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**On the Darboux problem for linear
hyperbolic functional-differential
equations**

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ON THE DARBOUX PROBLEM FOR LINEAR HYPERBOLIC FUNCTIONAL-DIFFERENTIAL EQUATIONS

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ABSTRACT. Theorems on the Fredholm alternative and well-posedness of the Darboux problem

$$\frac{\partial^2 u(t, x)}{\partial t \partial x} = \ell(u)(t, x) + q(t, x),$$

$$u(t, x_0) = \varphi(t) \quad \text{for } t \in [a, b], \quad u(t_0, x) = \psi(x) \quad \text{for } x \in [c, d]$$

are established, where $\ell: C(\mathcal{D}; \mathbb{R}) \rightarrow L(\mathcal{D}; \mathbb{R})$ is a linear bounded operator, $q \in L(\mathcal{D}; \mathbb{R})$, $t_0 \in [a, b]$, $x_0 \in [c, d]$, $\varphi: [a, b] \rightarrow \mathbb{R}$, $\psi: [c, d] \rightarrow \mathbb{R}$ are absolutely continuous functions, and $\mathcal{D} = [a, b] \times [c, d]$. New sufficient conditions are also given for the existence and uniqueness of a Carathéodory solution to the problem considered. The general results are applied to a hyperbolic equation with argument deviations and, moreover, for the equation without argument deviations an integral representation of solutions to the Darboux problem is derived in this preprint.

1. INTRODUCTION

On the rectangle $\mathcal{D} = [a, b] \times [c, d]$, we consider the linear partial functional-differential equation

$$\frac{\partial^2 u(t, x)}{\partial t \partial x} = \ell(u)(t, x) + q(t, x), \tag{1.1}$$

where $\ell: C(\mathcal{D}; \mathbb{R}) \rightarrow L(\mathcal{D}; \mathbb{R})$ is a linear bounded operator and $q \in L(\mathcal{D}; \mathbb{R})$. As usual, $C(\mathcal{D}; \mathbb{R})$ and $L(\mathcal{D}; \mathbb{R})$ denote the Banach spaces of continuous and Lebesgue integrable functions, respectively, equipped with the standard norms.

A function $u: \mathcal{D} \rightarrow \mathbb{R}$ absolutely continuous on \mathcal{D} in the sense of Carathéodory (see Proposition 2.1) is said to be a solution to equation (1.1) if it satisfies equality (1.1) almost everywhere on the set \mathcal{D} .

Various initial and boundary value problems for hyperbolic differential equations and their systems are studied in literature (see, e.g., [3, 7–12, 15, 22, 24, 25] and references therein). We shall consider the characteristic initial value problem, usually called Darboux problem. In this case, the values of a solution u to equation (1.1) are prescribed on both characteristics $t = t_0$ and $x = x_0$, i.e., the initial conditions are

$$u(t, x_0) = \varphi(t) \quad \text{for } t \in [a, b], \quad u(t_0, x) = \psi(x) \quad \text{for } x \in [c, d], \tag{1.2}$$

where $t_0 \in [a, b]$, $x_0 \in [c, d]$, and $\varphi: [a, b] \rightarrow \mathbb{R}$, $\psi: [c, d] \rightarrow \mathbb{R}$ are absolutely continuous functions such that $\varphi(t_0) = \psi(x_0)$.

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A particular case of problem (1.1), (1.2) (if $t_0 = a$ and $x_0 = c$) is studied in the paper [19]. The aim of this preprint is to generalize the paper mentioned and prove theorems on the Fredholm alternative and well-posedness of problem (1.1), (1.2) (see Sections 4 and 7). Moreover, some solvability conditions for the problem considered are given in Section 6, and equations with the so-called Volterra operators are studied as well. The results obtained are applied to the equation with deviating arguments

$$\frac{\partial^2 u(t, x)}{\partial t \partial x} = p(t, x)u(\tau(t, x), \mu(t, x)) + q(t, x), \quad (1.1')$$

where $p, q \in L(\mathcal{D}; \mathbb{R})$ and $\tau: \mathcal{D} \rightarrow [a, b]$, $\mu: \mathcal{D} \rightarrow [c, d]$ are measurable functions.

2. NOTATIONS AND PRELIMINARY RESULTS

The following notation is used throughout the paper.

- (i) \mathbb{N} is the set of all natural numbers. \mathbb{R} is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$. $\text{Ent}(x)$ denotes the entire part of the number $x \in \mathbb{R}$.
- (ii) $\mathcal{D} = [a, b] \times [c, d]$, where $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$.
- (iii) The first and the second order partial derivatives of a function $v: \mathcal{D} \rightarrow \mathbb{R}$ at the point $(t, x) \in \mathcal{D}$ are denoted by $v'_{[1]}(t, x)$ (or $v_t(t, x)$, $\frac{\partial v(t, x)}{\partial t}$), $v'_{[2]}(t, x)$ (or $v_x(t, x)$, $\frac{\partial v(t, x)}{\partial x}$), $v''_{[1,2]}(t, x)$ (or $v_{tx}(t, x)$, $\frac{\partial^2 v(t, x)}{\partial t \partial x}$), and $v''_{[2,1]}(t, x)$ (or $v_{xt}(t, x)$, $\frac{\partial^2 v(t, x)}{\partial x \partial t}$).
- (iv) $C(\mathcal{D}; \mathbb{R})$ is the Banach space of continuous functions $v: \mathcal{D} \rightarrow \mathbb{R}$ equipped with the norm $\|v\|_C = \max \{|v(t, x)| : (t, x) \in \mathcal{D}\}$.
- (v) $AC([\alpha, \beta]; \mathbb{R})$, where $-\infty < \alpha < \beta < +\infty$, is the set of absolutely continuous functions $u: [\alpha, \beta] \rightarrow \mathbb{R}$.
- (vi) $C^*(\mathcal{D}; \mathbb{R})$ is the set of functions $v: \mathcal{D} \rightarrow \mathbb{R}$ admitting the representation
$$v(t, x) = e + \int_a^t k(s)ds + \int_c^x l(\eta)d\eta + \int_a^t \int_c^x f(s, \eta)d\eta ds \quad \text{for } (t, x) \in \mathcal{D},$$
where $e \in \mathbb{R}$, $k \in L([a, b]; \mathbb{R})$, $l \in L([c, d]; \mathbb{R})$, and $f \in L(\mathcal{D}; \mathbb{R})$. Equivalent definitions of the class $C^*(\mathcal{D}; \mathbb{R})$ are presented in Proposition 2.1 below.
- (vii) $L(\mathcal{D}; \mathbb{R})$ is the Banach space of Lebesgue integrable functions $p: \mathcal{D} \rightarrow \mathbb{R}$ equipped with the norm $\|p\|_L = \iint_{\mathcal{D}} |p(t, x)| dt dx$.
- (viii) $\mathcal{L}(\mathcal{D})$ is the set of linear bounded operators $\ell: C(\mathcal{D}; \mathbb{R}) \rightarrow L(\mathcal{D}; \mathbb{R})$.
- (ix) $\text{meas } A$ denotes the Lebesgue measure of the set $A \subset \mathbb{R}^m$, $m = 1, 2$.
- (x) If X, Y are Banach spaces and $T: X \rightarrow Y$ is a linear bounded operator then $\|T\|$ denotes the norm of the operator T , i. e.,

$$\|T\| = \sup \{\|T(z)\|_Y : z \in X, \|z\|_X \leq 1\}.$$

The following proposition dealing with the equivalent characterizations of functions absolutely continuous in the sense of Carathéodory plays very important role in our investigation.

Proposition 2.1 ([18, Theorem 2.1]). *The following three statements are equivalent:*

- (1) *the function $v: \mathcal{D} \rightarrow \mathbb{R}$ is absolutely continuous on \mathcal{D} in the sense of Carathéodory¹;*

¹This notion is introduced in [2] (see also [18]).

- (2) $v \in C^*(\mathcal{D}; \mathbb{R})$;
(3) the function $v: \mathcal{D} \rightarrow \mathbb{R}$ satisfies the conditions:
(a) $v(\cdot, x) \in AC([a, b]; \mathbb{R})$ for every $x \in [c, d]$, $v(a, \cdot) \in AC([c, d]; \mathbb{R})$;
(b) $v'_{[1]}(t, \cdot) \in AC([c, d]; \mathbb{R})$ for almost every $t \in [a, b]$;
(c) $v''_{[1,2]} \in L(\mathcal{D}; \mathbb{R})$.

Remark 2.1. It is clear that the conditions (3a)–(3c) stated in the previous proposition can be replaced by the symmetric ones, i. e.,

- (3) the function $v: \mathcal{D} \rightarrow \mathbb{R}$ satisfies the conditions:
(A) $v(\cdot, c) \in AC([a, b]; \mathbb{R})$, $v(t, \cdot) \in AC([c, d]; \mathbb{R})$ for every $t \in [a, b]$;
(B) $v'_{[2]}(\cdot, x) \in AC([a, b]; \mathbb{R})$ for almost every $x \in [c, d]$;
(C) $v''_{[2,1]} \in L(\mathcal{D}; \mathbb{R})$.

Remark 2.2 ([18, Remark 2.2]). For an arbitrary function $v \in C^*(\mathcal{D}; \mathbb{R})$, we have $v'_{[1]}, v'_{[2]} \in L(\mathcal{D}; \mathbb{R})$, the equality

$$v''_{[1,2]}(t, x) = v''_{[2,1]}(t, x) \quad \text{for a. e. } (t, x) \in \mathcal{D}$$

holds, and

$$v'_{[1]}(t, x) \leq \alpha(t), \quad v'_{[2]}(t, x) \leq \beta(x) \quad \text{for a. e. } (t, x) \in \mathcal{D},$$

where $\alpha \in L([a, b]; \mathbb{R})$ and $\beta \in L([c, d]; \mathbb{R})$.

3. AUXILIARY STATEMENTS

The following proposition plays a crucial role in the proofs of statements given in Sections 4, 6, and 7.

Proposition 3.1. *Let $t_0 \in [a, b]$, $x_0 \in [c, d]$, and $\ell \in \mathcal{L}(\mathcal{D})$. Then the operator $T: C(\mathcal{D}; \mathbb{R}) \rightarrow C(\mathcal{D}; \mathbb{R})$ defined by the formula*

$$T(v)(t, x) = \int_{t_0}^t \int_{x_0}^x \ell(v)(s, \eta) d\eta ds \quad \text{for } (t, x) \in \mathcal{D}, \quad v \in C(\mathcal{D}; \mathbb{R}) \quad (3.1)$$

is completely continuous.

The above statement can be easily proved in the case where the operator ℓ is strongly bounded, i. e., if there exists a function $\eta \in L(\mathcal{D}; \mathbb{R}_+)$ such that

$$|\ell(v)(t, x)| \leq \eta(t, x) \|v\|_C \quad \text{for a. e. } (t, x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D}; \mathbb{R}). \quad (3.2)$$

However, H. H. Schaefer proved that there exists an operator $\ell \in \mathcal{L}(\mathcal{D})$ which is not strongly bounded (see [17]). To prove Proposition 3.1 without the additional requirement (3.2) we need several notions and statements from functional analysis.

Definition 3.1 ([5, Definition II.3.25]). Let X be a Banach space, X^* be its dual space.

We say that a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ is weakly convergent if there exists $x \in X$ such that $f(x) = \lim_{n \rightarrow +\infty} f(x_n)$ for every $f \in X^*$. The element x is said to be the weak limit of this sequence.

A set $M \subseteq X$ is said to be weakly sequentially compact if every sequence of elements from M contains a subsequence which is weakly convergent in X .

A sequence $\{x_n\}_{n=1}^{+\infty}$ of elements from X is called a weak Cauchy sequence if $\{f(x_n)\}_{n=1}^{+\infty}$ is a Cauchy sequence in \mathbb{R} for every $f \in X^*$.

We say that the space X is weakly complete if every weak Cauchy sequence of elements from X possesses a weak limit in X .

Definition 3.2 ([5, Definition Vi.4.1]). Let X and Y be Banach spaces, $T : X \rightarrow Y$ be a linear bounded operator. The operator T is said to be weakly compact if it maps bounded sets in X into weakly sequentially compact subsets of Y .

Definition 3.3. We say that a set $M \subseteq L(\mathcal{D}; \mathbb{R})$ has a property of absolutely continuous integral if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that the relation

$$\left| \iint_E p(t, x) dt dx \right| < \varepsilon \quad \text{for every } p \in M$$

holds whenever a measurable set $E \subseteq \mathcal{D}$ is such that $\text{meas } E < \delta$.

The following three lemmas can be found in [5].

Lemma 3.1 (Theorem IV.8.6). *The space $L(\mathcal{D}; \mathbb{R})$ is weakly complete.*

Lemma 3.2 (Theorem VI.7.6). *A linear bounded operator mapping the space $C(\mathcal{D}; \mathbb{R})$ into a weakly complete Banach space is weakly compact.*

Lemma 3.3 (Theorem IV.8.11). *If a set $M \subseteq L(\mathcal{D}; \mathbb{R})$ is weakly sequentially compact then it has the property of absolutely continuous integral.*

Proof of Proposition 3.1. Let $M \subseteq C(\mathcal{D}; \mathbb{R})$ be a bounded set. We will show that the set $T(M) = \{T(v) : v \in M\}$ is relatively compact in the space $C(\mathcal{D}; \mathbb{R})$. According to the Arzelà-Ascoli lemma, it is sufficient to show that the set $T(M)$ is bounded and equicontinuous.

Boundedness. It is clear that

$$|T(v)(t, x)| \leq \left| \int_{t_0}^t \int_{x_0}^x |\ell(v)(s, \eta)| d\eta ds \right| \leq \|\ell(v)\|_L \leq \|\ell\| \|v\|_C$$

for $(t, x) \in \mathcal{D}$ and every $v \in M$. Therefore, the set $T(M)$ is bounded in $C(\mathcal{D}; \mathbb{R})$.

Equicontinuity. Let $\varepsilon > 0$ be arbitrary. Lemmas 3.1 and 3.2 yield that the operator ℓ is weakly compact, that is, the set $\ell(M) = \{\ell(v) : v \in M\}$ is weakly sequentially compact subset of $L(\mathcal{D}; \mathbb{R})$. Therefore, Lemma 3.3 guarantees that there exists $\delta > 0$ such that the relation

$$\left| \iint_E \ell(v)(t, x) dt dx \right| < \frac{\varepsilon}{2} \quad \text{for } v \in M \quad (3.3)$$

holds for every measurable set $E \subseteq \mathcal{D}$ satisfying $\text{meas } E < \max\{b - a, d - c\}\delta$.

On the other hand, for $(t_1, x_1), (t_2, x_2) \in \mathcal{D}$ and $v \in M$, we have

$$\begin{aligned} |T(v)(t_2, x_2) - T(v)(t_1, x_1)| &= \\ &= \left| \int_{t_0}^{t_2} \int_{x_0}^{x_2} \ell(v)(s, \eta) d\eta ds - \int_{t_0}^{t_1} \int_{x_0}^{x_1} \ell(v)(s, \eta) d\eta ds \right| \leq \\ &\leq \left| \iint_{E_1} \ell(v)(s, \eta) ds d\eta \right| + \left| \iint_{E_2} \ell(v)(s, \eta) ds d\eta \right|, \end{aligned}$$

where measurable sets $E_1, E_2 \subseteq \mathcal{D}$ are such that $\text{meas } E_1 \leq (d - c)|t_2 - t_1|$ and $\text{meas } E_2 \leq (b - a)|x_2 - x_1|$. Hence, by virtue of relation (3.3), we get

$$\begin{aligned} |T(v)(t_2, x_2) - T(v)(t_1, x_1)| &< \varepsilon \\ &\text{for } (t_1, x_1), (t_2, x_2) \in \mathcal{D}, |t_2 - t_1| + |x_2 - x_1| < \delta, \text{ and } v \in M, \end{aligned}$$

i.e., the set $T(M)$ is equicontinuous in $C(\mathcal{D}; \mathbb{R})$. \square

4. FREDHOLM ALTERNATIVE

The main result of this section is the following statement on the Fredholmity of problem (1.1), (1.2).

Theorem 4.1. *For the unique solvability of problem (1.1), (1.2) it is sufficient and necessary that the homogeneous problem*

$$\frac{\partial^2 u(t, x)}{\partial t \partial x} = \ell(u)(t, x), \quad (1.1_0)$$

$$u(t, x_0) = 0 \quad \text{for } t \in [a, b], \quad u(t_0, x) = 0 \quad \text{for } x \in [c, d] \quad (1.2_0)$$

has only the trivial solution.

To prove this theorem we need a result stated in [18].

Lemma 4.1 ([18, Proposition 3.5]). *Let $f \in L(\mathcal{D}; \mathbb{R})$ and*

$$u(t, x) = \int_a^t \int_c^x f(s, \eta) d\eta ds \quad \text{for } (t, x) \in \mathcal{D}.$$

Then:

(i) *there exists a set $E \subseteq [a, b]$ such that $\text{meas } E = b - a$ and*

$$u'_{[1]}(t, x) = \int_c^x f(t, \eta) d\eta \quad \text{for } t \in E \text{ and } x \in [c, d];$$

(ii) *there exists a set $F \subseteq \mathcal{D}$ such that $\text{meas } F = (b - a)(d - c)$ and*

$$u''_{[1,2]}(t, x) = f(t, x) \quad \text{for } (t, x) \in F.$$

Proof of Theorem 4.1. Let u be a solution to problem (1.1), (1.2). Using Proposition 2.1, one can obtain that u is a solution to the equation

$$v = T(v) + f \quad (4.1)$$

in the space $C(\mathcal{D}; \mathbb{R})$, where the operator T is given by relation (3.1) and

$$f(t, x) = -\varphi(t_0) + \varphi(t) + \psi(x) + \int_{t_0}^t \int_{x_0}^x q(s, \eta) d\eta ds \quad \text{for } (t, x) \in \mathcal{D}. \quad (4.2)$$

Conversely, if $v \in C(\mathcal{D}; \mathbb{R})$ is a solution to equation (4.1) with f given by (4.2) then it is easy to verify that $v \in C^*(\mathcal{D}; \mathbb{R})$ (see Proposition 2.1) and, by virtue of Lemma 4.1(ii), v is a solution to problem (1.1), (1.2). Hence, problem (1.1), (1.2) and equation (4.1) are equivalent in this sense.

Note also that u is a solution to homogeneous problem (1.1₀), (1.2₀) if and only if u is a solution to the homogeneous equation

$$v = T(v) \quad (4.3)$$

in the space $C(\mathcal{D}; \mathbb{R})$.

According to Proposition 3.1, the operator T is completely continuous. It follows from the Riesz-Schauder theory that equation (4.1) is uniquely solvable for every $f \in C(\mathcal{D}; \mathbb{R})$ if and only if homogeneous equation (4.3) has only the trivial solution. Consequently, the assertion of the theorem holds. \square

Definition 4.1. Let problem (1.1₀), (1.2₀) have only the trivial solution. An operator $\Omega : L(\mathcal{D}; \mathbb{R}) \rightarrow C(\mathcal{D}; \mathbb{R})$ which assigns to every $q \in L(\mathcal{D}; \mathbb{R})$ the solution u to problem (1.1), (1.2₀) is called the Darboux operator of problem (1.1₀), (1.2₀).

Remark 4.1. It is clear that the Darboux operator Ω is linear.

If homogeneous problem (1.1₀), (1.2₀) has a nontrivial solution then, by virtue of Theorem 4.1, there exist functions q , φ , and ψ such that the problem (1.1), (1.2) has either no solution or infinitely many solutions. However, as follows from the proof of Theorem 4.1, a stronger assertion can be shown in this case.

Proposition 4.1. *Let problem (1.1₀), (1.2₀) have a nontrivial solution. Then for arbitrary $\varphi \in AC([a, b]; \mathbb{R})$ and $\psi \in AC([c, d]; \mathbb{R})$ satisfying $\varphi(t_0) = \psi(x_0)$, there exists a function $q \in L(\mathcal{D}; \mathbb{R})$ such that the problem (1.1), (1.2) has no solution.*

Proof. Let u_0 be a nontrivial solution to problem (1.1₀), (1.2₀) and $\varphi \in AC([a, b]; \mathbb{R})$, $\psi \in AC([c, d]; \mathbb{R})$ be arbitrary functions such that $\varphi(t_0) = \psi(x_0)$.

It follows from the proof of Theorem 4.1 that u_0 is also a nontrivial solution to homogeneous equation (4.3) in the space $C(\mathcal{D}; \mathbb{R})$. Therefore, by the Riesz-Schauder theory, there exists a function $f \in C(\mathcal{D}; \mathbb{R})$ such that equation (4.1) has no solution.

Then the problem (1.1), (1.2) has no solution for $q \equiv \ell(z)$, where

$$z(t, x) = f(t, x) + \varphi(t_0) - \varphi(t) - \psi(x) \quad \text{for } (t, x) \in \mathcal{D}.$$

Indeed, if the problem indicated had a solution u then the function $u + z$ would be a solution to equation (4.1), which would lead to a contradiction. \square

5. VOLTERRA OPERATORS

The following definitions introduce notions of Volterra operators which are useful in the question on the solvability of problem (1.1), (1.2) (see, e. g., Theorems 6.2 and 6.3 below).

Definition 5.1. Let $t_0 \in [a, b]$ and $x_0 \in [c, d]$. We say that $\ell \in \mathcal{L}(\mathcal{D})$ is a t_0 -Volterra operator (resp., an x_0 -Volterra operator) if the relation

$$\begin{aligned} \ell(v)(t, x) = 0 \quad & \text{for a. e. } (t, x) \in [a_0, b_0] \times [c, d] \\ & \text{(resp., for a. e. } (t, x) \in [a, b] \times [c_0, d_0]) \end{aligned}$$

holds for an arbitrary interval $[a_0, b_0] \subseteq [a, b]$ (resp., $[c_0, d_0] \subseteq [c, d]$) and every function $v \in C(\mathcal{D}; \mathbb{R})$ such that $t_0 \in [a_0, b_0]$ (resp., $x_0 \in [c_0, d_0]$) and

$$v(t, x) = 0 \quad \text{for } (t, x) \in [a_0, b_0] \times [c, d] \quad \text{(resp., for } (t, x) \in [a, b] \times [c_0, d_0]).$$

Definition 5.2. Let $t_0 \in [a, b]$ and $x_0 \in [c, d]$. We say that $\ell \in \mathcal{L}(\mathcal{D})$ is a (t_0, x_0) -Volterra operator if the relation

$$\ell(v)(t, x) = 0 \quad \text{for a. e. } (t, x) \in \mathcal{D}_0$$

is satisfied for an arbitrary rectangle $\mathcal{D}_0 \subseteq \mathcal{D}$ and every function $v \in C(\mathcal{D}; \mathbb{R})$ such that $(t_0, x_0) \in \mathcal{D}_0$ and

$$v(t, x) = 0 \quad \text{for } (t, x) \in \mathcal{D}_0.$$

Remark 5.1. If the operator ℓ in equation (1.1) is a t_0 -Volterra one (resp., x_0 -Volterra one, resp. (t_0, x_0) -Volterra one), then problem (1.1), (1.2) can be restricted to an arbitrary rectangle $[a_0, b_0] \times [c, d]$ (resp., $[a, b] \times [c_0, d_0]$, resp., \mathcal{D}_0) contained in \mathcal{D} and such that $t_0 \in [a_0, b_0]$ (resp., $x_0 \in [c_0, d_0]$, resp., $(t_0, x_0) \in \mathcal{D}_0$).

Definitions 5.1 and 5.2 immediately yield

Proposition 5.1. *Let $t_0 \in [a, b]$ and $x_0 \in [c, d]$. Then $\ell \in \mathcal{L}(\mathcal{D})$ is a (t_0, x_0) -Volterra operator if and only if ℓ is both t_0 -Volterra one and x_0 -Volterra one.*

Let an operator ℓ be defined by the formula

$$\ell(v)(t, x) = p(t, x)v(\tau(t, x), \mu(t, x)) \quad \text{for a. e. } (t, x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D}; \mathbb{R}), \quad (5.1)$$

where $p \in L(\mathcal{D}; \mathbb{R})$ and $\tau: \mathcal{D} \rightarrow [a, b]$, $\mu: \mathcal{D} \rightarrow [c, d]$ are measurable functions.

Then clearly $\ell \in \mathcal{L}(\mathcal{D})$. Moreover, the following statements hold.

Proposition 5.2. *Let $t_0 \in [a, b]$. Then the operator ℓ defined by formula (5.1) is a t_0 -Volterra one if and only if the condition*

$$|p(t, x)|(\tau(t, x) - t)(\tau(t, x) - t_0) \leq 0 \quad \text{for a. e. } (t, x) \in \mathcal{D} \quad (5.2)$$

is fulfilled.

To prove this proposition we need the following lemma.

Lemma 5.1. *Let $\ell \in \mathcal{L}(\mathcal{D})$ be a positive² t_0 -Volterra operator. Then for any non-decreasing function $\gamma \in C([a, b]; \mathbb{R})$, the relation*

$$\gamma(\alpha(t))\ell(1)(t, x) \leq \ell(\gamma)(t, x) \leq \gamma(\beta(t))\ell(1)(t, x)^3 \quad (5.3)$$

holds for a. e. $(t, x) \in \mathcal{D}$, where

$$\alpha(t) = \min\{t, t_0\}, \quad \beta(t) = \max\{t, t_0\} \quad \text{for } t \in [a, b].$$

Proof. Let $\gamma \in C([a, b]; \mathbb{R})$ be a non-decreasing function. We first show that the relation

$$\begin{aligned} \gamma(\alpha(t))\ell(1)(s, x) &\leq \ell(\gamma)(s, x) \\ &\leq \gamma(\beta(t))\ell(1)(s, x) \quad \text{for a. e. } (s, x) \in [\alpha(t), \beta(t)] \times [c, d] \end{aligned} \quad (5.4)$$

holds for every $t \in [a, b]$. Indeed, let $t \in [a, b]$, $t \neq t_0$, be arbitrary. Put

$$\gamma_0(s, x) = \begin{cases} \gamma(\alpha(t)) & \text{for } (s, x) \in \mathcal{D}, s \leq \alpha(t), \\ \gamma(s) & \text{for } (s, x) \in \mathcal{D}, \alpha(t) < s < \beta(t), \\ \gamma(\beta(t)) & \text{for } (s, x) \in \mathcal{D}, \beta(t) \leq s. \end{cases}$$

Then obviously $\gamma_0 \in C(\mathcal{D}; \mathbb{R})$ and

$$\gamma(\alpha(t)) \leq \gamma_0(s, x) \leq \gamma(\beta(t)) \quad \text{for } (s, x) \in \mathcal{D}.$$

Since the operator ℓ is positive, we obtain

$$\gamma(\alpha(t))\ell(1)(s, x) \leq \ell(\gamma_0)(s, x) \leq \gamma(\beta(t))\ell(1)(s, x) \quad \text{for a. e. } (s, x) \in \mathcal{D}.$$

On the other hand, the operator ℓ is supposed to be a t_0 -Volterra one which guarantees the equality

$$\ell(\gamma_0)(s, x) = \ell(\gamma)(s, x) \quad \text{for a. e. } (s, x) \in [\alpha(t), \beta(t)] \times [c, d]$$

²It maps the set $C(\mathcal{D}; \mathbb{R}_+)$ into the set $L(\mathcal{D}; \mathbb{R}_+)$.

³Here, $\ell(\bar{\gamma})$ means $\ell(\bar{\gamma})$ in which $\bar{\gamma}(t, x) = \gamma(t)$ for $(t, x) \in \mathcal{D}$.

and thus desired relation (5.4) holds for every $t \in [a, b]$.

Now we put

$$u(t, x) = \int_a^t \int_c^x \ell(\gamma)(s, \eta) d\eta ds, \quad v(t, x) = \int_a^t \int_c^x \ell(1)(s, \eta) d\eta ds \quad \text{for } (t, x) \in \mathcal{D}.$$

It follows from Lemma 4.1 that there exists a set $E \subseteq [a, b]$ such that $\text{meas } E = b - a$ and

$$\begin{aligned} u'_{[1]}(t, x) &= \int_c^x \ell(\gamma)(t, \eta) d\eta \quad \text{for } t \in E, \quad x \in [c, d], \\ v'_{[1]}(t, x) &= \int_c^x \ell(1)(t, \eta) d\eta \quad \text{for } t \in E, \quad x \in [c, d] \end{aligned} \quad (5.5)$$

and, moreover, there is a set $F \subseteq E \times]c, d]$ with $\text{meas } F = (b - a)(d - c)$ such that

$$u''_{[1,2]}(t, x) = \ell(\gamma)(t, x), \quad v''_{[1,2]}(t, x) = \ell(1)(t, x) \quad \text{for } (t, x) \in F. \quad (5.6)$$

Let $(t, x) \in F$, $t > t_0$, be arbitrary. Then $\alpha(t) = t_0$, $\beta(t) = t$, and relation (5.4) yields that

$$\begin{aligned} \frac{\gamma(t_0)}{hk} \int_{t-h}^t \int_{x-k}^x \ell(1)(s, \eta) d\eta ds &\leq \frac{1}{hk} \int_{t-h}^t \int_{x-k}^x \ell(\gamma)(s, \eta) d\eta ds \\ &\leq \frac{\gamma(t)}{hk} \int_{t-h}^t \int_{x-k}^x \ell(1)(s, \eta) d\eta ds \end{aligned}$$

for all $h \in]0, t - t_0]$ and $k \in]0, x - c]$, whence we get

$$\begin{aligned} \frac{\gamma(t_0)}{k} \left[\frac{v(t, x) - v(t - h, x)}{h} - \frac{v(t, x - k) - v(t - h, x - k)}{h} \right] \\ \leq \frac{1}{k} \left[\frac{u(t, x) - u(t - h, x)}{h} - \frac{u(t, x - k) - u(t - h, x - k)}{h} \right] \\ \leq \frac{\gamma(t)}{k} \left[\frac{v(t, x) - v(t - h, x)}{h} - \frac{v(t, x - k) - v(t - h, x - k)}{h} \right] \end{aligned}$$

for all $h \in]0, t - t_0]$ and $k \in]0, x - c]$. For any $k \in]0, x - c]$ fixed we pass to the limit $h \rightarrow 0+$ in the latter relation and, in view of equalities (5.5), we get

$$\begin{aligned} \frac{\gamma(t_0)}{k} \left[v'_{[1]}(t, x) - v'_{[1]}(t, x - k) \right] &\leq \frac{1}{k} \left[u'_{[1]}(t, x) - u'_{[1]}(t, x - k) \right] \\ &\leq \frac{\gamma(t)}{k} \left[v'_{[1]}(t, x) - v'_{[1]}(t, x - k) \right] \end{aligned}$$

for every $k \in]0, x - c]$. Now letting $k \rightarrow 0+$ in the previous inequalities and using equalities (5.6), we obtain

$$\begin{aligned} \gamma(t_0) \ell(1)(t, x) = \gamma(t_0) v''_{[1,2]}(t, x) &\leq u''_{[1,2]}(t, x) = \ell(\gamma)(t, x) \\ &\leq \gamma(t) v''_{[1,2]}(t, x) = \gamma(t) \ell(1)(t, x), \end{aligned}$$

i. e., desired relation (5.3) holds for every $(t, x) \in F$ such that $t > t_0$.

Let now $(t, x) \in F$, $t < t_0$, be arbitrary. Then $\alpha(t) = t$, $\beta(t) = t_0$, and relation (5.4) implies that

$$\frac{\gamma(t)}{hk} \int_t^{t+h} \int_{x-k}^x \ell(1)(s, \eta) d\eta ds \leq \frac{1}{hk} \int_t^{t+h} \int_{x-k}^x \ell(\gamma)(s, \eta) d\eta ds$$

$$\leq \frac{\gamma(t)}{hk} \int_t^{t+h} \int_{x-k}^x \ell(1)(s, \eta) d\eta ds$$

for all $h \in]0, t_0 - t]$ and $k \in]0, x - c]$, whence we get

$$\begin{aligned} & \frac{\gamma(t)}{k} \left[\frac{v(t+h, x) - v(t, x)}{h} - \frac{v(t+h, x-k) - v(t, x-k)}{h} \right] \\ & \leq \frac{1}{k} \left[\frac{u(t+h, x) - u(t, x)}{h} - \frac{u(t+h, x-k) - u(t, x-k)}{h} \right] \\ & \leq \frac{\gamma(t_0)}{k} \left[\frac{v(t+h, x) - v(t, x)}{h} - \frac{v(t+h, x-k) - v(t, x-k)}{h} \right] \end{aligned}$$

for all $h \in]0, t_0 - t]$ and $k \in]0, x - c]$. For any $k \in]0, x - c]$ fixed we pass to the limit $h \rightarrow 0+$ in the latter relation and, in view of equalities (5.5), we get

$$\begin{aligned} \frac{\gamma(t)}{k} \left[v'_{[1]}(t, x) - v'_{[1]}(t, x-k) \right] & \leq \frac{1}{k} \left[u'_{[1]}(t, x) - u'_{[1]}(t, x-k) \right] \\ & \leq \frac{\gamma(t_0)}{k} \left[v'_{[1]}(t, x) - v'_{[1]}(t, x-k) \right] \end{aligned}$$

for every $k \in]0, x - c]$. Now letting $k \rightarrow 0+$ in the previous inequalities and using equalities (5.6), we obtain

$$\begin{aligned} \gamma(t)\ell(1)(t, x) = \gamma(t)v''_{[1,2]}(t, x) & \leq u''_{[1,2]}(t, x) = \ell(\gamma)(t, x) \\ & \leq \gamma(t_0)v''_{[1,2]}(t, x) = \gamma(t_0)\ell(1)(t, x), \end{aligned}$$

i. e., desired relation (5.3) holds for every $(t, x) \in F$ such that $t < t_0$.

Consequently, relation (5.3) is fulfilled for every $(t, x) \in F$ with $t \neq t_0$ and thus we have proved that this relation holds for a. e. $(t, x) \in \mathcal{D}$. \square

Proof of Proposition 5.2. We first note that inequality (5.2) is equivalent to the condition

$$\begin{aligned} |p(t, x)| \min\{t, t_0\} & \leq |p(t, x)|\tau(t, x) \\ & \leq |p(t, x)| \max\{t, t_0\} \quad \text{for a. e. } (t, x) \in \mathcal{D}. \end{aligned} \quad (5.7)$$

Assume that the operator ℓ is a t_0 -Volterra one and put

$$\bar{\ell}(v)(t, x) = |p(t, x)|v(\tau(t, x), \mu(t, x)) \quad \text{for a. e. } (t, x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D}; \mathbb{R}).$$

Then $\bar{\ell} \in \mathcal{L}(\mathcal{D})$ and it is a positive⁴ t_0 -Volterra operator. Indeed, linearity and positivity of $\bar{\ell}$ are obvious. Moreover, let $[a_0, b_0] \subseteq [a, b]$ and $v \in C(\mathcal{D}; \mathbb{R})$ be arbitrary such that $t_0 \in [a_0, b_0]$ and

$$v(t, x) = 0 \quad \text{for } (t, x) \in [a_0, b_0] \times [c, d].$$

Then we have

$$p(t, x)v(\tau(t, x), \mu(t, x)) = 0 \quad \text{for a. e. } (t, x) \in [a_0, b_0] \times [c, d]$$

because the operator ℓ is a t_0 -Volterra one and thus the relation

$$|p(t, x)|v(\tau(t, x), \mu(t, x)) = 0 \quad \text{for a. e. } (t, x) \in [a_0, b_0] \times [c, d]$$

holds as well. However it means that $\bar{\ell}$ is also a t_0 -Volterra operator.

⁴It maps the set $C(\mathcal{D}; \mathbb{R}_+)$ into the set $L(\mathcal{D}; \mathbb{R}_+)$.

Now we put

$$\gamma(t) = t \quad \text{for } t \in [a, b].$$

The function $\gamma \in C([a, b]; \mathbb{R})$ is increasing and therefore, it follows from Lemma 5.1 that

$$\begin{aligned} |p(t, x)| \min\{t, t_0\} &= |p(t, x)| \alpha(t) = \alpha(t) \bar{\ell}(1)(t, x) \\ &\leq \bar{\ell}(\gamma)(t, x) = |p(t, x)| \tau(t, x) \\ &\leq \beta(t) \bar{\ell}(1)(t, x) = |p(t, x)| \max\{t, t_0\} \quad \text{for a. e. } (t, x) \in \mathcal{D}, \end{aligned}$$

i. e., relation (5.7) is satisfied.

Conversely, if relation (5.7) holds then, by using Definition 5.1, we easily show that the operator ℓ is a t_0 -Volterra one. \square

Proposition 5.3. *Let $x_0 \in [c, d]$. Then the operator ℓ defined by formula (5.1) is an x_0 -Volterra one if and only if the condition*

$$|p(t, x)|(\mu(t, x) - x)(\mu(t, x) - x_0) \leq 0 \quad \text{for a. e. } (t, x) \in \mathcal{D} \quad (5.8)$$

holds.

Proof. It can be proved similarly as Proposition 5.2 by exchanging the role of the variables t and x . \square

Proposition 5.4. *Let $t_0 \in [a, b]$ and $x_0 \in [c, d]$. Then the operator ℓ defined by formula (5.1) is a (t_0, x_0) -Volterra one if and only if conditions (5.2) and (5.8) are both satisfied.*

Proof. The assertion follows immediately from Propositions 5.1–5.3. \square

6. EXISTENCE AND UNIQUENESS THEOREMS

In this section, we give some efficient condition guaranteeing the unique solvability of problems (1.1), (1.2) as well as (1.1'), (1.2). We prove, in particular, that problem (1.1), (1.2) with Volterra type operator ℓ has a unique solution without any additional assumptions. We first formulate all the results, their proofs being postponed till Section 6.1 below.

Introduce the following notation.

Notation 6.1. Let $\ell \in \mathcal{L}(\mathcal{D})$. Define the operators $\vartheta_k : C(\mathcal{D}; \mathbb{R}) \rightarrow C(\mathcal{D}; \mathbb{R})$, $k = 0, 1, 2, \dots$, by setting

$$\vartheta_0(v) = v, \quad \vartheta_k(v) = T(\vartheta_{k-1}(v)) \quad \text{for } v \in C(\mathcal{D}; \mathbb{R}), \quad k \in \mathbb{N}, \quad (6.1)$$

where the operator T is given by formula (3.1).

Theorem 6.1. *Let there exist $m \in \mathbb{N}$ and $\alpha \in [0, 1[$ such that the inequality*

$$\|\vartheta_m(u)\|_C \leq \alpha \|u\|_C \quad (6.2)$$

is satisfied for every solution u to homogeneous problem (1.1₀), (1.2₀). Then problem (1.1), (1.2) is uniquely solvable.

Remark 6.1. The assumption $\alpha \in [0, 1[$ in the previous theorem cannot be replaced by the assumption $\alpha \in [0, 1]$ (see Example 9.1).

Corollary 6.1. *Let there exist a number $j \in \mathbb{N}$ such that the inequalities*

$$\begin{aligned} \int_a^{t_0} \int_c^{x_0} p_j(s, \eta) d\eta ds < 1, & \quad \int_a^{t_0} \int_{x_0}^d p_j(s, \eta) d\eta ds < 1, \\ \int_{t_0}^b \int_c^{x_0} p_j(s, \eta) d\eta ds < 1, & \quad \int_{t_0}^b \int_{x_0}^d p_j(s, \eta) d\eta ds < 1 \end{aligned} \quad (6.3)$$

are satisfied, where $p_1 \equiv |p|$ and

$$\begin{aligned} p_{k+1}(t, x) &= \\ &= |p(t, x)| \operatorname{sgn}((\tau(t, x) - t_0)(\mu(t, x) - x_0)) \int_{t_0}^{\tau(t, x)} \int_{x_0}^{\mu(t, x)} p_k(s, \eta) d\eta ds \\ &\quad \text{for a. e. } (t, x) \in \mathcal{D}, \quad k \in \mathbb{N}. \end{aligned} \quad (6.4)$$

Then problem (1.1'), (1.2) is uniquely solvable.

Remark 6.2. Example 9.1 shows that neither of strict inequalities (6.3) in Corollary 6.1 can be replaced by the nonstrict one.

Theorem 6.2. *Let ℓ be a t_0 -Volterra operator. Then problem (1.1), (1.2) has a unique solution.*

Theorem 6.3. *Let ℓ be an x_0 -Volterra operator. Then problem (1.1), (1.2) has a unique solution.*

Corollary 6.2. *Let ℓ be a (t_0, x_0) -Volterra operator. Then problem (1.1), (1.2) has a unique solution.*

Corollary 6.3. *Let at least one of conditions (5.2) and (5.8) be satisfied. Then problem (1.1'), (1.2) has a unique solution.*

6.1. Proofs. Now we prove statements formulated above.

Proof of Theorem 6.1. According to Theorem 4.1, it is sufficient to show that homogeneous problem (1.1₀), (1.2₀) has only the trivial solution.

Let u be a solution to problem (1.1₀), (1.2₀). Then, in view of Proposition 2.1(3), the function u satisfies

$$u(t, x) = \int_{t_0}^t \int_{x_0}^x \ell(u)(s, \eta) d\eta ds = T(u)(t, x) = \vartheta_1(u)(t, x) \quad \text{for } (t, x) \in \mathcal{D}.$$

Using the last relation, we get

$$u(t, x) = T(\vartheta_1(u))(t, x) = \vartheta_2(u)(t, x) \quad \text{for } (t, x) \in \mathcal{D}$$

and thus $u = \vartheta_k(u)$ for every $k \in \mathbb{N}$. Therefore, (6.2) implies

$$\|u\|_C = \|\vartheta_m(u)\|_C \leq \alpha \|u\|_C,$$

which guarantees that $u \equiv 0$ because $\alpha \in [0, 1[$. □

Proof of Corollary 6.1. It is clear that equation (1.1') is a particular case of equation (1.1) in which the operator ℓ is defined by formula (5.1). It is not difficult to verify that

$$|\vartheta_k(v)(t, x)| \leq$$

$$\begin{aligned} &\leq \operatorname{sgn}((t-t_0)(x-x_0)) \int_{t_0}^t \int_{x_0}^x |p(s,\eta) \vartheta_{k-1}(v)(\tau(s,\eta), \mu(s,\eta))| d\eta ds \\ &\leq \|v\|_C \operatorname{sgn}((t-t_0)(x-x_0)) \int_{t_0}^t \int_{x_0}^x p_k(s,\eta) d\eta ds \end{aligned}$$

for $(t, x) \in \mathcal{D}$, $k \in \mathbb{N}$, $v \in C(\mathcal{D}; \mathbb{R})$.

Therefore, the assumptions of Theorem 6.1 are satisfied with $m = j$ and

$$\alpha = \max \left\{ \operatorname{sgn}((t-t_0)(x-x_0)) \int_{t_0}^t \int_{x_0}^x p_j(s,\eta) d\eta ds : (t, x) \in \mathcal{D} \right\}.$$

□

To prove Theorem 6.2 we need the following lemma.

Lemma 6.1. *Let $t_0 \in [a, b]$ and $\ell \in \mathcal{L}(\mathcal{D})$ be a t_0 -Volterra operator. Then for any $x_0 \in [c, d]$, the relation*

$$\lim_{k \rightarrow +\infty} \|\vartheta_k\| = 0 \quad (6.5)$$

holds, where the operators ϑ_k are defined by formula (6.1).

Proof. Let $x_0 \in [c, d]$ and $\varepsilon \in]0, 1[$ be arbitrary. According to Proposition 3.1, the operator ϑ_1 is completely continuous. Therefore, by virtue of the Arzelà-Ascoli lemma, there exists $\delta > 0$ such that

$$\left| \int_{t_0}^{y_2} \int_{x_0}^{z_2} \ell(w)(s,\eta) d\eta ds - \int_{t_0}^{y_1} \int_{x_0}^{z_1} \ell(w)(s,\eta) d\eta ds \right| \leq \varepsilon \|w\|_C$$

for $(y_1, z_1), (y_2, z_2) \in \mathcal{D}$, $|y_2 - y_1| + |z_2 - z_1| < \delta$, $w \in C(\mathcal{D}; \mathbb{R})$,

and consequently we have

$$\left| \int_{y_1}^{y_2} \int_{x_0}^x \ell(w)(s,\eta) d\eta ds \right| \leq \varepsilon \|w\|_C$$

for $(y_1, x), (y_2, x) \in \mathcal{D}$, $|y_2 - y_1| < \delta$, $w \in C(\mathcal{D}; \mathbb{R})$. (6.6)

Let

$$n = \max \left\{ \operatorname{Ent} \left(\frac{t_0 - a}{\delta} \right), \operatorname{Ent} \left(\frac{b - t_0}{\delta} \right) \right\} + 1.$$

Choose $y_{n+1} \in [a, t_0]$ and $y_{n+2} \in [t_0, b]$ such that $y_{n+2} - y_{n+1} < \delta$, and put

$$y_k = \begin{cases} y_{n+1} - (n+1-k) \frac{y_{n+1}-a}{n} & \text{for } k = 1, 2, \dots, n, \\ y_k = y_{n+2} + (k-n-2) \frac{b-y_{n+2}}{n} & \text{for } k = n+3, n+4, \dots, 2n+2, \end{cases}$$

and

$$\mathcal{D}_k = [y_{n+2-k}, y_{n+1+k}] \times [c, d] \quad \text{for } k = 1, 2, \dots, n+1.$$

It is clear that

$$|t_2 - t_1| < \delta \quad \text{for } t_1, t_2 \in [y_j, y_{j+1}], \quad j = 1, 2, \dots, 2n+1. \quad (6.7)$$

Having $w \in C(\mathcal{D}; \mathbb{R})$, we denote

$$\|w\|_i = \|w\|_{C(\mathcal{D}_i; \mathbb{R})} \quad \text{for } i = 1, 2, \dots, n+1.$$

Let $v \in C(\mathcal{D}; \mathbb{R})$ be arbitrary. We shall show that the relation

$$\|\vartheta_k(v)\|_i \leq \alpha_i(k) \varepsilon^k \|v\|_C \quad \text{for } k \in \mathbb{N} \quad (6.8)$$

holds for every $i = 1, 2, \dots, n+1$, where

$$\alpha_i(k) = \alpha_i k^{i-1} \quad \text{for } k \in \mathbb{N}, \quad i = 1, 2, \dots, n+1 \quad (6.9)$$

and

$$\alpha_1 = 1, \quad \alpha_{i+1} = i + 1 + i\alpha_i \quad \text{for } i = 1, 2, \dots, n. \quad (6.10)$$

By virtue of (6.6) and (6.7), it is easy to verify that, for any $w \in C(\mathcal{D}; \mathbb{R})$ and $i = 1, 2, \dots, n+1$, we have

$$\left| \int_{t_0}^t \int_{x_0}^x \ell(w)(s, \eta) d\eta ds \right| \leq i \varepsilon \|w\|_C \quad \text{for } (t, x) \in \mathcal{D}_i. \quad (6.11)$$

Observe that the previous relation immediately implies

$$\|\vartheta_1(v)\|_i \leq i \varepsilon \|v\|_C \quad \text{for } i = 1, 2, \dots, n+1. \quad (6.12)$$

Furthermore, on account of (6.6), (6.7), and the fact that ℓ is a t_0 -Volterra operator, we obtain

$$|\vartheta_{k+1}(v)(t, x)| = \left| \int_{t_0}^t \int_{x_0}^x \ell(\vartheta_k(v))(s, \eta) d\eta ds \right| \leq \varepsilon \|\vartheta_k(v)\|_1 \quad \text{for } (t, x) \in \mathcal{D}_1, \quad k \in \mathbb{N}.$$

Hence, by virtue of (6.12), we get

$$\|\vartheta_k(v)\|_1 \leq \varepsilon^k \|v\|_C \quad \text{for } k \in \mathbb{N}$$

and thus relation (6.8) holds for $i = 1$.

Now suppose that relation (6.8) holds for some $i \in \{1, 2, \dots, n\}$. We shall show that the relation indicated is also true for $i+1$. With respect to (6.7), we obtain

$$\begin{aligned} \|\vartheta_{k+1}(v)\|_{i+1} &= \max \left\{ \left| \int_{t_0}^t \int_{x_0}^x \ell(\vartheta_k(v))(s, \eta) d\eta ds \right| : (t, x) \in \mathcal{D}_{i+1} \right\} \\ &= \left| \int_{t_0}^{t_k^*} \int_{x_0}^{x_k^*} \ell(\vartheta_k(v))(s, \eta) d\eta ds \right| \\ &\leq \left| \int_{t_0}^{\hat{t}_k} \int_{x_0}^{x_k^*} \ell(\vartheta_k(v))(s, \eta) d\eta ds \right| \\ &\quad + \left| \int_{\hat{t}_k}^{t_k^*} \int_{x_0}^{x_k^*} \ell(\vartheta_k(v))(s, \eta) d\eta ds \right| \quad \text{for } k \in \mathbb{N}, \end{aligned}$$

where $(t_k^*, x_k^*) \in \mathcal{D}_{i+1}$, $(\hat{t}_k, x_k^*) \in \mathcal{D}_i$, and $|t_k^* - \hat{t}_k| < \delta$ for $k \in \mathbb{N}$. Therefore, on account of (6.6), (6.11), and the fact that ℓ is a t_0 -Volterra operator, we get

$$\|\vartheta_{k+1}(v)\|_{i+1} \leq \varepsilon \|\vartheta_k(v)\|_{i+1} + i \varepsilon \|\vartheta_k(v)\|_i \leq \varepsilon \|\vartheta_k(v)\|_{i+1} + i \alpha_i(k) \varepsilon^{k+1} \|v\|_C$$

for $k \in \mathbb{N}$. Consequently,

$$\begin{aligned} \|\vartheta_{k+1}(v)\|_{i+1} &\leq \varepsilon \left(\varepsilon \|\vartheta_{k-1}(v)\|_{i+1} + i \alpha_i(k-1) \varepsilon^k \|v\|_C \right) \\ &\quad + i \alpha_i(k) \varepsilon^{k+1} \|v\|_C \quad \text{for } k \in \mathbb{N}. \end{aligned}$$

Continuing this procedure, on account of (6.12), we obtain

$$\|\vartheta_{k+1}(v)\|_{i+1} \leq \left(i + 1 + i (\alpha_i(1) + \dots + \alpha_i(k)) \right) \varepsilon^{k+1} \|v\|_C \quad \text{for } k \in \mathbb{N}. \quad (6.13)$$

By using (6.9) and (6.10), it is easy to verify that

$$\begin{aligned} i + 1 + i(\alpha_i(1) + \dots + \alpha_i(k)) &= i + 1 + i\alpha_i(1^{i-1} + \dots + k^{i-1}) \\ &\leq i + 1 + i\alpha_i k k^{i-1} = i + 1 + i\alpha_i k^i \\ &\leq (i + 1 + i\alpha_i) k^i = \alpha_{i+1} k^i \leq \alpha_{i+1} (k + 1). \end{aligned}$$

Therefore, (6.12) and (6.13) imply that

$$\|\vartheta_k(v)\|_{i+1} \leq \alpha_{i+1}(k) \varepsilon^k \|v\|_C \quad \text{for } k \in \mathbb{N}.$$

Hence, by induction, we have proved that relation (6.8) holds for every $i = 1, 2, \dots, n+1$.

Now it is already clear that, for any $k \in \mathbb{N}$, the estimate

$$\|\vartheta_k(v)\|_C = \|\vartheta_k(v)\|_{n+1} \leq \alpha_{n+1} k^n \varepsilon^k \|v\|_C \quad \text{for } v \in C(\mathcal{D}; \mathbb{R})$$

is fulfilled and thus

$$\|\vartheta_k\| \leq \alpha_{n+1} k^n \varepsilon^k \quad \text{for } k \in \mathbb{N}.$$

Since we suppose $\varepsilon \in]0, 1[$, the last relation yields the validity of desired relation (6.5). \square

Proof of Theorem 6.2. According to Lemma 6.1, there exists a number $m_0 \in \mathbb{N}$ such that $\|\vartheta_{m_0}\| < 1$. Moreover, it is clear that

$$\|\vartheta_{m_0}(v)\|_C \leq \|\vartheta_{m_0}\| \|v\|_C \quad \text{for } v \in C(\mathcal{D}; \mathbb{R})$$

because the operator ϑ_{m_0} is bounded. Therefore, the assumptions of Theorem 6.1 are satisfied with $m = m_0$ and $\alpha = \|\vartheta_{m_0}\|$. \square

Proof of Theorem 6.3. It can be proved analogously to Theorem 6.2 by exchanging the role of the variables t and x . \square

Proof of Corollary 6.2. The assertion follows immediately from Theorem 6.2 and Proposition 5.1. \square

Proof of Corollary 6.3. It is clear that equation (1.1') is a particular case of equation (1.1) in which the operator ℓ is defined by formula (5.1). According to Propositions 5.2 and 5.3, the assumptions (5.2) and (5.8) guarantee that the operator ℓ is a t_0 -Volterra one and an x_0 -Volterra one, respectively. Therefore, the assertion of the corollary follows immediately from Theorems 6.2 and 6.3. \square

7. WELL-POSEDNESS

In this section, the well-posedness of problems (1.1), (1.2) and (1.1'), (1.2) is studied. We first formulate all the results, their proofs being given in Section 7.1 below.

For any $k \in \mathbb{N}$, along with problem (1.1), (1.2) we consider the perturbed problem

$$\frac{\partial^2 u(t, x)}{\partial t \partial x} = \ell_k(u)(t, x) + q_k(t, x), \quad (1.1_k)$$

$$u(t, x_k) = \varphi_k(t) \quad \text{for } t \in [a, b], \quad u(t_k, x) = \psi_k(x) \quad \text{for } x \in [c, d], \quad (1.2_k)$$

where $\ell_k \in \mathcal{L}(\mathcal{D})$, $q_k \in L(\mathcal{D}; \mathbb{R})$, $t_k \in [a, b]$, $x_k \in [c, d]$, and $\varphi_k \in AC([a, b]; \mathbb{R})$, $\psi_k \in AC([c, d]; \mathbb{R})$ are such that $\varphi_k(t_k) = \psi_k(x_k)$.

Introduce the following notation.

Notation 7.1. Let $\Lambda \in \mathcal{L}(\mathcal{D})$, $t^* \in [a, b]$, and $x^* \in [c, d]$. Denote by $M(\Lambda, t^*, x^*)$ the set of all functions $y \in C^*(\mathcal{D}; \mathbb{R})$ admitting the representation

$$y(t, x) = \int_{t^*}^t \int_{x^*}^x \Lambda(z)(s, \eta) d\eta ds \quad \text{for } (t, x) \in \mathcal{D},$$

where $z \in C(\mathcal{D}; \mathbb{R})$ and $\|z\|_C = 1$.

Theorem 7.1. Let problem (1.1), (1.2) have a unique solution u and

$$\lim_{k \rightarrow +\infty} \lambda_k = 0, \quad (7.1)$$

where

$$\lambda_k = \sup_{\substack{(t, x) \in \mathcal{D} \\ y \in M(\ell_k, t_k, x_k)}} \left\{ \left| \int_{t_k}^t \int_{x_k}^x \ell_k(y)(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x \ell(y)(s, \eta) d\eta ds \right| \right\} \quad (7.2)$$

for $k \in \mathbb{N}$. Let, moreover,

$$\lim_{k \rightarrow +\infty} \varrho_k \left[\int_{t_k}^t \int_{x_k}^x \ell_k(y)(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x \ell(y)(s, \eta) d\eta ds \right] = 0$$

uniformly on \mathcal{D} for every $y \in C^*(\mathcal{D}; \mathbb{R})$, (7.3)

$$\lim_{k \rightarrow +\infty} \varrho_k \left[\int_{t_k}^t \int_{x_k}^x q_k(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x q(s, \eta) d\eta ds \right] = 0$$

uniformly on \mathcal{D} , (7.4)

$$\lim_{k \rightarrow +\infty} \varrho_k \|\varphi_k - \varphi\|_C = 0, \quad \lim_{k \rightarrow +\infty} \varrho_k \|\psi_k - \psi\|_C = 0, \quad (7.5)$$

and

$$\lim_{k \rightarrow +\infty} \varrho_k |\varphi_k(t_k) - \varphi(t_0)| = 0, \quad (7.6)$$

where

$$\varrho_k = 1 + \|\ell_k\| \quad \text{for } k \in \mathbb{N}. \quad (7.7)$$

Then there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$ problem (1.1_k), (1.2_k) has a unique solution u_k and

$$\lim_{k \rightarrow +\infty} \|u_k - u\|_C = 0. \quad (7.8)$$

Remark 7.1. It is clear that condition (7.6) is equivalent to the condition

$$\lim_{k \rightarrow +\infty} \varrho_k |\psi_k(x_k) - \psi(x_0)| = 0.$$

Note also that the sequences $\{t_k\}$ and $\{x_k\}$ in Theorem 7.1 do not necessarily converge to t_0 and x_0 , respectively. Indeed, let $\ell_k = \ell = 0^5$, $q_k \equiv q \equiv 0$, $a = c = 0$, $b = d = 1$, $t_0 = x_0 = 1$, $t_k = x_k = 1/k$, $\varphi_k \equiv \varphi \equiv \psi_k \equiv \psi \equiv \alpha$, where $\alpha \in AC([0, 1]; \mathbb{R})$ is such that $\alpha(0) = \alpha(1)$. Then the assumptions of Theorem 7.1 are satisfied whereas $t_k \rightarrow 0$ and $x_k \rightarrow 0$ when k tends to $+\infty$.

If we suppose that the operators ℓ_k are ‘‘uniformly bounded’’ in the sense of relation (7.9) then we obtain the following statement.

⁵The symbol 0 stands here for the zero operator.

Corollary 7.1. *Let problem (1.1), (1.2) have a unique solution u , let there exist a function $\omega \in L(\mathcal{D}; \mathbb{R}_+)$ such that*

$$|\ell_k(y)(t, x)| \leq \omega(t, x) \|y\|_C$$

for a. e. $(t, x) \in \mathcal{D}$ and all $y \in C(\mathcal{D}; \mathbb{R})$, $k \in \mathbb{N}$, (7.9)

and

$$\lim_{k \rightarrow +\infty} \int_{t_k}^t \int_{x_k}^x \ell_k(y)(s, \eta) d\eta ds = \int_{t_0}^t \int_{x_0}^x \ell(y)(s, \eta) d\eta ds$$

uniformly on \mathcal{D} for every $y \in C^(\mathcal{D}; \mathbb{R})$.* (7.10)

Moreover, let

$$\lim_{k \rightarrow +\infty} \int_{t_k}^t \int_{x_k}^x q_k(s, \eta) d\eta ds = \int_{t_0}^t \int_{x_0}^x q(s, \eta) d\eta ds \quad \text{uniformly on } \mathcal{D}, \quad (7.11)$$

$$\lim_{k \rightarrow +\infty} \|\varphi_k - \varphi\|_C = 0, \quad \lim_{k \rightarrow +\infty} \|\psi_k - \psi\|_C = 0, \quad (7.12)$$

and

$$\lim_{k \rightarrow +\infty} \varphi(t_k) = \varphi(t_0). \quad (7.13)$$

Then the conclusion of Theorem 7.1 holds.

Remark 7.2. Condition (7.13) is satisfied if and only if

$$\lim_{k \rightarrow +\infty} \psi(x_k) = \psi(x_0).$$

Remark 7.3. Assumption (7.9) in the previous corollary is essential and cannot be omitted (see Example 9.2).

Corollary 7.2. *Let problem (1.1), (1.2) have a unique solution u and let there exist a function $\omega \in L(\mathcal{D}; \mathbb{R}_+)$ such that relation (7.9) holds. Moreover, let condition (7.12) be satisfied,*

$$\lim_{k \rightarrow +\infty} \int_a^t \int_c^x [\ell_k(y)(s, \eta) - \ell(y)(s, \eta)] d\eta ds = 0$$

uniformly on \mathcal{D} for every $y \in C^(\mathcal{D}; \mathbb{R})$,* (7.14)

$$\lim_{k \rightarrow +\infty} \int_a^t \int_c^x [q_k(y)(s, \eta) - q(s, \eta)] d\eta ds = 0 \quad \text{uniformly on } \mathcal{D}, \quad (7.15)$$

and

$$\lim_{k \rightarrow +\infty} t_k = t_0, \quad \lim_{k \rightarrow +\infty} x_k = x_0. \quad (7.16)$$

Then the conclusion of Theorem 7.1 holds.

Corollary 7.2 immediately yields

Corollary 7.3. *Let homogeneous problem (1.1₀), (1.2₀) have only the trivial solution. Then the Darboux operator⁶ of problem (1.1₀), (1.2₀) is continuous.*

⁶The notion of the Darboux operator is introduced in Definition 4.1.

Now we give a statement on the well-posedness of problem (1.1'), (1.2). For any $k \in \mathbb{N}$, along with equation (1.1') we consider the perturbed equation

$$\frac{\partial^2 u(t, x)}{\partial t \partial x} = p_k(t, x)u(\tau_k(t, x), \mu_k(t, x)) + q_k(t, x), \quad (1.1'_k)$$

where $p_k, q_k \in L(\mathcal{D}; \mathbb{R})$ and $\tau_k: \mathcal{D} \rightarrow [a, b]$, $\mu_k: \mathcal{D} \rightarrow [c, d]$ are measurable functions.

Corollary 7.4. *Let problem (1.1'), (1.2) have a unique solution u , let there exist a function $\omega \in L(\mathcal{D}; \mathbb{R}_+)$ such that*

$$|p_k(t, x)| \leq \omega(t, x) \quad \text{for a. e. } (t, x) \in \mathcal{D}, \quad k \in \mathbb{N}, \quad (7.17)$$

and

$$\lim_{k \rightarrow +\infty} \int_a^t \int_c^x [p_k(s, \eta) - p(s, \eta)] ds d\eta = 0 \quad \text{uniformly on } \mathcal{D}. \quad (7.18)$$

Moreover, let conditions (7.12), (7.15), and (7.16) be satisfied, and

$$\lim_{k \rightarrow +\infty} \text{ess sup} \left\{ |\tau_k(t, x) - \tau(t, x)| : (t, x) \in \mathcal{D} \right\} = 0, \quad (7.19)$$

$$\lim_{k \rightarrow +\infty} \text{ess sup} \left\{ |\mu_k(t, x) - \mu(t, x)| : (t, x) \in \mathcal{D} \right\} = 0. \quad (7.20)$$

Then there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$ problem (1.1'_k), (1.2_k) has a unique solution u_k and relation (7.8) holds.

Remark 7.4. The assumption (7.17) in the previous statement is essential and cannot be omitted (see Example 9.2).

Finally, we consider the hyperbolic equation without argument deviations

$$u_{tx} = p(t, x)u + q(t, x) \quad (7.21)$$

in which $p, q \in L(\mathcal{D}; \mathbb{R})$. For any $k \in \mathbb{N}$, along with equation (7.21) we consider the perturbed equation

$$u_{tx} = p_k(t, x)u + q_k(t, x), \quad (7.21_k)$$

where $p_k, q_k \in L(\mathcal{D}; \mathbb{R})$.

The following statement can be derived from Theorem 7.1.

Corollary 7.5. *Let conditions (7.4)–(7.6) be satisfied,*

$$\lim_{k \rightarrow +\infty} \varrho_k \left[\int_{t_k}^t \int_{x_k}^x p_k(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x p(s, \eta) d\eta ds \right] = 0 \quad \text{uniformly on } \mathcal{D}, \quad (7.22)$$

and

$$\lim_{k \rightarrow +\infty} \varrho_k \int_{t_0}^{t_k} \int_c^d |p(s, \eta)| d\eta ds = 0, \quad \lim_{k \rightarrow +\infty} \varrho_k \int_{x_0}^{x_k} \int_a^b |p(s, \eta)| ds d\eta = 0, \quad (7.23)$$

where

$$\varrho_k = 1 + \|p_k\|_L. \quad (7.24)$$

Then relation (7.8) holds, where u and u_k are solutions to problems (7.21), (1.2) and (7.21_k), (1.2_k), respectively.

From Corollary 7.5 we get

Corollary 7.6. *Let conditions (7.12), (7.15), (7.16), and (7.18) be satisfied, and*

$$\sup \{ \|p_k\|_L : k \in \mathbb{N} \} < +\infty.$$

Then the conclusion of Corollary 7.5 holds.

Corollary 7.6 immediately yields

Corollary 7.7. *Let conditions (7.12) and (7.16) be satisfied,*

$$\lim_{k \rightarrow +\infty} \|p_k - p\|_L = 0, \quad (7.25)$$

and

$$\lim_{k \rightarrow +\infty} \|q_k - q\|_L = 0. \quad (7.26)$$

Then the conclusion of Corollary 7.5 holds.

7.1. Proofs. In order to prove Theorem 7.1, we need the following lemma.

Lemma 7.1. *Let problem (1.1₀), (1.2₀) have only the trivial solution and let condition (7.1) hold, where the numbers λ_k are defined by formula (7.2). Then, for any $z \in C^*(\mathcal{D}; \mathbb{R})$, there exist $r_0 > 0$ and $k_0 \in \mathbb{N}$ such that*

$$\|y - z\|_C \leq r_0(1 + \|\ell_k\|) \left[\|\Delta_k(y) - \Delta_0(z)\|_C + \|\Gamma_k(y, z)\|_C \right] \\ \text{for } k > k_0, y \in C^*(\mathcal{D}; \mathbb{R}), \quad (7.27)$$

where

$$\Delta_k(v)(t, x) = -v(t_k, x_k) + v(t, x_k) + v(t_k, x) \\ \text{for } (t, x) \in \mathcal{D}, v \in C^*(\mathcal{D}; \mathbb{R}), k \in \mathbb{N} \cup \{0\}, \quad (7.28)$$

and

$$\Gamma_k(v, w)(t, x) = \int_{t_k}^t \int_{x_k}^x [v''_{[1,2]}(s, \eta) - \ell_k(v - w)(s, \eta)] d\eta ds \\ - \int_{t_0}^t \int_{x_0}^x w''_{[1,2]}(s, \eta) d\eta ds \\ \text{for } (t, x) \in \mathcal{D}, v, w \in C^*(\mathcal{D}; \mathbb{R}), k \in \mathbb{N}. \quad (7.29)$$

Proof. Let the operators $T, T_k: C(\mathcal{D}; \mathbb{R}) \rightarrow C(\mathcal{D}; \mathbb{R})$ be defined by formula (3.1) and the relation

$$T_k(v)(t, x) = \int_{t_k}^t \int_{x_k}^x \ell_k(v)(s, \eta) d\eta ds \quad \text{for } (t, x) \in \mathcal{D}, v \in C(\mathcal{D}; \mathbb{R}), k \in \mathbb{N}.$$

Obviously, we have

$$\|T_k(y)\|_C \leq \|\ell_k(y)\|_L \leq \|\ell_k\| \|y\|_C \quad \text{for } y \in C(\mathcal{D}; \mathbb{R}), k \in \mathbb{N}.$$

Therefore, the operators T_k ($k \in \mathbb{N}$) are linear bounded ones, and the relation

$$\|T_k\| \leq \|\ell_k\| \quad \text{for } k \in \mathbb{N} \quad (7.30)$$

holds. Moreover, condition (7.1) with λ_k given by (7.2) can be rewritten in the form

$$\sup \left\{ \|T_k(y) - T(y)\|_C : y \in M(\ell_k, t_k, x_k) \right\} \rightarrow 0 \quad \text{as } k \rightarrow +\infty. \quad (7.31)$$

Assume that, on the contrary, the assertion of the lemma is not true. Then there exist $z \in C^*(\mathcal{D}; \mathbb{R})$, an increasing sequence $\{k_m\}_{m=1}^{+\infty}$ of natural numbers, and a sequence $\{y_m\}_{m=1}^{+\infty}$ of functions from $C^*(\mathcal{D}; \mathbb{R})$ such that the relation

$$\|y_m - z\|_C > m(1 + \|\ell_{k_m}\|) \left[\|\Delta_{k_m}(y_m) - \Delta_0(z)\|_C + \|\Gamma_{k_m}(y_m, z)\|_C \right] \quad (7.32)$$

holds for every $m \in \mathbb{N}$. For any $m \in \mathbb{N}$ and all $(t, x) \in \mathcal{D}$, we put

$$z_m(t, x) = \frac{y_m(t, x) - z(t, x)}{\|y_m - z\|_C}, \quad (7.33)$$

$$v_m(t, x) = \frac{1}{\|y_m - z\|_C} \left[\Delta_{k_m}(y_m)(t, x) - \Delta_0(z)(t, x) + \Gamma_{k_m}(y_m, z)(t, x) \right], \quad (7.34)$$

$$z_{0,m}(t, x) = z_m(t, x) - v_m(t, x), \quad (7.35)$$

$$w_m(t, x) = T_{k_m}(z_{0,m})(t, x) - T(z_{0,m})(t, x) + T_{k_m}(v_m)(t, x). \quad (7.36)$$

Obviously,

$$\|z_m\|_C = 1 \quad \text{for } m \in \mathbb{N}. \quad (7.37)$$

Using (7.28)–(7.29) in relation (7.34), by virtue of conditions (a)–(c) of Proposition 2.1, we get

$$z_{0,m}(t, x) = T_{k_m}(z_m)(t, x) \quad \text{for } (t, x) \in \mathcal{D}, \quad m \in \mathbb{N}, \quad (7.38)$$

and thus

$$z_{0,m}(t, x) = T(z_{0,m})(t, x) + w_m(t, x) \quad \text{for } (t, x) \in \mathcal{D}, \quad m \in \mathbb{N}. \quad (7.39)$$

Moreover, it follows from (7.32) and (7.34) that

$$\|v_m\|_C \leq \frac{\|\Delta_{k_m}(y_m) - \Delta_0(z)\|_C + \|\Gamma_{k_m}(y_m, z)\|_C}{\|y_m - z\|_C} < \frac{1}{m(1 + \|\ell_{k_m}\|)} \quad (7.40)$$

for $m \in \mathbb{N}$. Now the relations (7.30) and (7.40) yield

$$\|T_{k_m}(v_m)\|_C \leq \|T_{k_m}\| \|v_m\|_C \leq \frac{\|\ell_{k_m}\|}{m(1 + \|\ell_{k_m}\|)} < \frac{1}{m} \quad \text{for } m \in \mathbb{N}. \quad (7.41)$$

Observe that expression (7.38) and condition (7.37) guarantee the validity of the inclusion $z_{0,m} \in M(\ell_{k_m}, t_{k_m}, x_{k_m})$ for $m \in \mathbb{N}$ and thus, in view of (7.31), we obtain

$$\lim_{m \rightarrow +\infty} \|T_{k_m}(z_{0,m}) - T(z_{0,m})\|_C = 0. \quad (7.42)$$

According to (7.41) and (7.42), it follows from relation (7.36) that

$$\lim_{m \rightarrow +\infty} \|w_m\|_C = 0 \quad (7.43)$$

and, by virtue of (7.37) and (7.40), equality (7.35) implies that $\|z_{0,m}\|_C < 2$ for $m \in \mathbb{N}$. Since the sequence $\{\|z_{0,m}\|_C\}_{m=1}^{+\infty}$ is bounded and the operator T is completely continuous (see Proposition 3.1), there exists a subsequence of $\{T(z_{0,m})\}_{m=1}^{+\infty}$ which is convergent. We can assume without loss of generality that the sequence $\{T(z_{0,m})\}_{m=1}^{+\infty}$ is convergent, i. e., that there exists $z_0 \in C(\mathcal{D}; \mathbb{R})$ such that

$$\lim_{m \rightarrow +\infty} \|T(z_{0,m}) - z_0\|_C = 0.$$

Then it is clear that

$$\lim_{m \rightarrow +\infty} \|z_{0,m} - z_0\|_C = 0, \quad (7.44)$$

because the functions $z_{0,m}$ admit representation (7.39) and relation (7.43) is satisfied. However, estimate (7.40) holds for v_m and thus, equality (7.35) yields that

$$\lim_{m \rightarrow +\infty} \|z_m - z_0\|_C = 0$$

which, together with (7.37), guarantees $\|z_0\|_C = 1$. Since the operator T is continuous and conditions (7.43) and (7.44) are fulfilled, relation (7.39) yields that $z_0 = T(z_0)$. Consequently, $z_0 \in C^*(\mathcal{D}; \mathbb{R})$ (see Proposition 2.1) and, by virtue of Lemma 4.1(ii), z_0 is a nontrivial solution to homogeneous problem (1.1₀), (1.2₀), which is a contradiction. \square

Proof of Theorem 7.1. Since problem (1.1), (1.2) has a unique solution, homogeneous problem (1.1₀), (1.2₀) has only the trivial solution. Therefore, the assumptions of Lemma 7.1 are satisfied and thus there exist $r_0 > 0$ and $k_0 \in \mathbb{N}$ such that

$$\|y\|_C \leq r_0(1 + \|\ell_k\|) \left[\|\Delta_k(y)\|_C + \|\Gamma_k(y, 0)\|_C \right] \quad \text{for } k > k_0, y \in C^*(\mathcal{D}; \mathbb{R}) \quad (7.45)$$

and

$$\|y - u\|_C \leq r_0(1 + \|\ell_k\|) \left[\|\Delta_k(y) - \Delta_0(u)\|_C + \|\Gamma_k(y, u)\|_C \right] \quad \text{for } k > k_0, y \in C^*(\mathcal{D}; \mathbb{R}), \quad (7.46)$$

where the operators Δ_k and Γ_k are given by formulas (7.28) and (7.29), respectively.

If for some $k \in \mathbb{N}$, u_0 is a solution to the problem

$$\frac{\partial^2 u(t, x)}{\partial t \partial x} = \ell_k(u)(t, x), \quad (7.47)$$

$$u(t, x_k) = 0 \quad \text{for } t \in [a, b], \quad u(t_k, x) = 0 \quad \text{for } x \in [c, d],$$

then $\Delta_k(u_0) \equiv 0$ and $\Gamma_k(u_0, 0) \equiv 0$. Therefore, relation (7.45) guarantees that for every $k > k_0$, homogeneous problem (7.47) has only the trivial solution. Hence, for every $k > k_0$, problem (1.1_k), (1.2_k) has a unique solution u_k (see Theorem 4.1). Clearly we have

$$\begin{aligned} \Delta_k(u_k)(t, x) &= -\varphi_k(t_k) + \varphi_k(t) + \psi_k(x) \quad \text{for } (t, x) \in \mathcal{D}, k > k_0, \\ \Delta_0(u)(t, x) &= -\varphi(t_0) + \varphi(t) + \psi(x) \quad \text{for } (t, x) \in \mathcal{D}, \end{aligned}$$

and

$$\begin{aligned} \Gamma_k(u, u_k)(t, x) &= \int_{t_k}^t \int_{x_k}^x \ell_k(u)(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x \ell(u)(s, \eta) d\eta ds \\ &\quad + \int_{t_k}^t \int_{x_k}^x q_k(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x q(s, \eta) d\eta ds \end{aligned}$$

for all $(t, x) \in \mathcal{D}$ and every $k > k_0$. Therefore, by using relations (7.3)–(7.6), we get

$$\lim_{k \rightarrow +\infty} (1 + \|\ell_k\|) \left[\|\Delta_k(u_k) - \Delta_0(u)\|_C + \|\Gamma_k(u_k, u)\|_C \right] = 0. \quad (7.48)$$

On the other hand, it follows from inequality (7.46) that

$$\|u_k - u\|_C \leq r_0(1 + \|\ell_k\|) \left[\|\Delta_k(u_k) - \Delta_0(u)\|_C + \|\Gamma_k(u_k, u)\|_C \right] \quad \text{for } k > k_0 \quad (7.49)$$

and thus, in view of limit (7.48), desired relation (7.8) holds. \square

Proof of Corollary 7.1. We shall show that the assumptions of Theorem 7.1 are satisfied. Indeed, relation (7.9) yields that $\|\ell_k\| \leq \|\omega\|_L$ for $k \in \mathbb{N}$. Therefore, by virtue of relations (7.10)–(7.13), assumptions (7.3)–(7.6) of Theorem 7.1 are fulfilled. It remains to show that condition (7.1) holds, where the numbers λ_k are given by formula (7.2).

Assume that, on the contrary, condition (7.1) does not hold. Then there exist $\varepsilon_0 > 0$, an increasing sequence $\{k_m\}_{m=1}^{+\infty}$ of natural numbers, and a sequence $\{y_m\}_{m=1}^{+\infty}$ such that

$$y_m \in M(\ell_{k_m}, t_{k_m}, x_{k_m}) \quad \text{for } m \in \mathbb{N} \quad (7.50)$$

and

$$\max_{(t,x) \in \mathcal{D}} \left\{ \left| \int_{t_{k_m}}^t \int_{x_{k_m}}^x \ell_{k_m}(y_m)(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x \ell(y_m)(s, \eta) d\eta ds \right| \right\} \geq \varepsilon_0 \quad (7.51)$$

for $m \in \mathbb{N}$.

In view of inclusion (7.50) and Notation 7.1, we get

$$y_m(t, x) = \int_{t_{k_m}}^t \int_{x_{k_m}}^x \ell_{k_m}(z_m)(s, \eta) d\eta ds \quad \text{for } (t, x) \in \mathcal{D}, \quad m \in \mathbb{N},$$

where $z_m \in C(\mathcal{D}; \mathbb{R})$ and $\|z_m\|_C = 1$ for $m \in \mathbb{N}$. Since we suppose that the operators ℓ_k are uniformly bounded in the sense of condition (7.9), we obtain $\|y_m\|_C \leq \|\omega\|_L$ for $m \in \mathbb{N}$ and thus the sequence $\{y_m\}_{m=1}^{+\infty}$ is bounded in the space $C(\mathcal{D}; \mathbb{R})$. We will show that the sequence indicated is also equicontinuous. Let $\varepsilon > 0$ be arbitrary. Since the function ω is integrable on \mathcal{D} , there exists $\delta > 0$ such that the relation

$$\iint_E \omega(t, x) dt dx < \frac{\varepsilon}{2} \quad (7.52)$$

holds for every measurable set $E \subseteq \mathcal{D}$ satisfying $\text{meas } E < \max\{b-a, d-c\}\delta$. Using condition (7.9), we get

$$\begin{aligned} & \left| \int_{t_{k_m}}^{t_2} \int_{x_{k_m}}^{x_2} \ell_{k_m}(z_m)(s, \eta) d\eta ds - \int_{t_{k_m}}^{t_1} \int_{x_{k_m}}^{x_1} \ell_{k_m}(z_m)(s, \eta) d\eta ds \right| \\ & \leq \sum_{k=1}^2 \iint_{E_k} \omega(s, \eta) ds d\eta \quad \text{for } (t_1, x_1), (t_2, x_2) \in \mathcal{D}, \quad m \in \mathbb{N}, \end{aligned}$$

where the measurable sets $E_1, E_2 \subseteq \mathcal{D}$ are such that $\text{meas } E_1 = (d-c)|t_2 - t_1|$ and $\text{meas } E_2 = (b-a)|x_2 - x_1|$. Therefore, by virtue of (7.52), we have

$$\begin{aligned} & |y_m(t_2, x_2) - y_m(t_1, x_1)| < \varepsilon \\ & \text{for } (t_1, x_1), (t_2, x_2) \in \mathcal{D}, \quad |t_2 - t_1| + |x_2 - x_1| < \delta, \quad m \in \mathbb{N}. \end{aligned}$$

Consequently, the sequence $\{y_m\}_{m=1}^{+\infty}$ is equicontinuous in the space $C(\mathcal{D}; \mathbb{R})$. Therefore, according to the Arzelà-Ascoli lemma, we can assume without loss of generality that the sequence indicated is convergent. Hence, there exists $p_0 \in \mathbb{N}$ such that

$$\|y_m - y_{p_0}\|_C < \frac{\varepsilon_0}{2(\|\omega\|_L + \|\ell\| + 1)} \quad \text{for } m \geq p_0. \quad (7.53)$$

Since $y_{p_0} \in C^*(\mathcal{D}; \mathbb{R})$ and relation (7.10) holds, there exists $p_1 \in \mathbb{N}$ such that

$$\max_{(t,x) \in \mathcal{D}} \left\{ \left| \int_{t_k}^t \int_{x_k}^x \ell_k(y_{p_0})(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x \ell(y_{p_0})(s, \eta) d\eta ds \right| \right\} < \frac{\varepsilon_0}{2} \quad (7.54)$$

for $k \geq p_1$.

Now we choose a number $M \in \mathbb{N}$ satisfying $M \geq p_0$ and $k_M \geq p_1$. It is clear that

$$\begin{aligned} & \left| \int_{t_{k_M}}^t \int_{x_{k_M}}^x \ell_{k_M}(y_M)(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x \ell(y_M)(s, \eta) d\eta ds \right| \\ & \leq \left| \int_{t_{k_M}}^t \int_{x_{k_M}}^x \ell_{k_M}(y_{p_0})(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x \ell(y_{p_0})(s, \eta) d\eta ds \right| \\ & \quad + \left| \int_{t_{k_M}}^t \int_{x_{k_M}}^x \ell_{k_M}(y_M - y_{p_0})(s, \eta) d\eta ds \right| \\ & \quad + \left| \int_{t_0}^t \int_{x_0}^x \ell(y_{p_0} - y_M)(s, \eta) d\eta ds \right| \quad \text{for } (t, x) \in \mathcal{D}. \end{aligned}$$

Therefore, by virtue of conditions (7.9), (7.53), and (7.54), the last relation yields that

$$\begin{aligned} & \max_{(t,x) \in \mathcal{D}} \left\{ \left| \int_{t_{k_M}}^t \int_{x_{k_M}}^x \ell_{k_M}(y_M)(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x \ell(y_M)(s, \eta) d\eta ds \right| \right\} \\ & \leq \frac{\varepsilon_0}{2} + \|\omega\|_L \|y_M - y_{p_0}\|_C + \|\ell\| \|y_{p_0} - y_M\|_C < \varepsilon_0, \quad (7.55) \end{aligned}$$

which contradicts condition (7.51).

The contradiction obtained proves the validity of condition (7.1) and thus all the assumptions of Theorem 7.1 are satisfied. \square

To prove Corollary 7.2 we need the following lemma.

Lemma 7.2. *Let condition (7.16) hold and let $\{\sigma_k\}_{k=1}^{+\infty}$ be a sequence of functions from $L(\mathcal{D}; \mathbb{R})$ such that*

$$\lim_{k \rightarrow +\infty} \int_a^t \int_c^x [\sigma_k(s, \eta) - \sigma(s, \eta)] d\eta ds = 0 \quad \text{uniformly on } \mathcal{D}, \quad (7.56)$$

where $\sigma \in L(\mathcal{D}; \mathbb{R})$. Then

$$\lim_{k \rightarrow +\infty} \int_{t_k}^t \int_{x_k}^x \sigma_k(s, \eta) d\eta ds = \int_{t_0}^t \int_{x_0}^x \sigma(s, \eta) d\eta ds \quad \text{uniformly on } \mathcal{D}. \quad (7.57)$$

Proof. It is easy to verify that

$$\begin{aligned} & \int_{t_k}^t \int_{x_k}^x \sigma_k(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x \sigma(s, \eta) d\eta ds \\ & = \int_a^{t_k} \int_c^{x_k} [\sigma_k(s, \eta) - \sigma(s, \eta)] d\eta ds + \int_a^t \int_c^x [\sigma_k(s, \eta) - \sigma(s, \eta)] d\eta ds \\ & \quad + \left(\int_a^{t_k} \int_c^{x_k} \sigma(s, \eta) d\eta ds - \int_a^{t_0} \int_c^{x_0} \sigma(s, \eta) d\eta ds \right) \end{aligned}$$

$$\begin{aligned}
& + \int_{t_k}^{t_0} \int_c^x \sigma(s, \eta) d\eta ds - \int_a^{t_k} \int_c^x [\sigma_k(s, \eta) - \sigma(s, \eta)] d\eta ds \\
& + \int_a^t \int_{x_k}^{x_0} \sigma(s, \eta) d\eta ds - \int_a^t \int_c^{x_k} [\sigma_k(s, \eta) - \sigma(s, \eta)] d\eta ds
\end{aligned}$$

for $(t, x) \in \mathcal{D}$. Therefore, by using assumptions (7.16) and (7.56), we get the validity of condition (7.57). \square

Proof of Corollary 7.2. We shall show that the assumptions of Corollary 7.1 are satisfied. Indeed, according to Lemma 7.2, assumptions (7.14)–(7.16) guarantee the validity of conditions (7.10) and (7.11). On the other hand, condition (7.13) is obviously satisfied, because the function φ is continuous and $t_k \rightarrow t_0$ when k tends to $+\infty$. \square

In order to prove Corollary 7.4 we need the following Krasnoselskii-Krein type lemma.

Lemma 7.3. *Let $p, p_k \in L(\mathcal{D}; \mathbb{R})$ and let $\alpha, \alpha_k: \mathcal{D} \rightarrow \mathbb{R}$ be measurable and essentially bounded functions ($k \in \mathbb{N}$). Assume that relations (7.17) and (7.18) are satisfied, and*

$$\lim_{k \rightarrow +\infty} \text{ess sup} \left\{ |\alpha_k(t, x) - \alpha(t, x)| : (t, x) \in \mathcal{D} \right\} = 0. \quad (7.58)$$

Then

$$\lim_{k \rightarrow +\infty} \int_a^t \int_c^x [p_k(s, \eta) \alpha_k(s, \eta) - p(s, \eta) \alpha(s, \eta)] d\eta ds = 0 \quad \text{uniformly on } \mathcal{D}. \quad (7.59)$$

Proof. Without loss of generality we can assume that

$$|p(t, x)| \leq \omega(t, x) \quad \text{for a. e. } (t, x) \in \mathcal{D}. \quad (7.60)$$

Let $\varepsilon > 0$ be arbitrary. According to assumption (7.58), there exists $k_0 \in \mathbb{N}$ such that

$$\iint_{\mathcal{D}} \omega(t, x) |\alpha_k(t, x) - \alpha(t, x)| dt dx < \frac{\varepsilon}{4} \quad \text{for } k \geq k_0. \quad (7.61)$$

Since the function α is measurable and essentially bounded, there exists a function $w \in C(\mathcal{D}; \mathbb{R})$, which has continuous derivatives up to the second order and such that

$$\iint_{\mathcal{D}} \omega(t, x) |\alpha(t, x) - w(t, x)| dt dx < \frac{\varepsilon}{4}. \quad (7.62)$$

For any $k \in \mathbb{N}$, we put

$$f_k(t, x) = \int_a^t \int_c^x [p_k(s, \eta) - p(s, \eta)] d\eta ds \quad \text{for } (t, x) \in \mathcal{D}.$$

Clearly, assumption (7.18) can be rewritten in the form

$$\lim_{k \rightarrow +\infty} \|f_k\|_C = 0. \quad (7.63)$$

It can be verified by direct calculation that

$$\int_a^t \int_c^x [p_k(s, \eta) - p(s, \eta)] w(s, \eta) d\eta ds$$

$$\begin{aligned}
&= f_k(t, x)w(t, x) - \int_a^t f_k(s, x)w'_{[1]}(s, x)ds - \int_c^x f_k(t, \eta)w'_{[2]}(t, \eta)d\eta \\
&\quad + \int_a^t \int_c^x f_k(s, \eta)w''_{[1,2]}(s, \eta)d\eta ds \quad \text{for } (t, x) \in \mathcal{D}, k \in \mathbb{N}.
\end{aligned}$$

Consequently, by using relation (7.63), we get

$$\lim_{k \rightarrow +\infty} \int_a^t \int_c^x [p_k(s, \eta) - p(s, \eta)]w(s, \eta)d\eta ds = 0 \quad \text{uniformly on } \mathcal{D}.$$

Hence, there exists a number $k_1 \geq k_0$ such that

$$\left| \int_a^t \int_c^x [p_k(s, \eta) - p(s, \eta)]w(s, \eta)d\eta ds \right| < \frac{\varepsilon}{4} \quad \text{for } (t, x) \in \mathcal{D}, k \geq k_1. \quad (7.64)$$

On the other hand, it is clear that

$$\begin{aligned}
&\int_a^t \int_c^x [p_k(s, \eta)\alpha_k(s, \eta) - p(s, \eta)\alpha(s, \eta)]d\eta ds \\
&= \int_a^t \int_c^x p_k(s, \eta)[\alpha_k(s, \eta) - \alpha(s, \eta)]d\eta ds \\
&\quad + \int_a^t \int_c^x [p_k(s, \eta) - p(s, \eta)]w(s, \eta)d\eta ds \\
&\quad + \int_a^t \int_c^x [p_k(s, \eta) - p(s, \eta)][\alpha(s, \eta) - w(s, \eta)]d\eta ds.
\end{aligned}$$

for all $(t, x) \in \mathcal{D}$ and every $k \in \mathbb{N}$. Therefore, in view of relations (7.17), (7.60)–(7.62), and (7.64), we get

$$\begin{aligned}
&\left| \int_a^t \int_c^x [p_k(s, \eta)\alpha_k(s, \eta) - p(s, \eta)\alpha(s, \eta)]d\eta ds \right| \\
&\leq \iint_{\mathcal{D}} \omega(s, \eta)|\alpha_k(s, \eta) - \alpha(s, \eta)|dsd\eta \\
&\quad + \left| \int_a^t \int_c^x [p_k(s, \eta) - p(s, \eta)]w(s, \eta)d\eta ds \right| \\
&\quad + 2 \iint_{\mathcal{D}} \omega(s, \eta)|\alpha(s, \eta) - w(s, \eta)|dsd\eta \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2 \frac{\varepsilon}{4} = \varepsilon \quad \text{for } (t, x) \in \mathcal{D}, k \geq k_1
\end{aligned}$$

and thus desired relation (7.59) holds. \square

Proof of Corollary 7.4. Let the operator ℓ be defined by formula (5.1). Put

$$\begin{aligned}
\ell_k(v)(t, x) &= p_k(t, x)v(\tau_k(t, x), \mu_k(t, x)) \\
&\quad \text{for a. e. } (t, x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D}; \mathbb{R}), k \in \mathbb{N}. \quad (7.65)
\end{aligned}$$

We will show that condition (7.14) is satisfied. Indeed, let $y \in C^*(\mathcal{D}; \mathbb{R})$ be arbitrary. It is clear that conditions (7.19) and (7.20) guarantee the validity of relation (7.58), where

$$\alpha_k(t, x) = y(\tau_k(t, x), \mu_k(t, x)), \quad \alpha(t, x) = y(\tau(t, x), \mu(t, x))$$

for a. e. $(t, x) \in \mathcal{D}$ and all $k \in \mathbb{N}$. Therefore, it follows from Lemma 7.3 that relation (7.59) holds and thus condition (7.14) is fulfilled. On the other hand, by virtue of relation (7.17), condition (7.9) is satisfied.

Consequently, the assertion of the corollary follows from Corollary 7.2. \square

Proof of Corollary 7.5. We first mention that, according to Corollary 6.3, problems (7.21), (1.2) and (7.21_k), (1.2_k) have unique solutions u and u_k , respectively.

Let the operators ℓ and ℓ_k be defined by the formulas

$$\ell(v)(t, x) = p(t, x)v(t, x) \quad \text{for a. e. } (t, x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D}; \mathbb{R}), \quad (7.66)$$

and

$$\ell_k(v)(t, x) = p_k(t, x)v(t, x) \quad \text{for a. e. } (t, x) \in \mathcal{D} \text{ all } v \in C(\mathcal{D}; \mathbb{R}), \quad k \in \mathbb{N}, \quad (7.67)$$

respectively. Clearly

$$\|\ell_k\| = \|p_k\|_L \quad \text{for } k \in \mathbb{N}. \quad (7.68)$$

Therefore, assumptions (7.4)–(7.6) of Theorem 7.1 are satisfied. In order to apply Theorem 7.1, it remains to show that conditions (7.1) and (7.3) are fulfilled.

It is easy to see that

$$\begin{aligned} & \left| \int_{t_k}^t \int_{x_k}^x [p_k(s, \eta) - p(s, \eta)] d\eta ds \right| \\ & \leq \left| \int_{t_k}^t \int_{x_k}^x p_k(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x p(s, \eta) d\eta ds \right| \\ & \quad + \left| \int_{t_0}^{t_k} \int_c^d |p(s, \eta)| d\eta ds \right| + \left| \int_{x_0}^{x_k} \int_a^b |p(s, \eta)| ds d\eta \right| \end{aligned}$$

for all $(t, x) \in \mathcal{D}$ and every $k \in \mathbb{N}$. Therefore, conditions (7.22) and (7.23) guarantee that

$$\lim_{k \rightarrow +\infty} \varrho_k \|f_k\|_C = 0, \quad (7.69)$$

where

$$f_k(t, x) = \int_{t_k}^t \int_{x_k}^x [p_k(s, \eta) - p(s, \eta)] d\eta ds \quad \text{for } (t, x) \in \mathcal{D}, \quad k \in \mathbb{N}. \quad (7.70)$$

Observe that for an arbitrary $y \in C(\mathcal{D}; \mathbb{R})$ we have

$$\begin{aligned} & \left| \int_{t_k}^t \int_{x_k}^x \ell_k(y)(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x \ell(y)(s, \eta) d\eta ds \right| \\ & \leq \left| \int_{t_k}^t \int_{x_k}^x [p_k(s, \eta) - p(s, \eta)] y(s, \eta) d\eta ds \right| \\ & \quad + \left| \int_{t_0}^{t_k} \int_c^d |p(s, \eta) y(s, \eta)| d\eta ds \right| + \\ & \quad + \left| \int_{x_0}^{x_k} \int_a^b |p(s, \eta) y(s, \eta)| ds d\eta \right| \quad \text{for } (t, x) \in \mathcal{D}, \quad k \in \mathbb{N}. \quad (7.71) \end{aligned}$$

Moreover, for an arbitrary $y \in C^*(\mathcal{D}; \mathbb{R})$, we can verify by direct calculation that

$$\begin{aligned} & \int_{t_k}^t \int_{x_k}^x [p_k(s, \eta) - p(s, \eta)] y(s, \eta) d\eta ds \\ &= f_k(t, x) y(t, x) - \int_{t_k}^t f_k(s, x) y'_{[1]}(s, x) ds - \int_{x_k}^x f_k(t, \eta) y'_{[2]}(t, \eta) d\eta \\ &+ \int_{t_k}^t \int_{x_k}^x f_k(s, \eta) y''_{[1,2]}(s, \eta) d\eta ds \quad \text{for } (t, x) \in \mathcal{D}, \quad k \in \mathbb{N}. \end{aligned} \quad (7.72)$$

Let $k \in \mathbb{N}$ and $y \in M(\ell_k, t_k, x_k)$ be arbitrary. Then, by virtue of Notation 7.1 and Lemma 4.1, we get

$$|y(t, x)| = \left| \int_{t_k}^t \int_{x_k}^x p_k(s, \eta) z(s, \eta) d\eta ds \right| \leq \varrho_k \quad \text{for } (t, x) \in \mathcal{D}, \quad (7.73)$$

$$|y'_{[1]}(t, x)| = \left| \int_{x_k}^x p_k(t, \eta) z(t, \eta) d\eta \right| \leq \int_c^d |p_k(t, \eta)| d\eta$$

for a. e. $t \in [a, b]$ and all $x \in [c, d]$, (7.74)

$$|y'_{[2]}(t, x)| = \left| \int_{t_k}^t p_k(s, x) z(s, x) ds \right| \leq \int_a^b |p_k(s, x)| ds$$

for all $t \in [a, b]$ and a. e. $x \in [c, d]$, (7.75)

and

$$|y''_{[1,2]}(t, x)| = |p_k(t, x) z(t, x)| \leq |p_k(t, x)| \quad \text{for a. e. } (t, x) \in \mathcal{D}. \quad (7.76)$$

By virtue of relations (7.73)–(7.76), it follows from inequalities (7.71) and (7.72) that

$$\begin{aligned} & \left| \int_{t_k}^t \int_{x_k}^x \ell_k(y)(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x \ell(y)(s, \eta) d\eta ds \right| \\ & \leq 4\varrho_k \|f_k\|_C + \varrho_k \left| \int_{t_0}^{t_k} \int_c^d |p(s, \eta)| d\eta ds \right| + \\ & \quad + \varrho_k \left| \int_{x_0}^{x_k} \int_a^b |p(s, \eta)| ds d\eta \right| \quad \text{for } (t, x) \in \mathcal{D}. \end{aligned}$$

Therefore, according to assumptions (7.23) and (7.69), condition (7.1) holds, where the numbers λ_k are given by formula (7.2).

Now let $y \in C^*(\mathcal{D}; \mathbb{R})$ be arbitrary. Put

$$\begin{aligned} \varrho_0 &= \|y\|_C + \max \left\{ \int_a^b |y'_{[1]}(s, x)| ds : x \in [c, d] \right\} \\ &+ \max \left\{ \int_c^d |y'_{[2]}(t, \eta)| d\eta : t \in [a, b] \right\} + \|y''_{[1,2]}\|_L. \end{aligned} \quad (7.77)$$

Then inequalities (7.71) and (7.72) imply that

$$\begin{aligned} & \left| \int_{t_k}^t \int_{x_k}^x \ell_k(y)(s, \eta) d\eta ds - \int_{t_0}^t \int_{x_0}^x \ell(y)(s, \eta) d\eta ds \right| \\ & \leq \varrho_0 \left(\|f_k\|_C + \left| \int_{t_0}^{t_k} \int_c^d |p(s, \eta)| d\eta ds \right| \right. \\ & \quad \left. + \left| \int_{x_0}^{x_k} \int_a^b |p(s, \eta)| ds d\eta \right| \right) \quad \text{for } (t, x) \in \mathcal{D}, \quad k \in \mathbb{N}. \end{aligned}$$

According to relations (7.23) and (7.69), the last inequality yields the validity of condition (7.3).

Consequently, the assertion of the corollary follows from Theorem 7.1. \square

Proof of Corollary 7.6. It follows from condition (7.25) that

$$\sup \{ \|p_k\|_L : k \in \mathbb{N} \} < +\infty.$$

Therefore, in view of relations (7.12) and (7.16), assumptions (7.5), (7.6), and (7.23) of Corollary 7.5 are satisfied. Moreover, by virtue of relations (7.25), (7.26), and (7.16), Lemma 7.2 guarantees the validity of assumptions (7.4) and (7.22) of Corollary 7.5.

Consequently, all the assumptions of Corollary 7.5 are satisfied. \square

8. ON INTEGRAL REPRESENTATION OF SOLUTIONS

The following proposition follows immediately from the linearity of problem (1.1), (1.2) and Definition 4.1.

Proposition 8.1. *Let problem (1.1₀), (1.2₀) have only the trivial solution. Then the unique solution u to problem (1.1), (1.2) admits the representation*

$$u = u_0 + \Omega(q), \tag{8.1}$$

where u_0 is a solution to problem (1.1₀), (1.2₀) and Ω is the Darboux operator of problem (1.1₀), (1.2₀).

For the equation without argument deviations

$$u_{tx} = p(t, x)u + q(t, x) \tag{8.2}$$

in which $p, q \in L(\mathcal{D}; \mathbb{R})$, representation (8.1) of solutions to the Darboux problem can be expressed in terms of the Riemann functions introduced in the following definition.

Definition 8.1. Let $t_0 \in [a, b]$ and $x_0 \in [c, d]$. The (unique) solution to the problem

$$u_{tx} = p(t, x)u, \tag{8.2_0}$$

$$u(t, x_0) = 1 \quad \text{for } t \in [a, b], \quad u(t_0, x) = 1 \quad \text{for } x \in [c, d] \tag{8.3}$$

is denoted by Z_{t_0, x_0} and called the Riemann function of equation (8.2₀).

Remark 8.1. In view of Corollary 6.3, the Riemann function Z_{t_0, x_0} is well defined.

Example 8.1. Let $t_0 \in]a, b]$, $x_0 \in]c, d]$, and $k \geq 0$. We shall show that the Riemann function Z_{t_0, x_0} of the equation

$$u_{tx} = -ku \quad (8.4)$$

satisfies the relation

$$Z_{t_0, x_0}(t, x) = J_0\left(2\sqrt{k(t_0 - t)(x_0 - x)}\right) \quad \text{for } (t, x) \in [a, t_0] \times [c, x_0], \quad (8.5)$$

where J_0 denotes the Bessel function of the first kind and order 0. Indeed, let us consider the transformation

$$u(t, x) = v(z), \quad \text{where } z = \sqrt{(t_0 - t)(x_0 - x)}. \quad (8.6)$$

Then problem (8.4), (8.3) on the rectangle $[a, t_0] \times [c, x_0]$ can be reduced to the problem

$$z^2 v''(z) + z v'(z) + 4kz^2 v(z) = 0, \quad v(0) = 1 \quad (8.7)$$

on the interval $[0, \sqrt{(t_0 - a)(x_0 - c)}]$ in the following way. If v is a solution to problem (8.7) on the interval $[0, \sqrt{(t_0 - a)(x_0 - c)}]$ then the function u defined by relation (8.6) is a solution to problem (8.4), (8.3) on the rectangle $[a, t_0] \times [c, x_0]$.

From the theory of Bessel functions it follows that problem (8.7) has a solution

$$v(z) = J_0(2\sqrt{kz}) \quad \text{for } z \in [0, \sqrt{(t_0 - a)(x_0 - c)}]$$

and thus the Riemann function Z_{t_0, x_0} satisfies condition (8.5).

Theorem 8.1. *For an arbitrary solution u to problem (1.1), (1.2), the integral representation*

$$\begin{aligned} u(t, x) &= Z_{t, x}(t_0, x_0)\varphi(t_0) \\ &+ \int_{t_0}^t Z_{t, x}(s, x_0)\varphi'(s)ds + \int_{x_0}^x Z_{t, x}(t_0, \eta)\psi'(\eta)d\eta \\ &+ \int_{t_0}^t \int_{x_0}^x Z_{t, x}(s, \eta)q(s, \eta)d\eta ds \end{aligned} \quad (8.8)$$

holds for $(t, x) \in \mathcal{D}$.

Remark 8.2. Integral representations of the type (8.8) in the case of continuous coefficients p and q are discussed in the monograph [21].

Proof. Let $u, z \in C^*(\mathcal{D}; \mathbb{R})$ be arbitrary. Then, according to Proposition 2.1, clearly $zu''_{[1, 2]}, uz''_{[1, 2]} \in L(\mathcal{D}; \mathbb{R})$. Therefore, by using the properties of absolutely continuous functions presented in Proposition 2.1 and Remarks 2.1 and 2.2, we get

$$\begin{aligned} &\int_{t_0}^t \int_{x_0}^x z(s, \eta)u''_{[1, 2]}(s, \eta)d\eta ds \\ &= \int_{t_0}^t \left[z(s, x)u'_{[1]}(s, x) - z(s, x_0)u'_{[1]}(s, x_0) - \int_{x_0}^x z'_{[2]}(s, \eta)u'_{[1]}(s, \eta)d\eta \right] ds \\ &= z(t, x)u(t, x) - z(t_0, x)u(t_0, x) - \int_{t_0}^t z'_{[1]}(s, x)u(s, x)ds \\ &\quad - \int_{t_0}^t z(s, x_0)u'_{[1]}(s, x_0)ds - \int_{t_0}^t \int_{x_0}^x z'_{[2]}(s, \eta)u'_{[1]}(s, \eta)d\eta ds \end{aligned} \quad (8.9)$$

and

$$\begin{aligned}
\int_{t_0}^t \int_{x_0}^x z''_{[1,2]}(s, \eta) u(s, \eta) d\eta ds &= \int_{x_0}^x \int_{t_0}^t z''_{[2,1]}(s, \eta) u(s, \eta) ds d\eta \\
&= \int_{x_0}^x \left[z'_{[2]}(t, \eta) u(t, \eta) - z'_{[2]}(t_0, \eta) u(t_0, \eta) - \int_{t_0}^t z'_{[2]}(s, \eta) u'_{[1]}(s, \eta) ds \right] d\eta \\
&= \int_{x_0}^x z'_{[2]}(t, \eta) u(t, \eta) d\eta - z(t_0, x) u(t_0, x) + z(t_0, x_0) u(t_0, x_0) \\
&\quad + \int_{x_0}^x z(t_0, \eta) u'_{[2]}(t_0, \eta) d\eta - \int_{x_0}^x \int_{t_0}^t z'_{[2]}(s, \eta) u'_{[1]}(s, \eta) ds d\eta
\end{aligned} \tag{8.10}$$

for $(t, x) \in \mathcal{D}$. It follows from Remark 2.2 that $z'_{[2]} u'_{[1]} \in L(\mathcal{D}; \mathbb{R})$ and thus

$$\int_{t_0}^t \int_{x_0}^x z'_{[2]}(s, \eta) u'_{[1]}(s, \eta) d\eta ds = \int_{x_0}^x \int_{t_0}^t z'_{[2]}(s, \eta) u'_{[1]}(s, \eta) ds d\eta \quad \text{for } (t, x) \in \mathcal{D}.$$

Therefore, comparing equalities (8.9) and (8.10), we obtain for every $(t, x) \in \mathcal{D}$ the equality

$$\begin{aligned}
z(t, x) u(t, x) &= z(t_0, x_0) u(t_0, x_0) \\
&\quad + \int_{t_0}^t z(s, x_0) u'_{[1]}(s, x_0) ds + \int_{t_0}^t z'_{[1]}(s, x) u(s, x) ds \\
&\quad + \int_{x_0}^x z(t_0, \eta) u'_{[2]}(t_0, \eta) d\eta + \int_{x_0}^x z'_{[2]}(t, \eta) u(t, \eta) d\eta \\
&\quad + \int_{t_0}^t \int_{x_0}^x \left[z(s, \eta) u''_{[1,2]}(s, \eta) - z''_{[1,2]}(s, \eta) u(s, \eta) \right] d\eta ds.
\end{aligned} \tag{8.11}$$

Now let u be a solution to problem (1.1), (1.2). Moreover, let $(t_*, x_*) \in \mathcal{D}$ be arbitrary. Then it follows from equality (8.11) with $z = Z_{t_*, x_*}$ that

$$\begin{aligned}
u(t_*, x_*) &= Z_{t_*, x_*}(t_0, x_0) u(t_0, x_0) \\
&\quad + \int_{t_0}^{t_*} Z_{t_*, x_*}(s, x_0) u'_{[1]}(s, x_0) ds + \int_{x_0}^{x_*} Z_{t_*, x_*}(t_0, \eta) u'_{[2]}(t_0, \eta) d\eta \\
&\quad + \int_{t_0}^{t_*} \int_{x_0}^{x_*} Z_{t_*, x_*}(s, \eta) q(s, \eta) d\eta ds.
\end{aligned}$$

Consequently, in view of initial conditions (1.2), desired equality (8.8) holds. \square

9. COUNTER-EXAMPLES

Example 9.1. Let $p \in L(\mathcal{D}; \mathbb{R}_+)$ be such that

$$\int_{t_0}^b \int_{x_0}^d p(s, \eta) d\eta ds = 1$$

and let the operator ℓ be defined by the relation

$$\ell(v)(t, x) = p(t, x) v(b, d) \quad \text{for a. e. } (t, x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D}; \mathbb{R}).$$

Then condition (6.2) with $\alpha = 1$ is satisfied for every $m \in \mathbb{N}$ and $v \in C(\mathcal{D}; \mathbb{R})$. Moreover,

$$\int_{t_0}^b \int_{x_0}^d p_j(s, \eta) d\eta ds = 1 \quad \text{for every } j \in \mathbb{N},$$

where the function p_j is given by formula (6.4).

On the other hand, problem (1.1₀), (1.2₀) has a nontrivial solution

$$u(t, x) = \int_a^t \int_c^x p(s, \eta) d\eta ds \quad \text{for } (t, x) \in \mathcal{D}.$$

This example shows that the assumption $\alpha \in [0, 1[$ in Theorem 6.1 cannot be replaced by the assumption $\alpha \in [0, 1]$, and the strict inequality

$$\int_{t_0}^b \int_{x_0}^d p_j(s, \eta) d\eta ds < 1$$

in Corollary 6.1 cannot be replaced by the nonstrict one. The optimality of the other strict inequalities in (6.3) can be justified analogously.

Example 9.2. Let

$$g_k(t) = k \cos(k^2 t), \quad h_k(t) = k \sin(k^2 t) \quad \text{for } t \geq 0, k \in \mathbb{N}, \quad (9.1)$$

and

$$y_k(t) = -k \int_0^t \exp\left(\frac{\sin(k^2 t)}{k} - \frac{\sin(k^2 s)}{k}\right) \sin(k^2 s) ds \quad \text{for } t \geq 0, k \in \mathbb{N}. \quad (9.2)$$

It is not difficult to verify that for every $k \in \mathbb{N}$ we have

$$y'_k(t) = g_k(t)y_k(t) + h_k(t) \quad \text{for } t \geq 0 \quad (9.3)$$

and

$$|y_k(t)| \leq 1 + e + te^2 \quad \text{for } t \geq 0, \quad (9.4)$$

because

$$\begin{aligned} y_k(t) &= \frac{1}{k} \cos(k^2 t) - \frac{1}{k} \exp\left(\frac{\sin(k^2 t)}{k}\right) \\ &\quad + \frac{1}{2} \int_0^t \exp\left(\frac{\sin(k^2 t)}{k} - \frac{\sin(k^2 s)}{k}\right) ds \\ &\quad + \frac{1}{2} \int_0^t \exp\left(\frac{\sin(k^2 t)}{k} - \frac{\sin(k^2 s)}{k}\right) \cos(2k^2 s) ds \quad \text{for } t \geq 0. \end{aligned}$$

Moreover,

$$\lim_{k \rightarrow +\infty} y_k(t) = \frac{t}{2} \quad \text{for } t \geq 0. \quad (9.5)$$

Now, let $p \equiv 0$ and $q \equiv 0$ on \mathcal{D} , $t_0 = a$, $x_0 = c$, $\varphi \equiv 0$ on $[a, b]$, $\psi \equiv 0$ on $[c, d]$, and

$$\tau(t, x) = t, \quad \mu(t, x) = x \quad \text{for } (t, x) \in \mathcal{D}.$$

For any $k \in \mathbb{N}$, we put $t_k = a$, $x_k = c$, $\varphi_k \equiv 0$ on $[a, b]$, $\psi_k \equiv 0$ on $[c, d]$,

$$p_k(t, x) = g_k(t - a)g_k(x - c) \quad \text{for } (t, x) \in \mathcal{D},$$

$$\begin{aligned} q_k(t, x) &= h_k(t - a)y'_k(x - c) + y'_k(t - a)h_k(x - c) \\ &\quad - h_k(t - a)h_k(x - c) \quad \text{for } (t, x) \in \mathcal{D}, \end{aligned}$$

and

$$\tau_k(t, x) = t, \quad \mu_k(t, x) = x \quad \text{for } (t, x) \in \mathcal{D}.$$

According to (9.1), (9.3), and (9.4), it is clear that the assumptions of Corollary 7.4 are satisfied except condition (7.17). Let the operators ℓ and ℓ_k be defined by formulas (5.1) and (7.65), respectively. Then, it is not difficult to verify that the assumptions of Corollary 7.1 are fulfilled except condition (7.9).

On the other hand,

$$u(t, x) = 0 \quad \text{for } (t, x) \in \mathcal{D}$$

and

$$u_k(t, x) = y_k(t - a)y_k(x - c) \quad \text{for } (t, x) \in \mathcal{D}, \quad k \in \mathbb{N}$$

are solutions to problems (1.1'), (1.2) and (1.1'_k), (1.2_k), respectively, as well as problems (1.1), (1.2) and (1.1_k), (1.2_k), respectively. However, in view of condition (9.5), we get

$$\begin{aligned} \lim_{k \rightarrow +\infty} \left(u_k(t, x) - u(t, x) \right) &= \lim_{k \rightarrow +\infty} y_k(t - a)y_k(x - c) = \\ &= \frac{(t - a)(x - c)}{4} \quad \text{for } (t, x) \in \mathcal{D} \end{aligned}$$

and thus relation (7.8) does not hold.

This example shows that assumption (7.17) in Corollary 7.4 and assumption (7.9) in Corollary 7.1 are essential and cannot be omitted.

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