|^{1/p}) \approx \ell_p \text{ isometrically.} \tag{5.15}$$

In particular, the result holds if $\mathcal{M}$ is the range of a monotone sequence of real numbers.

Proof of Theorem 5.1. Since d^p is also an ultrametric whenever d is, we need only show that $\mathcal{F}_p(\mathcal{M}, d^{1/p}) \approx \ell_p$. By [7, Proposition 12], there exists a closed subset $\mathcal{S}$ of a separable $\mathbb{R}$-tree $\mathcal{T}$ containing all its branching points in such a way that $\mathcal{S}$ has length measure zero and $(\mathcal{M}, d)$ is bi-Lipschitz isomorphic to a Lipschitz retract of $\mathcal{S}$. Denoting the metric on $\mathcal{S}$ by η, we have that $(\mathcal{M}, d^{1/p})$ is bi-Lipschitz isomorphic to a Lipschitz retract of $(\mathcal{S}, \eta^{1/p})$. By Proposition 5.4, Lemma 4.4, and Corollary 4.6, $\mathcal{F}_p(\mathcal{S}, \eta^{1/p}) \approx \ell_p$ hence, by Lemma 4.4 and Lemma 4.19, $\mathcal{F}_p(\mathcal{M}, d^{1/p})$ is isomorphic to an infinite-dimensional complemented subspace of ℓ_p. Since every infinite-dimensional complemented subspace of ℓ_p is isomorphic to ℓ_p by a classical result of Stiles [22], we infer that $\mathcal{F}_p((\mathcal{M}, d^{1/p})) \approx \ell_p$, and the proof is over. $\square$

Theorem 4.13 and Corollary 5.5 allow us to identify the free p-spaces over some subsets of the real line equipped with the "anti-snowflaking" quasimetric $|\cdot|^{1/p}$. However, identifying the free p-space over subsets of $\mathbb{R}$ equipped with the Euclidean distance seems to be a more challenging task, which we plan to address in a further publication. For the time being, let us just mention that since the Banach envelope of $\mathcal{F}_p(I)$ is $L_1(I)$ for any interval I with the Euclidean distance (apply Proposition 4.20(b) and Theorem 4.13 for $p = 1$), the spaces $\mathcal{F}_p(I)$ are a new class of p-Banach spaces.

6. STRUCTURE OF LIPSCHITZ FREE p-SPACES

Lipschitz free p-spaces over quasimetric spaces constitute a nice family of new p-Banach spaces which are easy to define but whose geometry seems to be difficult to understand. To carry out this task sucessfuly one hopes to be able to count on "natural" structural results involving free p-spaces over subsets of $\mathcal{M}$. In this section we analyse this premise and confirm an unfortunate recurrent pattern in quasi-Banach spaces: the lack of tools can be an important stumbling block in the development of the nonlinear theory. However, as we will also see, not everything is lost and we still can develop specific methods that permit to shed light into the structure of $\mathcal{F}_p(\mathcal{M})$.

6.1. **Linearizations of Lipschitz embeddings.** If $\mathcal{M}$ is a pointed p-metric space and $\mathcal{N}$ is a subset of $\mathcal{M}$ containing 0, the linearization process of Lemma 4.8 applies in particular to the canonical injection $\jmath\colon \mathcal{N} \to \mathcal{M}$. If $p = 1$, McShane's theorem [19] ensures that $L_\jmath\colon \mathcal{F}(\mathcal{N}) \to \mathcal{F}(\mathcal{M})$ is an isometric embedding. Thus $\mathcal{F}(\mathcal{N})$ can be naturally identified with a subspace of $\mathcal{F}(\mathcal{M})$. However, if $p < 1$ this argument crashes. In the case when $p = 1$, $\mathcal{F}(\mathcal{N})$ is isometric to a subspace of $\mathcal{F}(\mathcal{M})$ and so the study of Lipschitz free spaces over subsets is a powerful tool. We start by exhibiting that this argument breaks down when $p < 1$, solving this way a problem raised in [3].

Theorem 6.1. *For each $0 < p < 1$ and $p \leq q \leq 1$ there is a q-metric space $(\mathcal{M}, \rho)$ and a subset $\mathcal{N} \subseteq \mathcal{M}$ such that the inclusion map $\jmath \colon \mathcal{N} \to \mathcal{M}$ induces a non-isometric isomorphic embedding $L_\jmath : \mathcal{F}_p(N) \to \mathcal{F}_p(M)$ with $\|L_\jmath^{-1}\| \geq 2^{1/q}$.*

Proof. Let $\mathcal{N} = \{0\} \cup \mathbb{N}$ and $\mathcal{M} = \{0, z\} \cup \mathbb{N}$, where $z \notin \mathcal{N}$. Define $\rho \colon \mathcal{M} \times \mathcal{M} \to [0, \infty)$ by $\rho(x, x) = 0$ for all $x \in \mathcal{M}$,

$$\rho(j, z) = \rho(z, j) = 2^{-1/q} \text{ for all } j \in \mathbb{N}, \quad \rho(x, y) = 1 \text{ otherwise.}$$

It is clear that $(\mathcal{M}, \rho)$ is a bounded and uniformly separated q-metric space. Then, by Theorem 4.14, $L_\jmath$ is an isomorphism.

Given $\epsilon > 0$, there is $k \in \mathbb{N}$ and a k-tuple $(a_j)_{j=1}^k$ such that $a_j \geq 0$ for every $j = 1, \ldots, k$, $\sum_{j=1}^k a_j^p = 1$, and $\sum_{j=1}^k a_j \leq \epsilon$. Indeed, it suffices to choose $k \geq \varepsilon^{p/(p-1)}$ and put $a_j = k^{-1/p}$ for $1 \leq j \leq k$. On the one hand, since ρ is the $\{0, 1\}$-metric on $\mathcal{N}$, by Proposition 4.16,

$$\left\| \sum_{j=1}^k a_j \, \delta(j) \right\|_{\mathcal{F}_p(\mathcal{N})} = 1.$$

On the other hand, considering the decomposition

$$\sum_{j=1}^k a_j \, \delta(j) = \left(\sum_{j=1}^k a_j \right) \delta(z) + \sum_{j=1}^k \frac{a_j}{2^{1/q}} \, \frac{\delta(j) - \delta(z)}{2^{-1/q}}$$

and using Corollary 4.12, we have

$$\left\| \sum_{j=1}^k a_j \, \delta(j) \right\|_{\mathcal{F}_p(\mathcal{M})}^p \leq \left(\sum_{j=1}^k a_j \right)^p + \sum_{j=1}^k (2^{-1/q} a_j)^p \leq \epsilon^p + 2^{-p/q}.$$

Consequently,

$$\|L_\jmath^{-1}\| \geq \sup_{\epsilon > 0} \frac{1}{(\epsilon^p + 2^{-p/q})^{1/p}} = 2^{1/q}. \qquad \square$$

The following problem seems to be open.

Question 6.2. Let $0 < p < 1$ and $\mathcal{N} \subseteq \mathcal{M}$ be two p-metric (or metric) spaces in inclusion. Is the canonical linear map of $\mathcal{F}_p(\mathcal{N})$ into $\mathcal{F}_p(\mathcal{M})$ an isomorphic embedding?

The answer to Question 6.2 is positive in some special cases.

Proposition 6.3. *If $\mathcal{M}$ is a $\{0,1\}$-metric space, then the canonical map of $\mathcal{F}_p(\mathcal{N})$ into $\mathcal{F}_p(\mathcal{M})$ is isometric for every $\mathcal{N} \subseteq \mathcal{M}$.*

Proof. Notice that a map f from a $\{0,1\}$-metric space $\mathcal{M}$ into a quasi-Banach space X is 1-Lipschitz if and only if $\|f(x) - f(y)\| \le 1$ for every x, $y \in \mathcal{M}$. Then, a 1-Lipschitz map $f\colon \mathcal{N} \to X$ with $f(0) = 0$ extends by $f(x) = 0$ for $x \in \mathcal{M} \setminus \mathcal{N}$ to a 1-Lipschitz map from $\mathcal{M}$ into X. This gives $\|L_\jmath(\mu)\|_{\mathcal{F}_p(\mathcal{M})} \ge \|\mu\|_{\mathcal{F}_p(\mathcal{N})}$ for every $\mu \in \mathcal{P}(\mathcal{N})$. $\square$

Proposition 6.4. *Let $\mathcal{M}$ be a subset of $\mathbb{R}$ equipped with the quasimetric $\rho(x,y) = |x-y|^{1/p}$ for $0 < p \le 1$. If the closure of $\mathcal{M}$ has measure zero and $\mathcal{N} \subseteq \mathcal{M}$ then the canonical mapping $L_\jmath\colon \mathcal{F}_p(\mathcal{N}) \to \mathcal{F}_p(\mathcal{M})$ is an isometry.*

Proof. By Proposition 4.17 we can assume that both $\mathcal{N}$ and $\mathcal{M}$ are closed. Let $T_\mathcal{M}\colon \mathcal{F}_p(\mathcal{M}) \to \ell_p(\mathcal{M}_+)$ and $T_\mathcal{N}\colon \mathcal{F}_p(\mathcal{N}) \to \ell_p(\mathcal{N}_+)$ be the isomotries provided by Proposition 5.4. If $U = T_\mathcal{M} \circ L_\jmath \circ T_\mathcal{N}^{-1}$, where $\jmath\colon \mathcal{N} \to \mathcal{M}$ is the inclusion map, we have

$$U(\mathbf{e}_s) = \frac{1}{|s - \sigma_\mathcal{N}(s)|^{1/p}} \sum_{z \in (\sigma_\mathcal{N}(s), s] \cap \mathcal{M}_+} L_\mathcal{M}^{1/p}(z) \mathbf{e}_z, \quad s \in \mathcal{N}_+.$$

By Lemma 5.3, $\|U(\mathbf{e}_s)\|_p = 1$ for every $s \in \mathcal{N}_+$. Pick s, $t \in \mathcal{N}_+$ with $s < t$. Since $(\sigma_\mathcal{N}(s), s) \cap \mathcal{N} = \emptyset$, we have $s \le \sigma_\mathcal{N}(t)$. Then, $(\sigma_\mathcal{N}(s), s] \cap (\sigma_\mathcal{N}(t), t] = \emptyset$. Consequently, $(U(\mathbf{e}_s))_{s \in \mathcal{N}_+}$ is a disjointly supported family in $\ell_p(\mathcal{M}_+)$. We infer that $(U(\mathbf{e}_s))_{s \in \mathcal{N}_+}$ is isometrically equivalent to the unit vector basis of $\ell_p(\mathcal{M}_+)$. In other words, U is an isometric embedding. $\square$

Question 6.5. Let $0 < p < 1$. Can we identify the p-metric spaces $\mathcal{M}$ for which the canonical linear map of $\mathcal{F}_p(\mathcal{N})$ into $\mathcal{F}_p(\mathcal{M})$ is an isometry for every $\mathcal{N} \subseteq \mathcal{M}$?

6.2. Embeddability of ℓ_p in $\mathcal{F}_p(\mathcal{M})$ for $0 < p \le 1$.

The last question we shall address in this paper is whether, by analogy with the case $p = 1$, we can guarantee that ℓ_p embeds isomorphically in any $\mathcal{F}_p(\mathcal{M})$ for $0 < p < 1$. The answer is positive, but in order to prove it we must develop a completely new set of techniques, specific of the nonlocally convex case.

For a mapping $f\colon X \to Y$, where X is a set and Y a vector space, we shall denote the set $f^{-1}(Y \setminus \{0\})$ by $\operatorname{supp}_0(f)$.

Lemma 6.6. *Suppose that X is a p-normed space and that Y is a q-normed space, $0 < q, p \le 1$. Let Γ a be set, and $f_\gamma\colon X \to Y$ be an L-Lipschitz map for each $\gamma \in \Gamma$. Suppose that the sets in the family $\{\operatorname{supp}_0(f_\gamma)\}_{\gamma \in \Gamma}$ are disjoint. Then $f = \sum_{\gamma \in \Gamma} f_\gamma$ is $L2^{1/q-1}$-Lipschitz.*

Proof. For each $x \in X$ there is $\gamma(x) \in \Gamma$ such that $f_\gamma(x) = 0$ if $\gamma \neq \gamma(x)$. In particular, the sum defining $f(x)$ is pointwise finite and so f is well-defined. Pick $x, y \in X$ and set $\alpha = \gamma(x)$ and $\beta = \gamma(y)$. Since the line segment $[x, y]$ is connected and the sets $\operatorname{supp}_0(f_\alpha)$ and $\operatorname{supp}_0(f_\beta)$ are open and disjoint, there is $z \in [x, y] \setminus (\operatorname{supp}_0(f_\alpha) \cup \operatorname{supp}_0(f_\beta))$. Using the elementary fact that

$$\|y + z\|_Y \leq 2^{1/q-1}(\|y\|_Y + \|z\|_Y), \quad y, z \in Y,$$

we obtain

$$\begin{aligned}
\|f(x) - f(y)\|_Y &= \|f_\alpha(x) - f_\beta(y)\|_Y \\
&= \|f_\alpha(x) - f_\alpha(z) + f_\beta(z) - f_\beta(y)\|_Y \\
&\leq 2^{1/q-1}(\|f_\alpha(x) - f_\alpha(z)\| + \|f_\beta(z) - f_\beta(y)\|_Y) \\
&\leq L2^{1/q-1}(\|x - z\|_X + \|z - y\|_X) \\
&= 2^{1/q-1}L\|x - y\|_X,
\end{aligned}$$

from where the conclusion follows. □

Lemma 6.7. *Let $(\mathcal{M}, \rho)$ be a pointed p-metric space. Assume that $(x_n)_{n=1}^\infty$ in $\mathcal{M}$ and $(r_n)_{n=1}^\infty$ in $[0, +\infty)$ are such that the metric $d = \rho^p$ satisfies*

$$d(x_m, x_n) \geq r_m + r_n, \quad m, n \in \mathbb{N}, \, m \neq n. \tag{6.16}$$

Then there is a sequence $(f_n)_{n=1}^\infty$ of 1-Lipschitz maps from $(\mathcal{M}, \rho)$ into $L_p(\mathbb{R})$ such that $f_n(x_n) = \chi_{(0,r_n]}$ for every $n \in \mathbb{N}$ and $(\operatorname{supp}_0(f_n))_{n=1}^\infty$ is a pairwise disjoint sequence.

Proof. For $n \in \mathbb{N}$ define $g_n \colon \mathcal{M} \to \mathbb{R}$ by

$$g_n(x) = \max\{r_n - \rho^p(x, x_n), 0\}, \quad x \in \mathcal{M}. \tag{6.17}$$

It is straightforward to check that the sets $\{\operatorname{supp}_0(g_n)\}_{n=1}^\infty$ are pairwise disjoint, that $g(x_n) = r_n$, and that $g_n \colon (\mathcal{M}, \rho) \to (\mathbb{R}, |\cdot|^{1/p})$ is 1 Lipschitz. Then, if Φ is as in (4.9), the sequence $(\Phi \circ g_n)_{n=1}^\infty$ has the desired properties. □

Definition 6.8. Let $(\mathcal{M}, \rho)$ be a p-metric space, $0 < p \leq 1$, and consider the metric $d = \rho^p$. A sequence $(x_n)_{n=1}^\infty$ in $\mathcal{M}$ will be said to be *sequentially separated* if there is a monotone sequence $(r_n)_{n=1}^\infty$ in $[0, +\infty)$ such that (6.16) holds and, for some $\alpha > 0$ we have

$$\frac{|r_n - r_{n+1}|}{d(x_n, x_{n+1})} \geq \alpha, \quad n \in \mathbb{N}.$$

If this is the case, we will say that $(x_n)_{n=1}^\infty$ is *sequentially α-separated.*

Theorem 6.9. *Let $(\mathcal{M}, \rho)$ be a p-metric space, $0 < p \leq 1$, containing a sequentially α-separated sequence $\mathcal{N} = \{x_n \colon n \in \mathbb{N}\}$. Then:*

(i) *The space* $\mathcal{F}_p(\mathcal{N})$ *is* $2^{1/p-1}\alpha^{-1}$*-isomorphic to* ℓ_p*, and*
(ii) *The canonical map* $L_\jmath\colon \mathcal{F}_p(\mathcal{N}) \to \mathcal{F}_p(\mathcal{M})$ *is an isomorphic embedding, where* $\jmath\colon \mathcal{N} \to \mathcal{M}$ *is the inclusion. Quantitatively,* $\|L_\jmath^{-1}\| \le 2^{1/p-1}\alpha^{-1}$.

Proof. Suppose $(x_n)_{n=1}^\infty$ and $(r_n)_{n=1}^\infty$ verify the conditions of Definition 6.8. We may assume that $0 = x_1$. For each $n \in \mathbb{N}$ put

$$\begin{aligned} v_n &= \frac{\delta_{\mathcal{N}}(x_{n+1}) - \delta_{\mathcal{N}}(x_n)}{\rho(x_{n+1}, x_n)} \in \mathcal{F}_p(\mathcal{N}), \\ b_n &= \frac{\delta_{\mathcal{M}}(x_{n+1}) - \delta_{\mathcal{M}}(x_n)}{\rho(x_{n+1}, x_n)} \in \mathcal{F}_p(\mathcal{M}). \end{aligned}$$

There is a norm-one linear map $T\colon \ell_p \to \mathcal{F}_p(\mathcal{N})$ such that $T(\mathbf{e}_n) = v_n$ for all $n \in \mathbb{N}$. Since $L_\jmath(v_n) = b_n$ for $n \in \mathbb{N}$ and $\overline{\operatorname{span}}\{v_n \colon n \in \mathbb{N}\} = \mathcal{F}_p(\mathcal{N})$, it suffices to prove that $(b_n)_{n=1}^\infty$ is $2^{1/p-1}\alpha^{-1}$-dominated by canonical basis of ℓ_p.

We consider $\mathcal{M}$ as a subset of $X = \mathcal{F}_p(\mathcal{M})$. Let $(f_n)_{n=1}^\infty$ be the sequence of maps from X into $L_p(\mathbb{R})$ provided by Lemma 6.7. Pick any $a_1, \dots, a_N \in \mathbb{R}$ and consider $x = \sum_{n=1}^N a_n b_n$. Let $g\colon X \to L_p$ be defined for $z \in \mathcal{F}_p(\mathcal{M})$ by

$$g(z) = \sum_{n=1}^N f_n(z).$$

Lemma 6.6 yields that $g\colon X \to L_p$ is $2^{1/p-1}$-Lipschitz. Hence, $f = g|_{\mathcal{M}}$ is $2^{1/p-1}$-Lipschitz. Since $f(x_n) = \chi_{(0,r_n]}$ we have

$$\begin{aligned} \|x\| &\ge \frac{2}{2^{1/p}} \left\| \sum_{n=1}^N a_n \frac{f(x_{n+1}) - f(x_n)}{\rho(x_n, x_{n+1})} \right\|_{L_p} \\ &= \frac{2}{2^{1/p}} \left\| \sum_{n=1}^N a_n \frac{\chi_{(r_n, r_{n+1}]}}{\rho(x_n, x_{n+1})} \right\|_{L_p} \\ &= \frac{2}{2^{1/p}} \left(\sum_{n=1}^N |a_n|^p \frac{|r_{n+1} - r_n|}{\rho^p(x_n, x_{n+1})} \right)^{1/p} \\ &\ge \frac{2}{2^{1/p}} \alpha \left(\sum_{n=1}^N |a_n|^p \right)^{1/p}. \end{aligned}$$

□

Lemma 6.10. *Let* $(\mathcal{M}, \rho)$ *be an infinite* p*-metric space,* $0 < p \le 1$*. Let* $(x_n)_{n=1}^\infty$ *be a sequence in* $\mathcal{M}$*. Suppose there is a point* $x_0 \in \mathcal{M}$ *and*

$\alpha < 1$ (respectively, $\beta < 1$) so that

$$\alpha = \sup_{n \in \mathbb{N}} \frac{\rho(x_{n+1}, x_0)}{\rho(x_n, x_0)} < 1 \tag{6.18}$$

(respectively,

$$\beta = \sup_{n \in \mathbb{N}} \frac{\rho(x_n, x_0)}{\rho(x_{n+1}, x_0)} < 1). \tag{6.19}$$

Then $(x_n)_{n=1}^\infty$ is sequentially $(1-\alpha^p)^2(1+\alpha^p)^{-2}$-separated (respectively, $(1-\beta^p)^2(1+\beta^p)^{-2}$-separated).

Proof. For $n \in \mathbb{N}$ set $r_n = \rho^p(x_n, x_0)$. Let $\gamma = (1+\alpha^p)(1-\alpha^p)^{-1}$ (respectively, $\gamma = (1+\beta^p)(1-\beta^p)^{-1}$). In both cases we have

$$r_n + r_m \le \gamma |r_n - r_m|, \quad m \neq n.$$

Then

$$\rho^p(x_n, x_{n+1}) \le \rho^p(x_n, x_0) + \rho^p(x_{n+1}, x_0) = r_n + r_{n+1} \le \gamma |r_n - r_{n+1}|$$

and, if $n \neq m$,

$$\rho^p(x_n, x_m) \ge |\rho^p(x_n, x_0) - \rho^p(x_m, x_0)| = |r_n - r_m| \ge \gamma^{-1}(r_n + r_m)$$

Consequently, $(x_n)_{n=1}^\infty$ is sequentially γ^{-2}-separated. □

For further reference, we record a couple of easy consequences of the above lemma.

Lemma 6.11. *If $\mathcal{M}$ is an unbounded p-metric space, $0 < p \le 1$, then it contains a sequentially γ-separated sequence for every $\gamma < 1$.*

Proof. It is clear that given $x_0 \in \mathcal{M}$ and $\alpha < 1$ there is a sequence $(x_n)_{n=1}^\infty$ in $\mathcal{M}$ satisfying (6.19). Then, the result follows from applying Lemma 6.10 with α small enough. □

Lemma 6.12. *Let $\mathcal{M}$ be a p-metric space, $0 < p \le 1$. If the completion of $\mathcal{M}$ has a limit point then $\mathcal{M}$ contains a sequentially separated sequence for every $\gamma < 1$.*

Proof. Let $0 < \alpha < 1$ be small enough. If x_0 is a limit point, there is a sequence $(x_n)_{n=1}^\infty$ in $\mathcal{M}$ satisfying (6.18). □

In [7, 11] it has been shown, that although the space $\mathcal{F}(\mathcal{M})$ is isomorphic to ℓ_1 for any separable ultrametric space $\mathcal{M}$, it is never isometric to ℓ_1. In contrast, we have the follwoing result.

Proposition 6.13. *For each $\epsilon > 0$ there is an ultrametric space $\mathcal{M}$ such that the Banach-Mazur distance between $\mathcal{F}(\mathcal{M})$ and ℓ_1 is smaller than $1 + \epsilon$.*

Proof. By Lemma 6.11, Lemma 6.12, and Theorem 6.9, in each ultrametric space which is unbounded or contains a limit point we may find a subset $\mathcal{M}$ (which is also ultrametric) so that the Banach-Mazur distance between $\mathcal{F}(\mathcal{M})$ and ℓ_1 arbitrarily close to 1. □

Corollary 6.14. *Let $0 < p \leq 1$. Every infinite p-metric space $\mathcal{M}$ has a subset $\mathcal{N}$ with $\mathcal{F}_p(\mathcal{N}) \approx \ell_p$.*

Proof. By Lemma 6.11 and Theorem 6.9 it suffices to consider the case when $\mathcal{M}$ is bounded. If $\mathcal{M}$ contains a uniformly separated infinite set, then, by Theorem 4.14, we are done. Otherwise, the completion of $\mathcal{M}$ is compact and so there is a limit point in the completion of $\mathcal{M}$. Then, the results follows from combining Lemma 6.12 with Theorem 6.9. □

Definition 6.15. Let $(\mathcal{M}, \rho)$ be a p-metric space, $0 < p \leq 1$. We will say that a sequence $(x_n)_{n=1}^\infty$ in $\mathcal{M}$ is *sequentially pairwise separated* if there is a monotone sequence $(r_n)_{n=1}^\infty$ in $[0, +\infty)$ such that and (6.16) holds and for some $\alpha > 0$ we have

$$\frac{|r_{2n+1} - r_{2n}|}{d(x_{2n}, x_{2n+1})} \geq \alpha, \quad n \in \mathbb{N},$$

where $d = \rho^p$. We then say that $(x_n)_{n=1}^\infty$ is *sequentially pairwise α-separated.*

Theorem 6.16. *Let $(\mathcal{M}, \rho)$ be a p-metric space, $0 < p \leq 1$, which contains a sequentially pairwise α-separated sequence. Then $\mathcal{F}_p(\mathcal{M})$ contains a subspace $2^{1/p-1}\alpha^{-1}$-isomorphic to ℓ_p.*

Proof. Let $(x_n)_{n=1}^\infty$ and $(r_n)_{n=1}^\infty$ be as in Definition 6.15. Set $0 = x_1$. For $n \in \mathbb{N}$ put

$$b_n = \frac{\delta(x_{2n+1}) - \delta(x_{2n})}{\rho(x_{2n}, x_{2n+1})} \in \mathcal{F}_p(\mathcal{M}).$$

Let us verify that $(b_n)_{n=1}^\infty$ is equivalent to the canonical basis of ℓ_p. To that end, as in the proof of Theorem 6.9, we consider $\mathcal{M}$ as a subset of $X = \mathcal{F}_p(\mathcal{M})$ and we consider the sequence $(f_n)_{n=1}^\infty$ of maps from X into $L_p(\mathbb{R})$ provided by Lemma 6.7.

Pick any $a_1, \ldots, a_N \in \mathbb{R}$ and take $x = \sum_{n=1}^N a_n b_n$. On the one hand we clearly have $\|x\|_{\mathcal{F}_p} \leq (\sum_{n=1}^N |a_n|^p)^{1/p}$. On the other hand, toprove the reverse inequality, we choose $g \colon \mathcal{F}_p(\mathcal{M}) \to L_p$ defined for $z \in \mathcal{F}_p(\mathcal{M})$ as

$$g(z) = \sum_{n=2}^{2N+1} f_n.$$

Lemma 6.6 yields that $g\colon \mathcal{F}_p(\mathcal{M}) \to L_p$ is $2^{1/p-1}$-Lipschitz, and $g(0) = 0$. Hence, $f = g|_{\mathcal{M}}$ is $2^{1/p-1}$-Lipschitz. Then

$$\begin{aligned}\|x\| &\ge \frac{2}{2^{1/p}} \left\| \sum_{n=1}^{N} a_n \frac{f(x_{2n+1}) - f(x_{2n})}{\rho(x_{2n}, x_{2n+1})} \right\|_{L_p} \\ &= \frac{2}{2^{1/p}} \left\| \sum_{n=1}^{N} a_n \frac{\chi_{(r_{2n}, r_{2n+1}]}}{\rho(x_{2n}, x_{2n+1})} \right\|_{L_p} \\ &= \frac{2}{2^{1/p}} \left(\sum_{n=1}^{N} |a_n|^p \frac{|r_{2n+1} - r_{2n}|}{\rho^p(x_{2n}, x_{2n+1})} \right)^{1/p} \\ &\ge \frac{2}{2^{1/p}} \alpha \left(\sum_{n=1}^{N} |a_n|^p \right)^{1/p}. \qquad \square\end{aligned}$$

It is clear that any sequently α-separated sequence also is sequently pairwise α-separated. So Lemmas 6.11 and 6.12 provide sufficient conditions for a p-metric space to have a sequently pairwise separated sequence. Next, we introduce another sufficient condition for a p-metric space to have a sequentially pairwise separated sequence, which we will use later.

Definition 6.17. A p-metric space $(\mathcal{M}, \rho)$, $0 < p \le 1$, is said to have the *selective R-closeness property*, $0 < R < 1$, if for every $\epsilon > 0$ there are y, $z \in \mathcal{M}$ such that $0 < d(y, z) \le \epsilon$, and for all $x \in \mathcal{M} \setminus \{y, z\}$

$$R\, d(y, z) \le \min\{d(x, y), d(x, z)\},$$

where $d = \rho^p$.

Lemma 6.18. *If a p-metric space $(\mathcal{M}, \rho)$ has the selective R-closeness property for some $R < 1$, then $(\mathcal{M}, \rho)$ contains a sequentially pairwise α-separated sequence for every $\alpha < R$.*

Proof. Set $d = \rho^p$. Pick y_0, z_0 two distinct points in $\mathcal{M}$ and $0 < t < R$. We recursively construct $(y_k, z_k)_{k=1}^{\infty}$ such that, if $s_k = d(y_k, z_k)$, then, for every $k \in \mathbb{N}$, $0 < s_k \le t s_{k-1}$ and $R s_k \le \min\{d(y_k, x), d(z_k, x)\}$ whenever $x \notin \{x_k, y_k\}$.

Let $k < j$. Since $s_j < s_k$ we have $\{y_j, z_j\} \ne \{y_k, z_k\}$. Assume that $\{y_j, z_j\} \cap \{y_k, z_k\} \ne \emptyset$. Then there are $y \in \{y_j, z_j\} \cap \{y_k, z_k\}$ and $z \in \{y_j, z_j\} \setminus \{y_k, z_k\}$, so that

$$s_j = d(y, z) \ge \min\{d(y_k, z), d(z_k, z)\} \ge R s_k > t s_k \ge s_j.$$

This absurdity shows that $\{y_j, z_j\} \cap \{y_k, z_k\} = \emptyset$ for $k \neq j$. For $n \in \mathbb{N} \cup \{0\}$ put

$$x_n = \begin{cases} y_{n/2} & \text{if } n \text{ is even,} \\ z_{(n-1)/2} & \text{if } n \text{ is odd,} \end{cases}$$

$$r_n = \begin{cases} s_{n/2} & \text{if } n \text{ is even,} \\ ts_{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

For $k \in \mathbb{N} \cup \{0\}$,

$$d(x_{2k}, x_{2k+1}) = s_k = \frac{1}{1-t}(r_{2k} - r_{2k+1})$$

and

$$r_{2k+2} = s_{k+1} \leq ts_k = r_{2k+1}.$$

In particular, $(r_n)_{n=0}^\infty$ is non-increasing. Let $m < n$ and put $k = \lfloor m/2 \rfloor$. If $\lfloor n/2 \rfloor = k$ we have $m = 2k$ and $n = 2k+1$. Then

$$d(x_m, x_n) = s_k = \frac{1}{1+t}\left(s_k + ts_k\right) = \frac{1}{1+t}(r_{2k} + r_{2k+1}).$$

If the case when $k < \lfloor n/2 \rfloor$ we have $x_n \notin \{y_k, z_k\}$ so that

$$\begin{aligned} d(x_m, x_n) &\geq \min\{d(y_k, x_n), d(z_k, x_n)\} \\ &\geq Rs_k \\ &= \frac{R}{1+t}(r_{2k} + r_{2k+1}) \\ &\geq \frac{R}{1+t}(r_m + r_n). \end{aligned}$$

Hence $(x_n)_{n=1}^\infty$ is sequentially pairwise $R(1-t)(1+t)^{-1}$-separated. Choosing t close enough to zero we are done. □

Lemma 6.19. *Let* $0 < p \leq 1$ *and let* $(\mathcal{M}, \rho)$ *be a* p*-metric space which is not uniformly separated. Assume that there is* $R < 1$ *such that* $\mathcal{M}$ *does not have the* R*-closeness property. Then* $\mathcal{M}$ *contains a limit point in its completion.*

Proof. Set $d = \rho^p$. By the assumptions we have $\inf\{d(x,y) \colon x \neq y\} = 0$ and there is $c > 0$ such that for every $(y,z) \in \mathcal{M}^2$ with $0 < d(y,z) \leq c$ there is $x \in \mathcal{M} \setminus \{y,z\}$ satisfying either $d(y,x) < R\, d(y,z)$ or $d(z,x) < R\, d(y,z)$. We recursively construct an infinite sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ with a graph structure and infinite sequence $(c_n)_{n\in\mathbb{N}}$ of distances with $c_{i+1} < R\, c_i$, for $i \in \mathbb{N}$ as follows. We choose arbitrarily $x_1 \neq x_2 \in \mathcal{M}$

such that $d(x_1, x_2) = c_1 \leq c$ and we put an edge between them. Suppose we have produced points $x_1, \dots, x_n$ with graph structure and distances $c_1, \dots, c_{n-1}$. Let $y \neq z$ be the points that realized the distance c_{n-1} and find x_{n+1} with $R\, d(y, z) > \min\{d(y, x_{n+1}), d(z, x_{n+1})\} = c_n$. We put an edge between the points realizing the distance c_n. At the end, we have a produced an infinite graph-theoretic tree. Either there is an infinite branch $(z_n)_{n=1}^\infty$ in the tree, which is Cauchy since $d(z_{n+2}, z_{n+1}) < R\, d(z_{n+1}, z_n)$, for all n, or there is a point z connected by edge to infinitely many points. Clearly z is then a limit point of them. □

Theorem 6.20. *If $\mathcal{M}$ is an infinite p-metric space, $0 < p \leq 1$, then $\mathcal{F}_p(\mathcal{M})$ contains a subspace X isomorphic to ℓ_p.*

Proof. Let us first see that the result holds in each of the following three possible scenarios for the p-metric space $\mathcal{M}$:

(i) If $\mathcal{M}$ is bounded and uniformly separated, Theorem 4.14 tells us that $\mathcal{F}_p(\mathcal{M}) \approx \ell_p(\mathcal{M} \setminus \{0\})$.
(ii) If $\mathcal{M}$ is unbounded, Theorem 6.9 and Lemma 6.11 take care of the problem.
(iii) If $\mathcal{M}$ has the selective R-closeness property for some $R < 1$, we are done by an appeal to Lemma 6.18 and Theorem 6.16.

Now assume that $\mathcal{M}$ does not fall into any of those three categories. Then $\mathcal{M}$ is not uniformly separated nor has the R-closeness property for any $R < 1$. Combining Lemma 6.19 with Theorem 6.9 and Lemma 6.12 puts an end to the proof. □

As the dutiful reader will have noticed, although the tools we have developed allow us to estimate the Banach-Mazur distance between ℓ_p and the subspace of $\mathcal{F}_p(\mathcal{M})$ whose existence is guaranteed in Theorem 6.20, we have not included the quantitative version of this result. The reason is that the isomorphism constant plays no role now since it can be obtained from a more general theorem, namely, a non-locally convex version of James's ℓ_1 distortion theorem. To the best of our knowledge, this is the first time that the validity of this result is explicitly stated and so we include its proof for further reference.

Theorem 6.21 (James's ℓ_p distortion theorem for $0 < p \leq 1$)**.** *Let $(x_j)_{j=1}^\infty$ be a normalized basic sequence in a p-Banach space X which is equivalent to the canonical ℓ_p-basis, $0 < p \leq 1$. Then given $\epsilon > 0$ there is a normalized block basic sequence $(y_k)_{k=1}^\infty$ of $(x_j)_{j=1}^\infty$ such that*

$$\left\| \sum_{k=1}^\infty a_k y_k \right\| \geq (1 - \epsilon) \left(\sum_{k=1}^\infty |a_k|^p \right)^{1/p},$$

for any sequence of scalars $(a_k)_{k=1}^\infty \in c_{00}$.

Proof. By hypothesis, there is $M_0 < \infty$ such that

$$\left(\sum_{j=1}^\infty |a_j|^p\right)^{1/p} \le M_0 \left\|\sum_{j=1}^\infty a_j x_j\right\|$$

for every $(a_j)_{j=1}^\infty \in c_{00}$. Hence, for each integer n we can consider the least constant M_n so that if $(a_j)_{j=1}^\infty \in c_{00}$ with $a_j = 0$ for $j \le n$ then

$$\left(\sum_{j=1}^\infty |a_j|^p\right)^{1/p} \le M_n \left\|\sum_{j=1}^\infty a_j x_j\right\|.$$

The sequence $(M_n)_{n=1}^\infty$ is decreasing, and the inequality $\|\sum_{j=1}^\infty a_j x_j\| \le (\sum_{j=1}^\infty |a_j|^p)^{1/p}$ yields $M_n \ge 1$. Let $M = \lim_{n\to\infty} M_n \ge 1$. We recursively construct an increasing sequence $(n_k)_{k=0}^\infty$ of positive integers and a sequence $(y_k)_{k=1}^\infty$ in X. We start by choosing $n_0 \in \mathbb{N}$ such that

$$M_{n_0} \le (1-\epsilon)^{-1/2} M.$$

Let $k \in \mathbb{N}$ and assume that n_i and y_i are constructed for $i < k$. Since $(1-\epsilon)^{1/2} M < M_{n_{k-1}}$ there is $n_k > n_{k-1}$ and a norm-one vector $y_k = \sum_{j=1+n_{k-1}}^{n_k} b_j\, x_j \in X$ such that

$$\left(\sum_{j=1+n_{k-1}}^{n_k} |b_j|^p\right)^{1/p} \ge (1-\epsilon)^{1/2} M.$$

The normalized block basic sequence $(y_n)_{n=1}^\infty$ satisfies the desired property. Indeed, for any $(a_j)_{j=1}^\infty \in c_{00}$ we have

$$\begin{aligned}\left\|\sum_{k=1}^\infty a_k\, y_k\right\| &= \left\|\sum_{k=1}^\infty \sum_{j=1+n_{k-1}}^{n_k} a_k\, b_j\, x_j\right\| \\ &\ge M_{n_0}^{-1} \left(\sum_{k=1}^\infty \sum_{j=1+n_{k-1}}^{n_k} |a_k|^p\, |b_j|^p\right)^{1/p} \\ &\ge (1-\epsilon)^{1/2} M M_{n_0}^{-1} \left(\sum_{k=1}^\infty |a_k|^p\right)^{1/p} \\ &\ge (1-\epsilon) \left(\sum_{k=1}^\infty |a_k|^p\right)^{1/p}. \qquad \square\end{aligned}$$

We close with an easy consequence of Theorem 6.20.

Corollary 6.22. *If $\mathcal{M}$ is an infinite p-metric space, $0 < p < 1$, then $\mathcal{F}_p(\mathcal{M})$ is not q-convex for any $p < q \leq 1$.*

Acknowledgment. The authors would like to thank Prof. Antonín Procházka for the helpful discussions maintained during the Conference "Non Linear Functional Analysis" held at CIRM (Luminy, France) from March 5 to 9, 2018.

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