# SOME ESTIMATES FOR THE MINIMAL EIGENVALUE OF THE STURM-LIOUVILLE PROBLEM WITH THIRD-TYPE BOUNDARY CONDITIONS

Elena Karulina, Moskva

(Received October 15, 2009)

*Abstract.* We consider the Sturm-Liouville problem with symmetric boundary conditions and an integral condition. We estimate the first eigenvalue $\lambda_1$ of this problem for different values of the parameters.

*Keywords*: Sturm-Liouville problem, minimal eigenvalue

*MSC 2010*: 34B24, 34L15

## 1. Introduction

Consider the Sturm-Liouville problem

$$y''(x) - q(x)y(x) + \lambda y(x) = 0, \tag{1.1}$$

$$\begin{cases} y'(0) - k^2 y(0) = 0, \\ y'(1) + k^2 y(1) = 0, \end{cases} \tag{1.2}$$

where $q(x)$ is a non-negative bounded summable function on $[0,1]$ such that

$$\int_0^1 q^\gamma(x)\,\mathrm{d}x = 1, \quad \gamma \neq 0. \tag{1.3}$$

By $A_\gamma$ we denote the set of all such functions.

A function $y(x)$ is called a solution of problem (1.1)–(1.2) if it is defined on $[0,1]$, satisfies conditions (1.2), its derivative $y'(x)$ is absolutely continuous, and equation (1.1) holds almost everywhere on $(0,1)$.

The author was partially supported by the Russian Foundation for Basic Researches (Grant 11-01-00989).

We estimate the first eigenvalue $\lambda_1(q)$ of this problem for different values of $\gamma$ and $k$.

According to the variation principle $\lambda_1(q) = \inf_{y(x)\in H_1(0,1)\setminus\{0\}} R(q,y)$, where

$$R(q,y) = \frac{\int_0^1 y'^2(x)\,\mathrm{d}x + \int_0^1 q(x)y^2(x)\,\mathrm{d}x + k^2\big(y^2(0)+y^2(1)\big)}{\int_0^1 y^2(x)\,\mathrm{d}x}. \tag{1.4}$$

Put $m_\gamma = \inf_{q(x)\in A_\gamma} \lambda_1(q)$, $M_\gamma = \sup_{q(x)\in A_\gamma} \lambda_1(q)$.

Remark. The problem for the equation $y'' + \lambda q(x)y = 0$, $q(x) \in A_\gamma$, with conditions $y(0) = y(1) = 0$ was considered in [1]. The problem for equation (1.1), $q(x) \in A_\gamma$, with conditions $y(0) = y(1) = 0$ was considered in [2], [3]. In [4] the problem for the equation $y'' + \lambda q(x)y = 0$, $q(x) \in A_\gamma$, with conditions (1.2) was considered.

## 2. Results

**Theorem 2.1.**

(1) *If $\gamma \in (-\infty,0)\cup(0,1)$, then $M_\gamma = +\infty$.*

(2) *If $\gamma \geqslant 1$, then $M_\gamma \leqslant \pi^2 + 2$;*

(3) *if $\gamma \geqslant 1$ and $k = 0$, then $M_\gamma = 1$.*

(4) *If $\gamma = 1$ and $k \neq 0$, then $M_1 = \xi_*$, where $\xi_*$ is the solution to the equation*

$$\arctan\frac{k^2}{\sqrt{\xi}} = \frac{\xi-1}{2\sqrt{\xi}};$$

*$M_1 \in (1; \frac{1}{2}\pi^2 + 1 + \frac{1}{2}\pi\sqrt{\pi^2+4})$ for all $k \neq 0$.*

**Theorem 2.2.**

(1) *If $k = 0$, $\gamma > 1$, then $m_\gamma = 0$;*

(2) *if $k = 0$, $\gamma \leqslant 1$, then $m_\gamma \geqslant 1/4$.*

(3) *If $0 < k^2 < (-1+\sqrt{3})/2$, then $m_\gamma \geqslant k^2/(2k^2+2)$ for all $\gamma \neq 0$;*

(4) *if $k^2 \in [(-1+\sqrt{3})/2; \pi/2)$, then $m_\gamma > k^4$ for all $\gamma \neq 0$;*

(5) *if $k^2 = \pi/2$, then $m_\gamma \geqslant \pi^2/4$ for all $\gamma \neq 0$;*

(6) *if $k^2 > \pi/2$, then $m_\gamma > \pi^2/4$ for all $\gamma \neq 0$.*

## 3. Proofs

**Proposition.** *If* $\gamma \geqslant 1$, *then* $M_\gamma \leqslant 1 + 2k^2$.

Proof. Put $y_1(x) = \varepsilon$, then for any $q \in A_\gamma$ we have

$$\begin{aligned}\lambda_1(q) &= \inf_{y(x)\in H_1(0,1)\setminus\{0\}} R(q,y) \leqslant R(q,y_1)\\ &= \frac{\int_0^1 {y_1'}^2\,\mathrm{d}x + \int_0^1 q(x)y_1^2\,\mathrm{d}x + k^2\left(y_1^2(0)+y_1^2(1)\right)}{\int_0^1 y_1^2\,\mathrm{d}x}\\ &= \frac{\varepsilon^2\int_0^1 q(x)\,\mathrm{d}x + 2k^2\varepsilon^2}{\varepsilon^2} = \int_0^1 q(x)\,\mathrm{d}x + 2k^2.\end{aligned}$$

If $\gamma = 1$, then $\int_0^1 q(x)\,\mathrm{d}x = 1$. For $\gamma > 1$, using the Hölder inequality, we obtain

$$\int_0^1 q(x)\,\mathrm{d}x \leqslant \left(\int_0^1 q^\gamma(x)\,\mathrm{d}x\right)^{1/\gamma}\left(\int_0^1 1^{\gamma/(\gamma-1)}\,\mathrm{d}x\right)^{1-1/\gamma} = 1.$$

Hence $\lambda_1(q) \leqslant 1 + 2k^2$, and it follows that

$$M_\gamma = \sup_{q(x)\in A_\gamma} \lambda_1(q) \leqslant \sup_{q(x)\in A_\gamma} (1+2k^2) = 1 + 2k^2.$$

□

**Proposition.** *If* $\gamma \geqslant 1$ *and* $k = 0$, *then* $M_\gamma = 1$.

Proof. If $q(x) \equiv 1$, then problem (1.1)–(1.2) has the form

$$y'' - y + \lambda y = 0, \tag{3.1}$$

$$y'(0) = y'(1) = 0. \tag{3.2}$$

Note that $\lambda = 1$ is an eigenvalue of this problem. For $\lambda < 1$ the solution to equation (3.1) is $y = C_1 \cosh\left(\sqrt{1-\lambda}\,x\right) + C_2 \sinh\left(\sqrt{1-\lambda}\,x\right)$. Under condition (3.2) we have $C_2 = 0$, and $C_1 = 0$ or $\lambda = 1$. This means that problem (3.1)–(3.2) has no eigenvalues $\lambda < 1$. So $\lambda_1 = 1$ is the minimal eigenvalue of problem (1.1)–(1.2) with $q(x) \equiv 1$ and $k = 0$.

It now follows that $M_\gamma = \sup\limits_{q(x)\in A_\gamma} \lambda_1(q) \geqslant 1$. For $\gamma \geqslant 1$ we already got that $M_\gamma \leqslant 1 + 2k^2$, which means $M_\gamma \leqslant 1$ for $k = 0$. Combining these, we have the accurate estimate $M_\gamma = 1$. □

**Proposition.** *If* $\gamma = 1$ *and* $k \neq 0$, *then* $M_1 = \xi_*$, *where* $\xi_*$ *is the solution to the equation* $\arctan(k^2/\sqrt{\xi}) = (\xi - 1)/(2\sqrt{\xi})$.

Proof. 1. Consider the continuous function

$$y_\xi(x) = \begin{cases} \dfrac{\sqrt{\xi}}{k^2}\cos\sqrt{\xi}x + \sin\sqrt{\xi}x, & 0 \leqslant x < \tau, \\ \dfrac{\sqrt{\xi}}{k^2}\cos\sqrt{\xi}\tau + \sin\sqrt{\xi}\tau, & \tau \leqslant x < 1-\tau, \\ \dfrac{\sqrt{\xi}}{k^2}\cos\sqrt{\xi}(1-x) + \sin\sqrt{\xi}(1-x), & 1-\tau \leqslant x \leqslant 1. \end{cases}$$

If $\tau = \sqrt{\xi^{-1}}\arctan(k^2/\sqrt{\xi})$, then $y'_\xi(x)$ is continuous too, and $y_\xi(x)$ can be a solution to problem (1.1)–(1.2).

2. Now consider

$$L(y) = \frac{\int_0^1 y'^2\,\mathrm{d}x + \max\limits_{x\in[0,1]} y^2(x) + k^2\big(y^2(0) + y^2(1)\big)}{\int_0^1 y^2(x)\,\mathrm{d}x}. \tag{3.3}$$

Since

$$\int_0^1 q(x)y^2(x)\,\mathrm{d}x \leqslant \max_{x\in[0,1]} y^2(x)\int_0^1 q(x)\,\mathrm{d}x = \max_{x\in[0,1]} y^2(x),$$

we have

$$\lambda_1(q) = \inf_{y\in H_1(0,1)\setminus\{0\}} R(q,y) \leqslant \inf_{y\in H_1(0,1)\setminus\{0\}} L(y).$$

By $\xi_*$ denote the solution to the equation

$$L(y_\xi) = \xi.$$

Substituting $y_\xi(x)$ into (3.3), we obtain

(1) $y_\xi(0) = y_\xi(1) = \sqrt{\xi}/k^2$, $y_\xi(x) = \sqrt{\xi + k^4}/k^2$ for $\tau \leqslant x < 1-\tau$;

(2) since $y_\xi(x)$ is increasing for $x \in [0,\tau]$ and decreasing for $x \in [1-\tau, 1]$, we have $\max\limits_{x\in[0,1]} y_\xi^2(x) = (\xi + k^4)/k^4$;

(3)

$$\begin{aligned} &\int_0^1 (y'_\xi(x))^2\,\mathrm{d}x \\ &= \int_0^\tau \Big(-\frac{\xi}{k^2}\sin\sqrt{\xi}x + \sqrt{\xi}\cos\sqrt{\xi}x\Big)^2\,\mathrm{d}x \\ &\quad + \int_{1-\tau}^1 \Big(\frac{\xi}{k^2}\sin\sqrt{\xi}(1-x) - \sqrt{\xi}\cos\sqrt{\xi}(1-x)\Big)^2\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^\tau \Big(\frac{\xi^2}{k^4}\,\frac{1-\cos(2\sqrt{\xi}x)}{2} + \xi\,\frac{1+\cos(2\sqrt{\xi}x)}{2} - \frac{\xi\sqrt{\xi}}{k^2}\sin(2\sqrt{\xi}x)\Big)\,\mathrm{d}x\\
&= \frac{\xi^2}{k^4}\Big(x - \frac{\sin(2\sqrt{\xi}x)}{2\sqrt{\xi}}\Big)\Big|_0^\tau + \xi\Big(x + \frac{\sin(2\sqrt{\xi}x)}{2\sqrt{\xi}}\Big)\Big|_0^\tau + \frac{\xi}{k^2}\cos(2\sqrt{\xi}x)\Big|_0^\tau\\
&= \frac{\xi^2}{k^4}\Big(\tau - \frac{k^2}{\xi+k^4}\Big) + \xi\Big(\tau + \frac{k^2}{\xi+k^4}\Big) + \frac{\xi}{k^2}\Big(\frac{\xi-k^4}{\xi+k^4} - 1\Big)\\
&= \frac{1}{\sqrt{\xi}}\arctan\frac{k^2}{\sqrt{\xi}}\Big(\frac{\xi^2}{k^4} + \xi\Big) - \frac{\xi}{k^2};
\end{aligned}
$$

(4)

$$
\begin{aligned}
&\int_0^1 y_\xi^2(x)\,\mathrm{d}x\\
&\quad= \int_0^\tau \Big(\frac{\sqrt{\xi}}{k^2}\cos\sqrt{\xi}x + \sin\sqrt{\xi}x\Big)^2\mathrm{d}x + \int_\tau^{1-\tau}\frac{\xi+k^4}{k^4}\,\mathrm{d}x\\
&\quad\quad+ \int_{1-\tau}^1 \Big(\frac{\sqrt{\xi}}{k^2}\cos\sqrt{\xi}(1-x) + \sin\sqrt{\xi}(1-x)\Big)^2\mathrm{d}x\\
&\quad= \frac{\xi}{k^4}\Big(x + \frac{\sin(2\sqrt{\xi}x)}{2\sqrt{\xi}}\Big)\Big|_0^\tau + \Big(x - \frac{\sin(2\sqrt{\xi}x)}{2\sqrt{\xi}}\Big)\Big|_0^\tau - \frac{1}{k^2}\cos(2\sqrt{\xi}x)\Big|_0^\tau\\
&\quad\quad+ \Big(\frac{\xi}{k^4}+1\Big)(1-2\tau) = -\frac{1}{\sqrt{\xi}}\arctan\frac{k^2}{\sqrt{\xi}}\Big(\frac{\xi}{k^4}+1\Big) + \frac{1}{k^2} + \frac{\xi}{k^4} + 1.
\end{aligned}
$$

Finally, we have that $\xi_*$ is a solution to the equation $\arctan(k^2/\sqrt{\xi}) = \frac{1}{2}(\xi-1)/\sqrt{\xi}$.

Put $t = \sqrt{\xi} > 0$ and consider the equation $\arctan(k^2/t) = \frac{1}{2}(t^2-1)/t$ for $t \in (0, +\infty)$.

The function $\arctan(k^2/t)$ is decreasing for $t > 0$, tends to $\pi/2$ as $t \to 0+0$, to 0 as $t \to +\infty$ (see Fig. 1). The function $\frac{1}{2}(t^2-1)/t$ is increasing for $t > 0$, tends to $-\infty$ as $t \to 0+0$, to $+\infty$ as $t \to +\infty$, is equal to 0 for $t = 1$. It follows that this equation has a unique positive solution $t_*$, and $t_* > 1$.

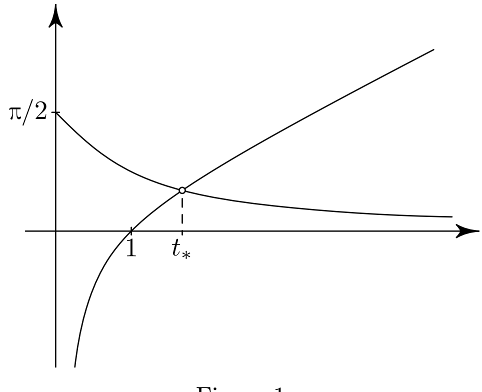


Figure 1.

Besides, though the solution depends on $k^2$, it is possible to indicate the interval which $t_*$ belongs to, where the bounds do not depend on $k^2$, and to estimate $t_*$ on these bounds. According to the behaviour of $\arctan(k^2/t)$, we get:

(1) if $k^2 \to 0$, then $t_* \to 1+0$;

(2) if $k^2 \to +\infty$, then $\arctan(k^2/t) \to \pi/2$, and $t_*$ tends to the positive solution of the equation $\frac{1}{2}(t^2-1)/t = \pi/2$, which means $t_* \to (\pi + \sqrt{\pi^2+4})/2$;

(3) $t_* \in (1, (\pi+\sqrt{\pi^2+4})/2)$ for all $k \neq 0$.

For $\xi_* = t_*^2$ we obtain:

(1) if $k^2 \to 0$, then $\xi_* \to 1+0$;

(2) if $k^2 \to +\infty$, then $\xi_* \to \frac{1}{2}\pi^2 + 1 + \frac{1}{2}\pi\sqrt{\pi^2+4}$;

(3) $\xi_* \in (1, \frac{1}{2}\pi^2 + 1 + \frac{1}{2}\pi\sqrt{\pi^2+4})$ for all $k \neq 0$.

3. Consider $y_*(x) = y_{\xi_*}(x)$. This function is a solution to the problems

$$\begin{aligned} &y'' + \lambda y = 0, \quad y'(0) - k^2 y(0) = 0 \quad \text{for } 0 \leqslant x < \tau, \\ &y'' - \xi_* y + \lambda y = 0, \quad \text{for} \quad \tau \leqslant x < 1-\tau, \\ &y'' + \lambda y = 0, \quad y'(1) + k^2 y(1) = 0 \quad \text{for } 1-\tau \leqslant x \leqslant 1 \end{aligned}$$

where $\lambda = \xi_*$. It follows that $y_*(x)$ is a solution to problem (1.1)–(1.2), where

$$q(x) = q_*(x) = \begin{cases} 0, & 0 \leqslant x < \tau, \\ \xi_*, & \tau \leqslant x < 1-\tau, \\ 0, & 1-\tau \leqslant x < 1 \end{cases}$$

(note that $q_*(x)$ satisfies condition (1.3)). Since $y_*(x) > 0$ on $(0,1)$, it is the first eigenfunction of problem (1.1)–(1.2), and $\xi_*$ is the first eigenvalue of this problem.

Finally, the following conditions hold:

$$\xi_* = \lambda_1(q_*) \leqslant M_1 = \sup_{q \in A_\gamma} \inf_{y \in H_1(0,1)\setminus\{0\}} R(q,y) \leqslant \inf_{y \in H_1(0,1)\setminus\{0\}} L(y) \leqslant L(y_*) = \xi_*.$$

Therefore $M_1 = \xi_*$. □

**Proposition.** *If $k = 0$, $\gamma > 1$, then $m_\gamma = 0$.*

P r o o f. Substituting $k = 0$ in (1.2), we have $y'(0) = y'(1) = 0$; similarly, from (1.4) we get

$$R(q,y) = \frac{\int_0^1 {y'}^2(x)\,\mathrm{d}x + \int_0^1 q(x) y^2(x)\,\mathrm{d}x}{\int_0^1 y^2(x)\,\mathrm{d}x}.$$

Put

$$y_1 = 1, \qquad q_\varepsilon(x) = \begin{cases} \varepsilon^{-1/\gamma}, & 0 < x < \varepsilon, \\ 0, & \varepsilon < x < 1. \end{cases}$$

Then, since $\gamma > 1$, we have

$$m_\gamma = \inf_{q \in A_\gamma} \Bigl( \inf_{y \in H_1(0,1) \setminus \{0\}} R(q,y) \Bigr) \leqslant R(q_\varepsilon, y_1) = \varepsilon^{1-1/\gamma} \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus we conclude that $m_\gamma = 0$. □

**Proposition.** *If $k = 0$, $\gamma \leqslant 1$, then $m_\gamma \geqslant 1/4$.*

Proof. Put $\Delta = \{y(x)\colon\ y(x) \in H_1(0,1) \setminus \{0\}, \int_0^1 y^2(x)\,\mathrm{d}x = 1, y(x) \geqslant 0\}$. Note that $\lambda_1 = \inf\limits_{y \in H_1(0,1) \setminus \{0\}} R(q,y) = \inf\limits_{y \in \Delta} R(q,y)$.

Put $\alpha = \int_0^1 {y'}^2(x)\,\mathrm{d}x$, $\beta = \min\limits_{y \in [0,1]} y = y(\xi)$, where $\xi \in [0,1]$.

Using $y(x) = y(\xi) + \int_\xi^x y'(s)\,\mathrm{d}s$ and the Hölder inequality, we obtain

$$y^2(x) \leqslant 2\beta^2 + 2\Bigl(\int_\xi^x y'(s)\,\mathrm{d}s\Bigr)^2 \leqslant 2\beta^2 + 2\int_\xi^x y'^2(s)\,\mathrm{d}s \leqslant 2\beta^2 + 2\alpha.$$

For $y(x) \in \Delta$ we get $2\beta^2 + 2\alpha \geqslant 1$. If follows that one of the following cases takes place: (a) $2\alpha \geqslant 1/2$; (b) $2\beta^2 \geqslant 1/2$.

(a) Suppose $\alpha \geqslant 1/4$. Hence for $y(x) \in \Delta$ and $q(x) \in A_\gamma$ we get

$$R(q,y) = \frac{\alpha + \int_0^1 q(x) y^2\,\mathrm{d}x}{1} \geqslant \frac{1}{4}.$$

(b) Suppose $\beta \geqslant 1/2$. Since $y(x) \geqslant \beta$ for all $y(x) \in [0,1]$, for $y(x) \in \Delta$ and $q(x) \in A_\gamma$ we get

$$R(q,y) = \frac{\int_0^1 {y'}^2(x)\,\mathrm{d}x + \int_0^1 q(x) y^2\,\mathrm{d}x}{1} \geqslant \int_0^1 q(x) y^2\,\mathrm{d}x \geqslant \frac{1}{4}\int_0^1 q(x)\,\mathrm{d}x.$$

Using the Hölder inequality, we have

$$\begin{aligned} 1 = \int_0^1 q^{\gamma/(\gamma-1)} q^{\gamma/(1-\gamma)}\,\mathrm{d}x &\leqslant \Bigl(\int_0^1 q(x)\,\mathrm{d}x\Bigr)^{\gamma/(\gamma-1)} \Bigl(\int_0^1 q^\gamma\,\mathrm{d}x\Bigr)^{1/(1-\gamma)} \\ &= \Bigl(\int_0^1 q(x)\,\mathrm{d}x\Bigr)^{\gamma/(\gamma-1)} \quad \text{for } \gamma < 0, \end{aligned}$$

and

$$\int_0^1 q^\gamma(x)\,\mathrm{d}x \leqslant \Bigl(\int_0^1 q(x)\,\mathrm{d}x\Bigr)^\gamma \Bigl(\int_0^1 1^{1/(1-\gamma)}\,\mathrm{d}x\Bigr)^{1-\gamma} \quad \text{for } \gamma \in (0,1],$$

whence $\int_0^1 q(x)\,\mathrm{d}x \geqslant 1$.

Hence, $R(q, y) \geqslant 1/4$ in both cases, and

$$m_\gamma = \inf_{q \in A_\gamma} \Big( \inf_{y \in H_1(0,1) \setminus \{0\}} R(q, y) \Big) = \inf_{q \in A_\gamma} \Big( \inf_{y \in \Delta} R(q, y) \Big) \geqslant \frac{1}{4}.$$

□

*References*


[1] *Yu. Egorov, V. Kondratiev*: On Spectral Theory of Elliptic Operators. Birkhäuser, Basel, 1996. zbl

[2] *O. V. Muryshkina*: On estimates for the first eigenvalue of the Sturm-Liouville problem with symmetric boundary conditions. Vestnik Molodyh Uchenyh. – 3'2005. Series: Applied Mathematics and Mechanics. – 1'2005, 36–52.

[3] *V. A. Vinokurov, V. A. Sadovnichii*: On the range of variation of an eigenvalue when the potential is varied. Dokl. Math. *68* (2003), 247–252; Translation from Dokl. Akad. Nauk, Ross. Akad. Nauk *392* (2003), 592–597. zbl

[4] *S. S. Ezhak*: On the estimates for the minimum eigenvalue of the Sturm-Liouville problem with integral condition. J. Math. Sci., New York *145* (2007), 5205–5218 (In English.); Translation from Sovrem. Mat. Prilozh. *36* (2005), 56–69. zbl



*Author's address*: *Elena Karulina*, Moscow State University of Economics, Statistics and Informatics, Moskva, 119501, Russia, e-mail: `KarulinaES@yandex.ru`.