

Introductory thermodynamics

(Phenomenological approach)

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and

Wikipedia (for pictures)

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Synopsis:

This series of three lectures presents an introductory explanation of main points of the phenomenological thermodynamics. As is usual, I declare no attempt on completeness, rigor, and details. The order of material is different from usual. The main points are the state functions representing the total energy, entropy and their spatially and temporarily varying densities. No attempt is made to “derive” these quantities and the corresponding equation/inequality from verbal statements of the first/second laws of thermodynamics. The emphasis is laid on the role of the entropy inequality and its distinction from the extremum principles based on asymptotic stability. Some controversies, which arose on the day of birth of thermodynamics and continue to trouble certain writers up to the present, are briefly mentioned. A very brief outline of the history is mentioned also, together with the “zoo” of researchers in thermodynamics.

List of symbols:

(a) Global quantities:

$$U = E + K$$

E, K

H

U^i, U^f and H^i, H^f

W and Q

\dot{W} and \dot{Q}

T

superimposed dot

P

W

T_o

Ψ

total energy

internal and kinetic energies

entropy

initial and final values of U and H in a process

total work and gain of heat in a process

rate of working and of gain of heat

absolute temperature

time derivative

canonical free energy

potential energy of loads

ambient temperature

free energy

(b) Bodies & densities:

P	continuous body or its part
dV	element of volume of P
∂P	boundary of P
dA	element of area of ∂P
\mathbf{n}	normal to ∂P
e, η, ψ	density of internal energy, entropy and free energy
$E = \int_P e dV$	internal energy
$H = \int_P \eta dV$	entropy
$\Psi = \int_P \psi dV$	free energy
$K = \frac{1}{2} \int_P \mathbf{v} ^2 dV$	kinetic energy
\mathbf{v}	velocity

(c) Fluxes, sources, geometry:

q	heat flux vector
r	rate of volumetric heating
ϕ	entropy flux
σ	the stress tensor
b	external force

(d) Harmonic oscillator:

u	vertical displacement
k	spring constant
c	damping coefficient
f	external force

(e) Stability theory: σ, τ

states of a dynamical system

 σ_0

initial state

 L

Liapunov function

(f) Van der Waals: p

pressure

 v

specific volume

 R

ideal gas constant

 a, b

positive constants

(g) Linear elasticity: ε

strain

 λ, μ

Lamé moduli

 u

displacement

Lecture I: Foundations and informal review

Synopsis *This lecture outlines the basic structure of thermodynamics. The central notions of energy and entropy are introduced. A clear distinction is made between the entropy inequality and the extremum principles. Both will be discussed in the subsequent lectures. Some comments on the interpretations and controversies are presented together with the main points of the history of thermodynamics. Only phenomenological thermodynamics is the concern here, no mention of statistical thermodynamics is made.*

I The two laws: their differences. The Clausius integral

- Cyclic processes

$$W = Q, \quad \int \frac{dQ}{T} \leq 0.$$

- General processes

$$U^f - U^i = W - Q, \quad H^f - H^i \geq \int \frac{dQ}{T}.$$

Entropy: Clausius (1854, 1865)

- Modern version: Duhem (1911)

$$\dot{H} \geq - \int_{\partial P} \frac{\mathbf{q} \cdot \mathbf{n}}{T} dA,$$

Truesdell-Toupin (1960)

$$\dot{H} \geq - \int_{\partial P} \frac{\mathbf{q} \cdot \mathbf{n}}{T} dA + \int_P \frac{r}{T} dV;$$

local version:

$$\dot{\eta} \geq - \operatorname{div} \left(\frac{\mathbf{q}}{T} \right) + \frac{r}{T}$$

⇒ the entropy flux is $\phi = \frac{q}{T}$,

- Müller's generalization (1967, 1972)

$$\dot{\eta} \geq \text{div } \phi$$

here ϕ is a general entropy flux, **given by an independent constitutive equation**. A detailed analysis shows that sometimes we have the proportionality

$$\phi = \Lambda q$$

and sometimes not.

2 Understanding the Second Law

- Carnot: for an engine working in a cycle, to produce positive work, is not only necessary to gain heat, but also to **emit some also**
- Classical formulations of the second law due to Clausius and Kelvin
- Gibbs:

“In thermodynamic problems, heat received at one temperature is by no means the equivalent of the same amount of heat received at another temperature. For example, a supply of a million calories at 150° is a very different thing from a supply of a million calories at 50° . But no such distinction exists in regard to work. This is a result of the general law, that heat can only pass from a hotter to a colder body, while work can be transferred by mechanical means from one fluid to any other, whatever may be the pressures. Hence, in thermodynamic problems, it is generally necessary to distinguish between the quantities of heat received or given out by the body at different temperatures, while as far as work is concerned, it is generally sufficient to ascertain the total amount performed.”

3 **Stability, minimum of energy, maximum of entropy**

- Clausius: for a thermally isolated system the entropy can only increase

$$\dot{H} \geq 0$$

until it reaches its maximum value at the final equilibrium state; if the system reaches the equilibrium, then

$$H_{\text{equilibrium}} \geq H$$

where H is the entropy of any other state. Principle of Maximum Entropy

- Duhem (1911): for a system that is not thermally isolated, but the temperature on its boundary is constant, the canonical free energy

(also called availability) P **decreases**

$$P := U + W - T_0 H, \quad W = \text{potential energy of loads}$$

if the system reaches the equilibrium, then

$$P_{\text{equilibrium}} \leq P$$

Principle of minimum energy

- Clausius:

“Die Energie der Welt ist constant.

Die Entropie der Welt strebt einem Maximum zu.”

the state of heat death

4 Restrictions on processes or on constitutive equations?

The mathematical role of the First law of thermodynamics is clear: the equation of balance of energy serves as an evolution equation for temperature. But what about the Second law? What is the mathematical role of the entropy *inequality*? Two extreme interpretations are possible:

(i) the response functions must be such as to satisfy the entropy inequality in every process

(ii) only those processes are admitted that satisfy the entropy inequality

It turns out that the nature has chosen a compromise: for smooth processes, Item (i) holds and for discontinuous ones (shock waves, propagating phase boundaries) (ii) holds true. (Subsequent lectures.)

5 Controversies

They exist from the early days of thermodynamics up to present. The main themes are the concept of state (equilibrium or non-equilibrium states), and the meaning of the entropy or absolute temperature away from equilibrium. It is often difficult to distinguish what is only a psychological ballast and conservativity from positive knowledge in these discussions. Notable participants in the discussions include G. Kichhoff, R. J. E. Clausius, M. Planck, C. Carathéodory, C. Truesdell, L. Onsager, I. Müller

6 Selected literature

- Classical thermodynamics Zemansky [10], Buchdahl [1].
- Classical nonequilibrium thermodynamics, De Groot-Mazur [2]
- Rational: Truesdell [9], Šilhavý [7]
- German extended: Müller [4], Müller-Ruggeri [6]
- Spanish extended: Jou, Casas-Vázquez, Lebon [3]

7 Brief history

- 1805 P. S. Laplace solves more than 100 year's problem of discrepancy between Newton's calculation of the speed of sound and experiment by assuming that the sound vibrations are adiabatic instead of Newton's assumption of isothermal nature of these (in modern terminology)
- 1824 Sadi Carnot's "Réflexions sur la puissance motrice du feu et sur les machines propres a développer cette puissance" (The second law)
- 1845–1850 R. Mayer, J. P. Joule, W. Thomson, H. Helmholtz conceive, partly independently, the first law of thermodynamics

- 1850–1865 R. J. E. Clausius publishes 9 memoirs in which he creates the equilibrium thermodynamics. The fourth memoir (1854) contains the germ of the notion of entropy. The sixth memoir (1862) coins the word “entropy” explicitly
- 1911 P. Duhem publishes a treatise in which he treats inhomogeneous processes and introduces the canonical free energy
- 1940–1947 C. Eckart, J. Meixner and I. Prigogine create the thermodynamics of irreversible processes based on the “principle of local equilibrium”
- 1960 C. Truesdell and R. A. Toupin formulate the Clausius-Duhem inequality

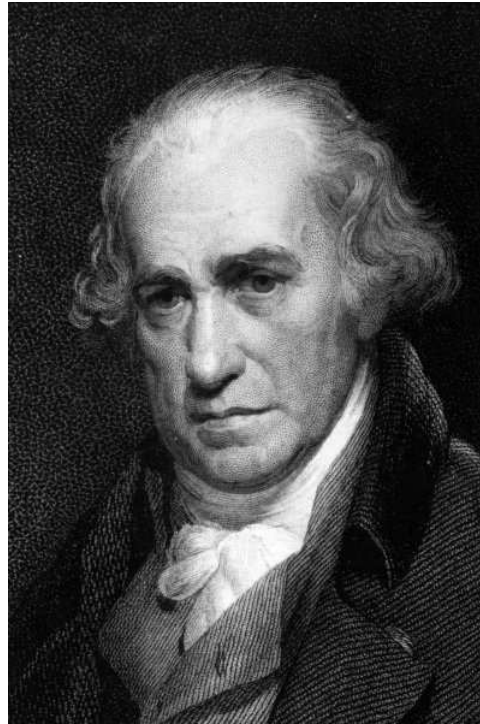
1963 B. D. Coleman and W. Noll publish “Elastic materials with heat conduction and viscosity”

The reader is referred to [8] for a detailed analysis of the early history of thermodynamics using present-day standards of rigor or to the gossipy account [5].

8 Picture gallery



Sir Isaac Newton (1643 Woolsthorpe-by-Colsterworth – 1728 London)
physicist, mathematician
“speed of sound”



Jamse Prescott Watt (1736 Greenock – 1819 Handsworth)

Scottish mechanist

“inventor steam engine”



Pierre Simon de Laplace (1749 Beaumont-en-Auge - 1827 Paris)

mathematician, physicist, astronomer

“correct calculation of the speed of sound”

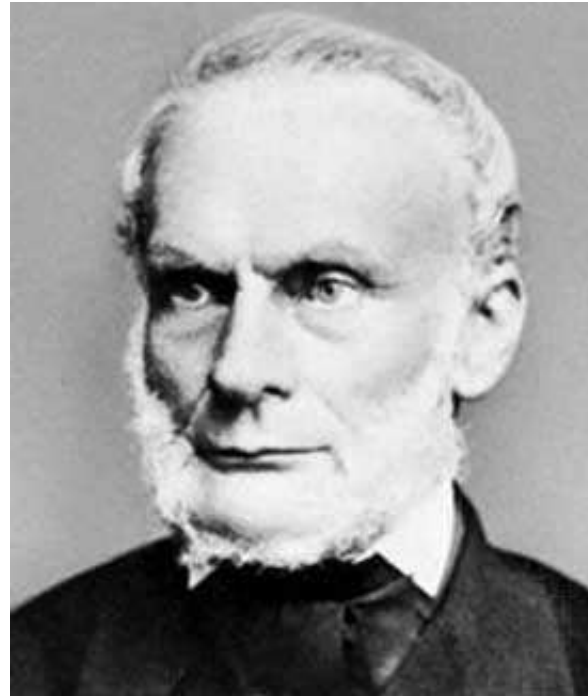


Nicolas Léonard Sadi Carnot (1796 Limoges – 1832 Lyon)
physicist, founder thermodynamics

“Carnot’s cycle; the second law without knowing the first law”



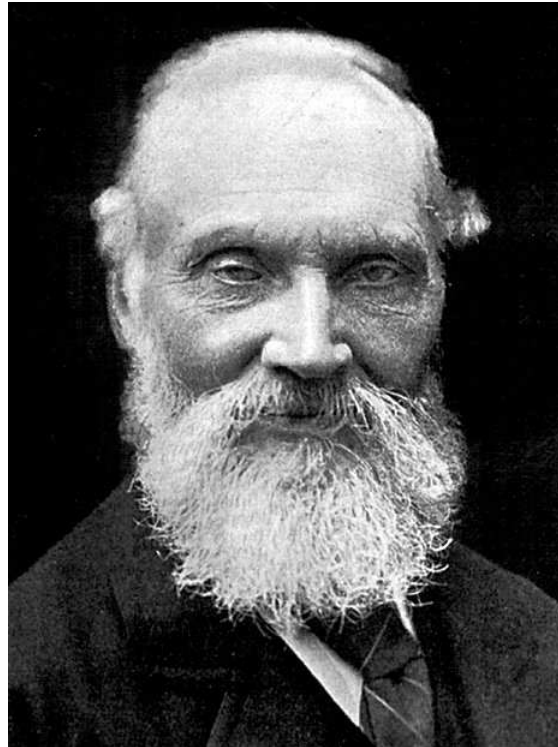
Hermann Ludwig Ferdinand von Helmholtz
(1821 Potsdam – 1894 Charlottenburg)
“Reichskanzler der Physik”
“The first law, Helmholtz free energy”



Rudolf Julius Emanuel Clausius (1822 – 1888)

physicist and mathematician

“founder of thermodynamics, entropy, heat death”



William Thomson, lord Kelvin of Largs (1824 Belfast – 1907 Netherhall)

Scottish physicist

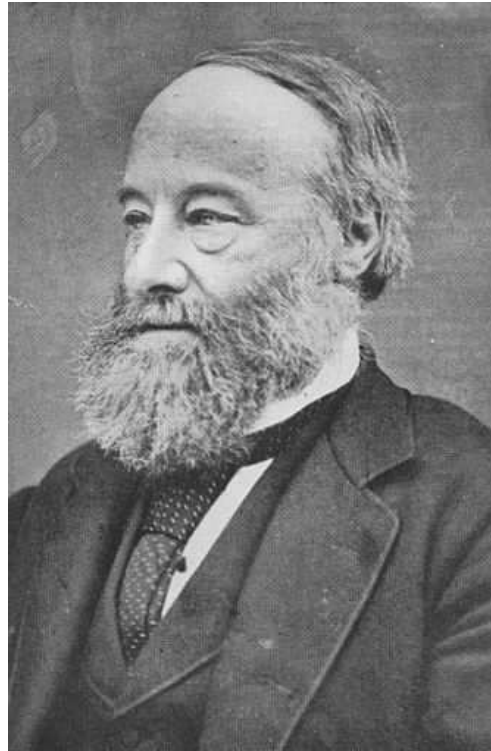
“absolute temperature”



Gustav Robert Kirchhoff (1824 Königsberg – 1887 Berlin)

physicist

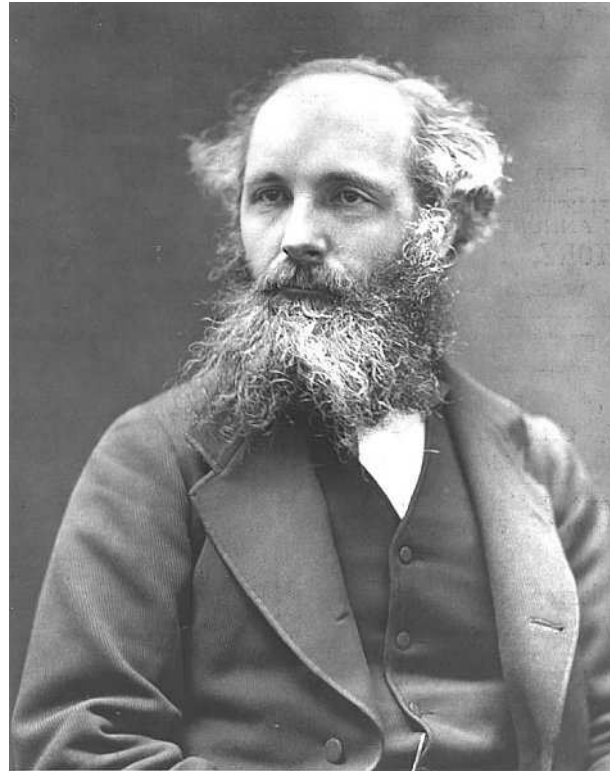
“interpretation of entropy”



James Prescott Joule (1825 Manchester – 1889 Sale)

physicist

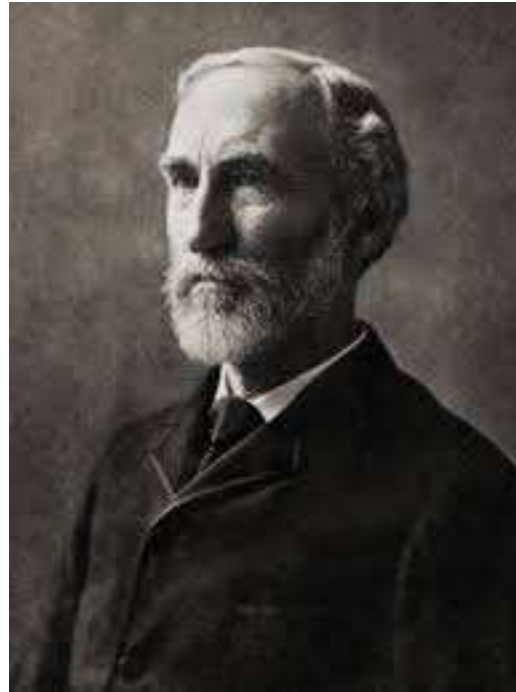
“the first law”



James Clerk Maxwell (1831 Edinburgh – 1879 Cambridge)

physicist

“statistical physics, Maxwell’s rule”



Josiah Willard Gibbs (1839 New Heaven – 1903 New Heaven)

mathematician, physicist and chemist

“Gibbs energy, phase rule, minimum and maximum principles”



Pierre Maurice Marie Duhem (1861 Paris – 1916 Cabrespine)

physicist, historian of science

“Clausius-Duhem inequality, canonical free energy”



Constantin Carathéodory (1873 Berlin – 1950 Munich)

Greek mathematician

“measure theory, Carathéodory’s axiomatics of thermodynamics”



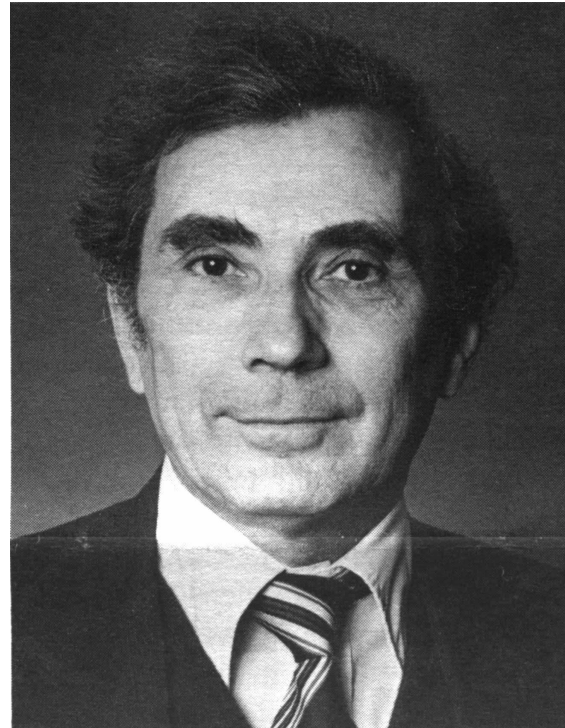
Lars Onsager (1903 Oslo – 1976 Coral Gables)

Norwegian physicist and chemist

“Nobel prize for chemistry, Onsager’s relations”



Clifford Ambrose Truesdell III (1919 Los Angeles – 2000 Baltimore)
natural philosopher and historian of science
“Clausius–Duhem inequality”



W. Noll (1925 Berlin –)

mathematician, continuum mechanist, thermodynamicist

“elastic materials with heat conduction and viscosity, phase rule”



James Burton Serrin (1926 Chicago – 2012 Minneapolis)
mathematician, continuum mechanist, thermodynamicist
“many areas, axiomatics of thermodynamics”



B. D. Coleman (1930 –)

mathematician, continuum mechanist, and chemist

“elastic materials with heat conduction and viscosity”

“materials with fading memory”

Lecture 2: Thermodynamics and material theory

Synopsis *This lecture pursues the interpretation of the entropy inequality as the means to derive the restrictions on the response functions of mathematical models of real bodies. This is demonstrated in detail on the sign of the damping coefficient of a spring. The same procedure applies to thermoelastic materials; the difference is only formal. The derivation is omitted, but the results are discussed in detail. Some other classes of materials are mentioned also.*

I The free energy

Defined by

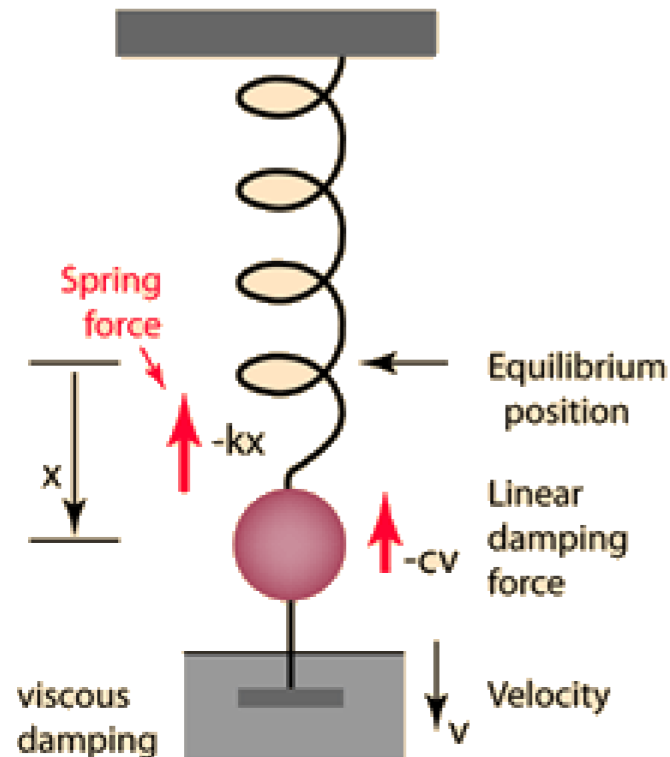
$$\Psi = U - T H;$$

then

$$\dot{\Psi} \leq \dot{W} - H\dot{T}.$$

2 Introductory example: the damping (viscosity) coefficient

Forced damped harmonic oscillator



Equation of motion

$$\ddot{u} = -ku - c\dot{u} + f,$$

free energy

$$\Psi = \frac{1}{2}(\dot{u}^2 + ku^2),$$

rate of working

$$\dot{W} = f\dot{u}.$$

2 The sign of the damping coefficient

The isothermal second law of thermodynamics: Inequality

$$\dot{\Psi} \leq \dot{W}$$

must be satisfied in every process, i.e., for every external force $f = f(t)$ (the Coleman-Noll interpretation).

Equation of motion \Rightarrow

$$\dot{\Psi} = \dot{u}\ddot{u} + ku\dot{u} = \dot{u}(-ku - c\dot{u} + f) + ku\dot{u} = -c\dot{u}^2 + f\dot{u}.$$

\Rightarrow second law of thermodynamics requires

$$\underbrace{-c\dot{u}^2 + f\dot{u}}_{\text{change of free energy}} \leq \underbrace{f\dot{u}}_{\text{rate of working}}$$

$$\begin{aligned} &\Downarrow \\ -c\dot{u}^2 &\leq 0 \\ &\Downarrow \\ c &\geq 0 \end{aligned}$$

Remark: no restriction on the sign of k !

3 Stability, instability and the heath death of the universe

Autonomous case (no force)

$$\ddot{u} = -ku - c\dot{u}, \quad c > 0 \quad (\text{the second law}).$$

Seek solutions in the form

$$u = e^{\lambda t} \quad \text{where } \lambda \text{ is a complex exponent;}$$

characteristic equation

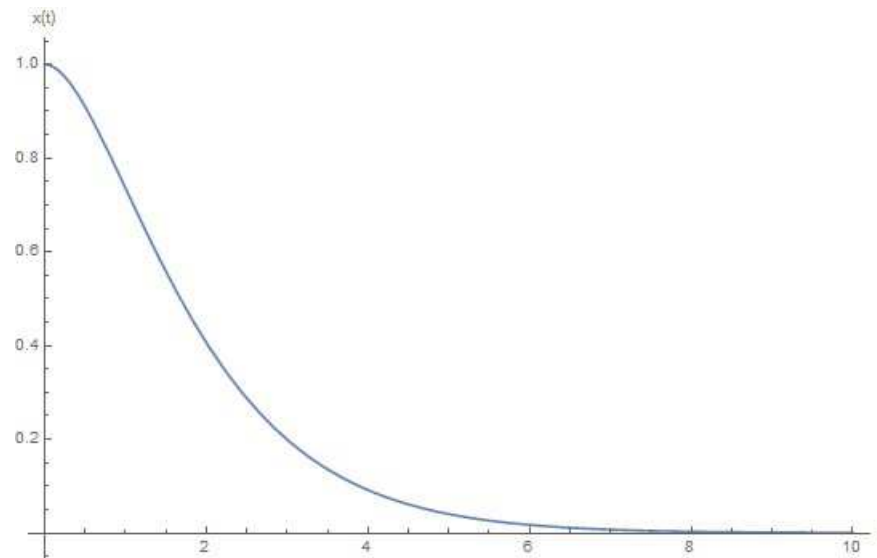
$$\lambda^2 + c\lambda + k = 0;$$
$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4k}}{2}.$$

Four cases:

1. Overdamped:

$$k > 0, \quad c^2 - 4k > 0$$

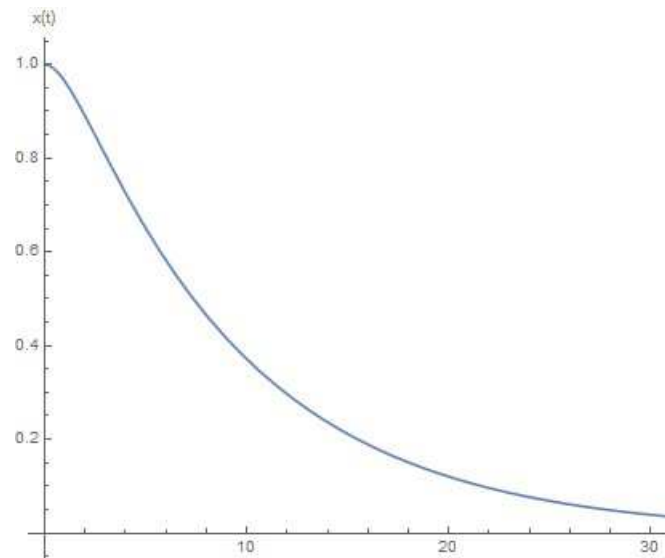
The system exponentially decays to steady state without oscillating



2. Critical damping

$$k > 0, \quad c^2 - 4k = 0$$

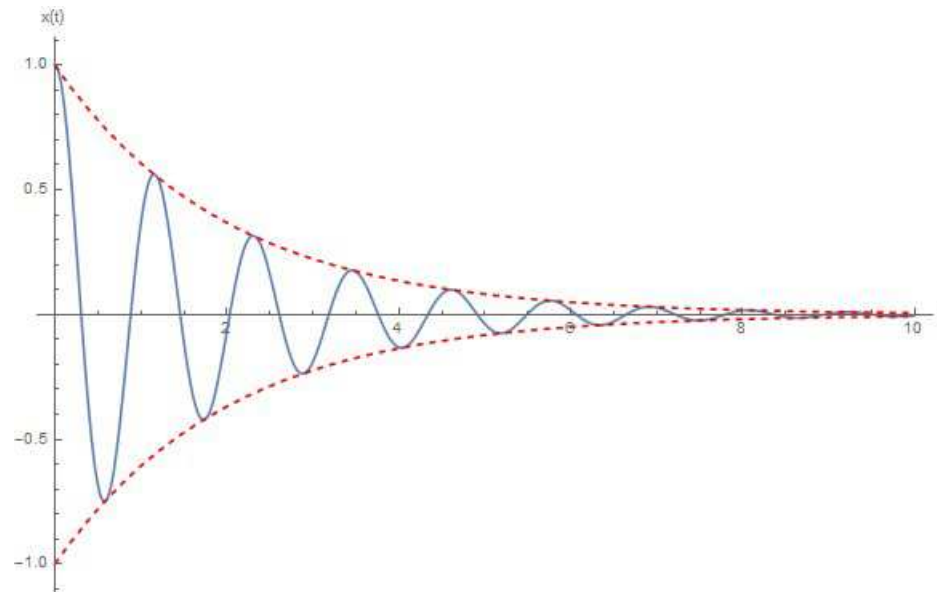
The system returns to steady state without oscillating



3. Underdamped

$$k > 0, \quad c^2 - 4k < 0$$

The system oscillates with the amplitude gradually decreasing to zero



4. Unstable

$$k < 0$$

the system exponentially escapes to infinity (fully compatible with the second law)

4 Continuum mechanics and its relation to thermodynamics

Basic equations:

$$(e + \frac{1}{2}v^2)' = \text{div}(\sigma^T v - q) + v \cdot b + r$$

$$\dot{v} = \text{div} \sigma + b,$$

$$\dot{\eta} \geq -\text{div} \left(\frac{q}{T} \right) + \frac{r}{T}.$$

5 Elastic materials with heat conduction and viscosity

Constitutive equations, with $\mathbf{F} = \nabla \mathbf{y}$ where \mathbf{y} is the deformation, and $\mathbf{G} = \nabla T$, read

$$e = \bar{e}(\mathbf{F}, T, \mathbf{G}, \dot{\mathbf{F}}), \quad \eta = \bar{\eta}(\mathbf{F}, T, \mathbf{G}, \dot{\mathbf{F}}),$$

$$\boldsymbol{\sigma} = \bar{\mathbf{S}}(\mathbf{F}, T, \mathbf{G}, \dot{\mathbf{F}}), \quad \mathbf{q} = \bar{\mathbf{q}}(\mathbf{F}, T, \mathbf{G}, \dot{\mathbf{F}}).$$

For example:

$$\eta = T \log T, \quad \mathbf{q} = -\kappa \nabla T.$$

The equilibrium and dynamic responses

$$\bar{\mathbf{S}}_e(\mathbf{F}, T) = \bar{\mathbf{S}}(\mathbf{F}, T, \mathbf{0}, \mathbf{0}),$$

the dynamic stress response functions

$$\bar{\mathbf{S}}_d(\mathbf{F}, T, \mathbf{G}, \dot{\mathbf{F}}) = \bar{\mathbf{S}}(\mathbf{F}, T, \mathbf{G}, \dot{\mathbf{F}}) - \bar{\mathbf{S}}_e(\mathbf{F}, T),$$

Define

$$\bar{\psi} = \bar{e} - T\eta.$$

Coleman & Noll 1963, “**Coleman-Noll procedure**” then:

the equilibrium thermostatic relations

$$\begin{aligned} \bar{\mathbf{S}}_e(\mathbf{F}, T) &= \partial_{\mathbf{F}} \bar{\psi}(\mathbf{F}, T), \\ \bar{\eta}(\mathbf{F}, T) &= -\partial_T \bar{\psi}(\mathbf{F}, T), \quad \bar{e}(\mathbf{F}, T) = -T^2 \partial_T (\bar{\psi}(\mathbf{F}, T)/T), \end{aligned}$$

the internal dissipation inequality

$$\bar{\mathbf{S}}_d(\mathbf{F}, T, \mathbf{G}, \dot{\mathbf{F}}) \cdot \dot{\mathbf{F}} - \bar{\mathbf{q}}(\mathbf{F}, T, \mathbf{G}, \dot{\mathbf{F}}) \cdot \mathbf{G}/T \geq 0,$$

For example:

$$\mathbf{q} = -\kappa \operatorname{grad} T, \quad \kappa \geq 0.$$

6 Types of materials: a review

- thermoelastic materials:

$$f = \bar{\psi}(\mathbf{F}, T, \nabla T),$$

- materials of rate type:

$$f = \bar{\psi}(\mathbf{F}, T, \nabla T, \dot{\mathbf{F}});$$

- materials with fading memory

$$f(t) = \int_{s=-\infty}^{s=t} \mathfrak{F}(F(s), T(s), \nabla T(s))$$

- Materials with fading memory (Coleman 1964) “**Coleman’s relations**”

$$\mathfrak{h} = -D_T \mathfrak{F}, \quad \mathfrak{G} = D_F \mathfrak{F}$$

plus the “**residual dissipation inequality.**”

- plastic materials ...

7 Researches on the foundations of thermodynamics W. A. Day (1968, 1969), B. D. Coleman & D. R. Owen (1974), J. Serrin, M. Šilhavý, L. Deseri & G. del Piero (1990), M. Fabrizio, etc. etc.

Lecture 3: Thermodynamics and stability

Synopsis *This lecture discusses the extremum principles of thermodynamics (maximum entropy, minimum canonical free energy) in more detail. Their relationship to asymptotic stability of equilibrium is established. Further, these principles are treated as problems of the calculus of variations as they are shown to imply the equilibrium equations for both continuous and discontinuous states. The extremum principles are illustrated on a Van der Waals fluid and on linear elasticity.*

I Stability theory

Behavior of evolving systems in a long interval of time under (small) perturbations of initial conditions.

Two types of stability of **equilibrium states**:

- Liapunov stability: a nearby initial state leads to an evolution which remains close to the equilibrium state? If “yes,” the state is Liapunov stable.
- Asymptotic stability: will the process emanating from a perturbed initial state converge to the equilibrium state? If “yes,” the state is asymptotically stable.

Harmonic oscillator:

- if ideal (undamped) then Liapunov stable but not asymptotically stable;
- if damped then both Liapunov and asymptotically stable.

2 Example: a contractive map

Discrete time, $k = 0, 1, \dots$, states: elements of \mathbb{R}^n , evolution law:

$$\sigma_{k+1} = f(\sigma_k)$$

where f is a possibly nonlinear map from \mathbb{R}^n into itself; σ is an equilibrium state if

$$\sigma = f(\sigma);$$

f is **contractive** if

$$|f(\sigma) - f(\tau)| \leq c |\sigma - \tau|, \quad \sigma, \tau \in \mathbb{R}^n,$$

where $c < 1$.

Theorem If f is contractive then the equilibrium state is asymptotically stable.

Example Let us identify the state space with the complex plane \mathbb{C} , and put

$$f(\sigma) = \alpha\sigma, \quad \sigma \in \mathbb{C}$$

where $\alpha \in \mathbb{C}$ with $|\alpha| < 1$. Then f is contractive. If $\sigma_0 \in \mathbb{C}$ is the initial state then

$$\sigma_1 = \alpha\sigma_0, \quad \sigma_2 = \alpha^2\sigma_0, \quad \dots, \quad \sigma_k = \alpha^k\sigma_0, \quad \dots$$

and

$$\sigma_k \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

3 Liapunov functions

- A function L of state of an evolving system is said to be a Liapunov function if it decreases (or stays constant) along any trajectory. **A system can, but need not, have a Liapunov function.**

Example: in a discrete time system with a contractive map, the distance from the equilibrium state is a Liapunov function.

Theorem A: (A minimum of a Liapunov function) If an evolving system has a Liapunov function and the equilibrium state σ is asymptotically stable then

$$L(\tau) \geq L(\sigma)$$

for any state y , where σ is the equilibrium state and $\alpha > 0$, then σ is Liapunov stable.

Theorem B: If an evolving system has a Liapunov function such that

$$L(\tau) - L(\sigma) \geq \alpha |\tau - \sigma|$$

for any state y , where σ is the equilibrium state and $\alpha > 0$, then σ is Liapunov stable.

Theorem C: If an evolving system has a Liapunov function such that

$$L(\tau) - L(\sigma) \geq \alpha |\tau - \sigma|$$

for any state y , where σ is the equilibrium state and $\alpha > 0$, and

$$\dot{L} \leq -\beta L$$

along any trajectory where $\beta > 0$, then σ is asymptotically stable.

4 Thermodynamic Liapunov functions

How is all this related to thermodynamics? *The thermodynamics provides natural Liapunov functions of bodies in various types of environments:*

- The negative of entropy, $-H$, is a Liapunov function for a thermally insulated body since

$$\dot{H} \geq 0,$$

- If the boundary of a body is has a fixed temperature T_0 (in time and space over the boundary), then the canonical free energy $P = U + W - T_0 H$, is a Liapunov function since

$$\dot{P} \leq 0$$



if the equilibrium state is asymptotically stable then

$$H_{\text{equilibrium}} \geq H \quad \text{or} \quad P_{\text{equilibrium}} \leq P;$$

i.e., minimum principles.

5 Example: Van der Waals fluid The state equation

$$p(v) = \frac{RT}{v-b} - \frac{a}{v^2}, \quad v > b$$

The energy ψ defined by

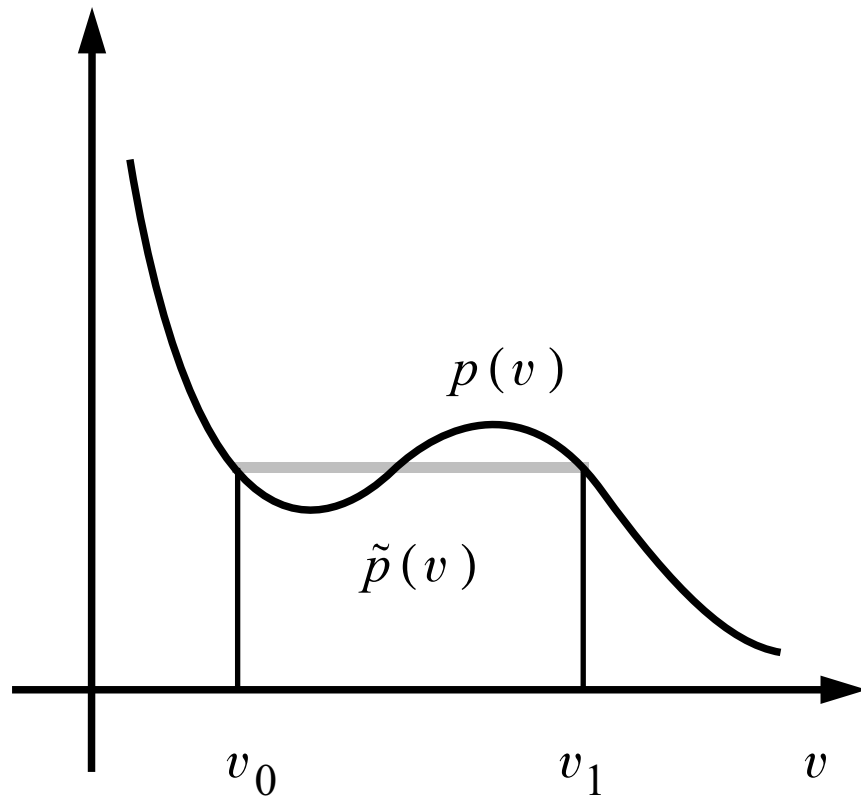
$$p = -\frac{d\psi}{dv}$$

$$\psi(v) = -RT \ln(v-b) - a/v, \quad v > b.$$

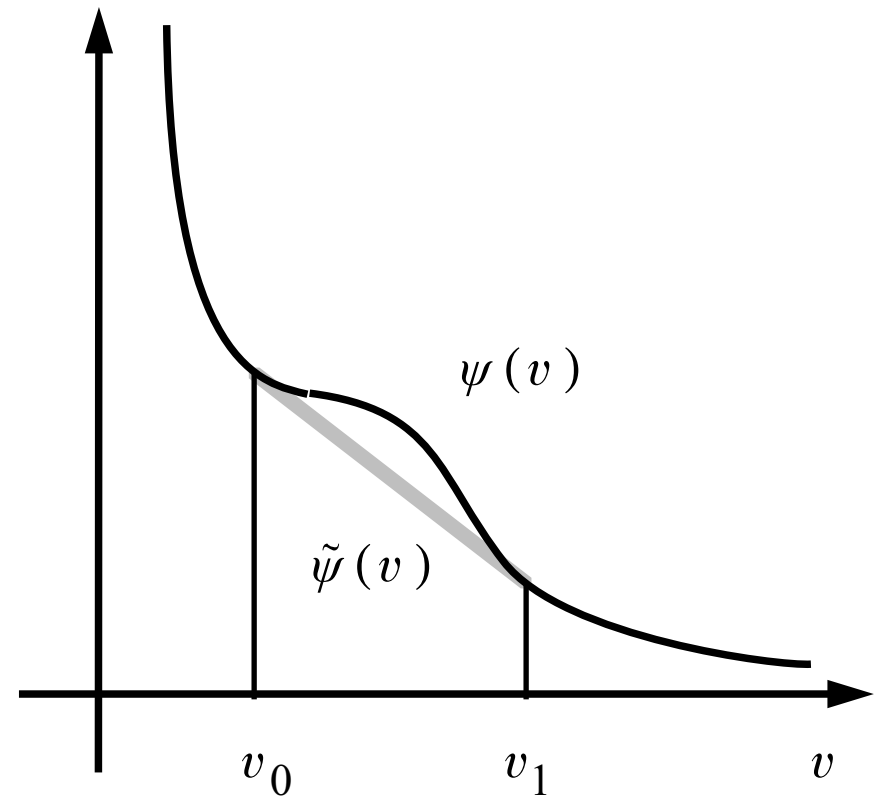
If

$$T < T_c := \frac{8a}{27Rb}$$

then the pressure function is nonmonotone and ψ nonconvex:



(a) Pressure



(b) (Free) energy

Inhomogeneous states in the one-dimensional situation, described by the specific volume $v : [0, 1] \rightarrow (b, \infty)$. The total volume and energy

$$V(v) := \int_0^1 v \, dx, \quad \Psi(v) := \int_0^1 \psi(v) \, dx.$$

Stable equilibrium states: minimizers of energy at fixed total volume.

Thus we are led to

$$\tilde{\psi}(w) := \inf \{ \Psi(v) : V(v) = w \}.$$

Conclusions:

(i) *the problem has a minimizer for each $w > b$, i.e., there exists a state v such that*

$$V(v) = w \quad \text{and} \quad \tilde{\psi}(w) = \Psi(v);$$

(ii) *exists a unique pair v_0, v_1 of points such that*

$$p(v_0) = p(v_1), \quad g(v_0) = g(v_1),$$

where

$$g(v) = \psi(v) + p(v)v$$

is the Gibbs function;

(iii) *for w outside the interval (v_0, v_1) the minimizer is unique, namely,*

$$v \equiv w = \text{const};$$

for w in the interval (v_0, v_1) infinitely many minimizers: all states v such that

$$v(x) \in \{v_0, v_1\} \text{ on } [0, 1] \text{ and } V(v) = w;$$

(iv) the minimum energy $\tilde{\psi}$ is given by

$$\tilde{\psi}(w) = \begin{cases} \psi(w) \\ \frac{\psi(v_0)(v_1 - w) + \psi(v_1)(w - v_0)}{v_1 - v_0} \end{cases}$$

if w is outside or inside the interval (v_0, v_1) , respectively.

(v) the total energy $\tilde{\psi}$ is the convexification of ψ , i.e., the largest convex function not exceeding ψ .

Morals:

- Interpretation as a phase transition (Maxwell, Gibbs):
 - (a) Maxwell's equal area rule
 - (a) Gibbs' common tangent construction

- in higher dimensions we need the semiconvexity notions: quasiconvexity, rank 1 convexity, polyconvexity
- $\tilde{\psi}$: the relaxed functional
- in no conflict with the balance of energy. $\tilde{\psi}$ does not occur in the equation of balance of energy.

6 Stability analysis of an isotropic linearly elastic body

Hooke's law

$$\begin{aligned}\boldsymbol{\sigma} &= \lambda (\operatorname{tr} \boldsymbol{\varepsilon}) \mathbf{1} + 2\mu \boldsymbol{\varepsilon}, \\ \psi(\boldsymbol{\varepsilon}) &= \frac{1}{2}(\lambda (\operatorname{tr} \boldsymbol{\varepsilon})^2 + 2\mu |\boldsymbol{\varepsilon}|^2)\end{aligned}$$

The total energy of the body P is a **functional**

$$P(\mathbf{u}) = \int_P \psi(\boldsymbol{\varepsilon}) dV + \text{the potential energy of the loads}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is a (generally inhomogeneous) displacement of the point $\mathbf{x} \in P$, with $\boldsymbol{\varepsilon}$ varying on P .

Homogeneous states \mathbf{u}_0 : $\boldsymbol{\varepsilon}$ constant over P .

Stability of a homogeneous state \mathbf{u}_0

$$P(\mathbf{u}) \geq P(\mathbf{u}_0)$$

for any state \mathbf{u} satisfying the same boundary conditions as \mathbf{u}_0 .

Since Ψ is a quadratic functional,

one state is stable \Leftrightarrow any state is stable:

suffices to examine the stability of the undeformed state $\mathbf{u}_0 = \mathbf{0}$.

Three situations:

Dirichlet's boundary conditions The boundary is fixed;

$$P(\mathbf{u}) \geq P(\mathbf{u}_0) = 0 \quad \text{for all } \mathbf{u} \text{ satisfying } \mathbf{u} = 0 \text{ on } \partial P$$

Necessary and sufficient conditions for the stability of \mathbf{u}_0 is

$$\mu \geq 0, \quad \lambda + 2\mu \geq 0.$$

If these conditions are violated then the body reduces its energy by creating a microstructure (Biot's instabilities)

Mixed boundary conditions Part \mathcal{D} of the boundary fixed and part \mathcal{S} is free

$$P(\mathbf{u}) \geq P(\mathbf{u}_0) = 0 \quad \text{for all } \mathbf{u} \text{ satisfying } \begin{cases} \boldsymbol{\sigma} \mathbf{v} = \mathbf{0} & \text{on } \mathcal{S}, \\ \mathbf{u} = \mathbf{0} & \text{on } \mathcal{D}. \end{cases}$$

The stability conditions depend strongly on the shape of P and on \mathcal{S} and \mathcal{D} ; necessary conditions are

$$\mu \geq 0, \quad \lambda + \mu \geq 0.$$

If violated, the free part of the boundary exhibits surface wrinkling for the body to reduce the energy.

Neumann's boundary condition The whole boundary is free.

$$P(\mathbf{u}) \geq P(\mathbf{u}_0) = 0 \quad \text{for all } \mathbf{u} \text{ satisfying } \boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{0} \text{ on } \partial P$$

Necessary and sufficient:

$$\mu \geq 0, \quad 3\lambda + 2\mu \geq 0.$$

If violated, all the preceding instabilities can occur.

7 States with discontinuous strain: equilibrium conditions

The Van der Waals fluid exhibits equilibrium states with an interface \mathcal{I} across which the specific volume (\Leftrightarrow density) has a jump discontinuity on \mathcal{I} ,

$$v^+ \neq v^-.$$

Equilibrium of forces

$$[p] = 0 \quad \text{on } \mathcal{I}$$

where for any quantity h

$$[h] := h^+ - h^-$$

where h^\pm are the limits of h from the two sides of \mathcal{I} . The requirement of minimum energy gives an **extra condition** on the equality of Gibbs' function

$$g(v) = \psi(v) + p(v)v,$$

viz,

$$[g] = 0.$$

Solids: interfaces of discontinuity of the strain exist also (martensite/austenite interface, shape memory alloys). What are the conditions of equilibrium and stability?

Equilibrium of forces

$$[\sigma]v = \mathbf{0} \quad \text{on } \mathcal{I}$$

Minimum of energy (stability):

$$[\gamma] \mathbf{v} = \mathbf{0} \quad \text{on } \mathcal{I}$$

where

$$\gamma = \psi(\boldsymbol{\varepsilon}) \mathbf{1} - \boldsymbol{\varepsilon} \boldsymbol{\sigma}$$

is the Eshelby tensor.

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