NON-STEADY STOKES FLOW AND FINITE DIFFERENCES

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Introduction

In the present paper we apply elementary energy estimates to prove optimal convergence properties of an implicit time stepping procedure for the non-stationary Stokes equations

$$\partial_t v - \nu\Delta v + \nabla p = F, \qquad \text{div}\, v = 0 \quad \text{in} \quad (0,T)\times G, \qquad v\big|_{\partial G} = 0\,, \quad v\big|_{t=0} = v_0\,. \tag{1}$$

These equations are important in hydrodynamics. They describe the motion of a viscous incompressible fluid, if the nonlinear term $v\cdot\nabla v$ of the corresponding Navier-Stokes equations is ignorable small. We consider (1) on a fixed cylindric domain $(0,T)\times G$, where $T>0$ is given and $G\subset\mathbb{R}^3$ is a bounded domain with a sufficiently smooth compact boundary ∂G.

Second Order Approximation

Setting $h = T/N > 0,\ t_k = k\,h\,(k = 0,1,\dots,N)$ we want to approximate the solution v, p of (1) at time t_k by the solution v^k, p^k $(k = 1,2,\dots,N)$ of the second order Crank-Nicholson-type procedure

$$\begin{aligned} \frac{v^k - v^{k-1}}{h} - \frac{\nu}{2}\,\Delta(v^k + v^{k-1}) + \frac{1}{2}\,\nabla(p^k + p^{k-1}) \ &= \ \frac{1}{h}\int\limits_{(k-1)h}^{kh} F(t)dt\,, \\ \text{div}\, v^k \ = \ 0\,, \qquad v^k|_{\partial G} \ = \ 0\,, \qquad v^0 \ &= \ v_0 \quad \text{in } G\,. \end{aligned} \tag{2}$$

This scheme is implicit for the sum $(v^k + v^{k-1})$, and we can prove similar convergence statements as for the standard first order method. Moreover, we can prove (see [1] for the notation we use)

$$\max_k ||v^k - v(t_k)||_{H^{2-i}(G)} \ = \ 0(h^{1+\frac{i}{2}}) \quad \text{as } h\to 0\,, \quad i = 0,1,2,$$

uniformly provided $v\in C([0,T],H^{2+i}\cap V),\ i = 0,1,2$. It is known, however, that such an assumption is not realistic in general, not even if the data are smooth: Any solution $v\in C([0,T],H^3\cap V)$ of (1) has to satisfy a non-local compatibility condition at time $t=0$, which is uncheckable for given data. Nevertheless, we can prove the above assertions by prescribing the initial acceleration $\partial_t v_{|t=0} = a_0$ instead of the initial velocity $v_{|t=0} = v_0$ in a suitable way. In this case, the corresponding Stokes solution has the above required continuity properties.

Main Results

Proposition 1: *Let $v_0\in H^2\cap V$ and $F\in H^1(0,T,H)$. Then there is a unique solution v of (1) such that $v\in C([0,T],H^2\cap V)$ and $\partial_t v\in C([0,T],H)\cap L^2(0,T,H^1)$. Moreover, there is some constant $K_1 = K_1(G,\nu,F,v_0)$ independent of $t\in[0,T]$ with*

$$\int\limits_0^T ||\nabla\partial_\sigma v(\sigma)||^2 d\sigma \ \le \ K_1\,, \qquad ||v(t)||_2 \ \le \ K_1\,, \qquad ||\partial_t v(t)|| \ \le \ K_1 \quad (t\in[0,T])\,.$$

The property $v\in C([0,T],H^2\cap V)$ is the highest spatial regularity uniformly in time, which is possible for any solution v of (1), if integer order Sobolev (Hilbert) spaces are used. Higher order spatial regularity uniformly in time is possible only, if an additional compatibility condition is satisfied:

Proposition 2: *Let v_0 and F be given as in Proposition* 1*, and let v denote the solution of the Stokes equations* (1) *from Proposition* 1*. If in addition $v_0 \in H^4 \cap V$ and $F \in H^1(0,T,H^2 \cap H)$ with $\partial_t^2 F \in L^2(0,T,H)$, then $v \in C([0,T],H^4 \cap V)$ with $\partial_t v \in C([0,T],H^2 \cap V)$ and $\partial_t^2 v \in C([0,T],H) \cap L^2(0,T,H^1)$ if and only if*

$$\partial_t v(0)\big|_{\partial G} \;=\; 0\,. \tag{3}$$

In this case there is a constant $K_2 = K_2(G,\nu,F,v_0)$ independent of $t \in [0,T]$ with

$$\int_0^T ||\nabla \partial_t^2 v(t)||^2 dt \le K_2\,, \quad ||v(t)||_4 \le K_2\,, \quad ||\partial_t v(t)||_2 \le K_2\,, \quad ||\partial_t^2 v(t)|| \le K_2 \quad (t \in [0,T])\,.$$

The condition (3) corresponds to the condition $v(0)|_{\partial G} = 0$ (we always require $v_0 \in V$), if we differentiate the Stokes equations (1) with respect to t and take the resulting equations as an initial value problem for the acceleration $\partial_t v$. Thus (3) is satisfied, if we prescribe an initial acceleration $a_0 \in V$:

Proposition 3: *Let $F \in H^1(0,T,H^2 \cap H)$ with $\partial_t^2 F \in L^2(0,T,H)$ as in Proposition* 2*, and let $a_0 \in H^2 \cap V$. Then there is a unique solution v_0 of the stationary Stokes equations*

$$-\nu P \Delta v_0 \;=\; F(0) - a_0 \quad \text{in} \quad G \tag{4}$$

such that $v_0 \in H^4 \cap V$ (here P denotes the Helmholtz projection, see [1]). The corresponding solution v of the non-stationary equations (1) *satisfies the compatibility condition* (3)*, hence it has all regularity properties asserted in Proposition* 2 *and satisfies all estimates given there.*

Proposition 4: **(a)** *Let $v_0 \in H^2 \cap V$ and let $F \in L^2(0,T,H)$. Then there is a unique solution $v^k \in H^2 \cap V$ of* (2) *for all $k = 1,2,\ldots,N$.*
(b) *In addition, if $F \in L^2(0,T,H^2 \cap H)$, then $(v^k + v^{k-1}) \in H^4 \cap V$ for all $k = 1,2,\ldots,N$.*
(c) *If even $v_0 \in H^4 \cap V$ and $F \in L^2(0,T,H^2 \cap H)$, then $v^k \in H^4 \cap V$ for all $k = 1,2,\ldots,N$.*

Theorem 1: *Let $v_0 \in H^2 \cap V$ and $F \in H^1(0,T,H)$ be given. Let v denote the solution of the non-stationary Stokes equation* (1) *on $[0,T]$ from Proposition* 1*. Let $h = T/N > 0$ $(N \in \mathbb{N})$ and let v^k for $k = 1,2,\ldots,N$ $(v^0 = v_0)$ be the solution of* (2)*, constructed in Proposition* 4 (a)*. Setting $t_k = kh$, let $w^k = v^k - v(t_k)$ $(k = 0,1,\ldots,N)$ denote the discretization error. Then*

$$\max ||w^k|| \;=\; 0(h)\,, \quad \max ||(w^k + w^{k-1})||_1 \;=\; 0(h^{\frac{1}{2}})\,, \quad \max ||(w^k + w^{k-1})||_2 \;=\; \circ(1)$$

as $h \to 0$ (or $N \to \infty$).

Theorem 2: Let $v \in C([0,T],H^4 \cap H)$ denote the Stokes solution constructed in Proposition 3. Let $h = T/N > 0$ $(N \in \mathbb{N})$ and let v^k for $k = 1,2,\ldots,N$ $(v^0 = v_0)$ be the solution of (2), constructed in Proposition 4 *(c)*. Setting $t_k = kh$, let $w^k = v^k - v(t_k)$ $(k = 0,1,\ldots,N)$ denote the discretization error. Then

$$\max ||w^k|| \;=\; 0(h^2)\,, \quad \max ||w^k||_1 \;=\; 0(h^{\frac{3}{2}})\,, \quad \max ||w^k||_2 \;=\; 0(h) \quad as \quad h \to 0\,.$$

[1] Varnhorn, W.: *Numerical Methods for Non-Stationary Stokes Flow.* In: Advances in Mathematical Fluid Mechanics, R. Rannacher, A. Sequeira (eds.), Springer-Verlag (2010) 600 - 622