REMARKS ON STATISTICAL AND I-CONVERGENCE OF SERIES

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Abstract. In this paper we investigate the relationship between the statistical (or generally I-convergence) of a series and the usual convergence of its subseries. We also give a counterexample which shows that Theorem 1 of the paper by B. C. Tripathy "On statistically convergent series", Punjab. Univ. J. Math. 32 (1999), 1–8, is not correct.

Keywords: statistical convergence, I-convergence, I-convergent series

MSC 2000: 40A05, 54A20

1. Introduction

The concept of the statistical convergence was introduced in [5], [12] and has been developed in several directions in [2], [3], [4], [7], [8], [10], [13] and by many other authors. We will deal mainly with the generalization of the statistical convergence introduced in the [8].

Let $A \subseteq \mathbb{N}$. Put $A(n) := \operatorname{card}(\{1, 2, \ldots, n\} \cap A)$. Then the numbers

$$\underline{d}(A) := \liminf_{n\to\infty} \frac{A(n)}{n}, \qquad \overline{d}(A) := \limsup_{n\to\infty} \frac{A(n)}{n}$$

are called respectively the *lower* and *upper asymptotic density* of the set A. If $\underline{d}(A) = \overline{d}(A) =: d(A)$ then A is said to have the *asymptotic density* $d(A)$.

Similarly one can define the *logarithmic density* $\delta(A)$ of a set A if we take

$$\frac{1}{\ln n} \sum_{k=1}^{n} \frac{\chi_A(k)}{k} \qquad (n = 2, 3, \ldots)$$

instead of $A(n)/n$, χ_A being the characteristic function of A (cf. [6, XIX]).

A sequence $x = (x_n)_{n=1}^{\infty}$ of real numbers is said to be *statistically convergent* to $L \in \mathbb{R}$ if for every $\varepsilon > 0$ we have $d(A_\varepsilon) = 0$, where $A_\varepsilon := \{n;\ |x_n - L| \geqslant \varepsilon\}$. In this case we write stat-lim $x_n = L$.

A class $I \subseteq 2^{\mathbb{N}}$ is said to be an *ideal* on $\mathbb{N}$ if I is additive $(A, B \in I \Longrightarrow A \cup B \in I)$ and hereditary $(A \subseteq B \in I \Longrightarrow A \in I)$. An ideal I is *proper* if $I \neq 2^{\mathbb{N}}$. It is *admissible* provided it is proper and $I \supseteq I_f$, I_f being the ideal of all finite subsets of $\mathbb{N}$.

A sequence $x = (x_n)_{n=1}^{\infty}$ is said to be *I-convergent to L* (we will also say that x I-converges to L) if for each $\varepsilon > 0$ we have $A_\varepsilon \in I$. In this case we write I-lim $x_n = L$.

Obviously the I-convergence coincides with the usual convergence if $I = I_f$ and with the statistical convergence if $I = I_d = \{A \subseteq \mathbb{N};\ d(A) = 0\}$. The I-convergence can be regarded as a generalization of the statistical convergence.

Further particular cases of the I-convergence can be obtained by choosing

$$I = I_\delta := \{A \subseteq \mathbb{N};\ \delta(A) = 0\} \text{ or } I = I_c := \Big\{A \subseteq \mathbb{N};\ \sum_{a \in A} a^{-1} < +\infty\Big\}.$$

Obviously I_δ and I_c are admissible ideals on $\mathbb{N}$.

If $A \in I_c$ then $d(A) = 0$ (cf. [9], [11, p. 100]), hence $I_c \subseteq I_d$ and so

$$I_c\text{-}\lim x_n = L \Longrightarrow I_d\text{-}\lim x_n = L.$$

Note that I_d-lim x_n = stat-lim x_n.

In what follows we will use also the concept of the uniform density of a set (cf. [1]). Let $A \subseteq \mathbb{N}$, let t, s be integers, $t \geqslant 0$, $s \geqslant 1$. Denote by $A(t+1, t+s)$ the cardinality of the set $A \cap [t+1, t+s]$. Put

$$\alpha_s = \liminf_{t \to \infty} A(t+1, t+s), \qquad \alpha^s = \limsup_{t \to \infty} A(t+1, t+s).$$

Then $\alpha_s \leqslant \alpha^s$ and there exist

$$\underline{u}(A) = \lim_{s \to \infty} \frac{\alpha_s}{s}, \quad \overline{u}(A) = \lim_{s \to \infty} \frac{\alpha^s}{s}.$$

If $\underline{u}(A) = \overline{u}(A) =: u(A)$ then the common value $u(A)$ is called the *uniform density* of the set A. Obviously

$$\text{(1)} \qquad \underline{u}(A) \leqslant \underline{d}(A) \leqslant \overline{d}(A) \leqslant \overline{u}(A).$$

The uniform density of sets yields the I_u-convergence, where $I_u := \{A \subseteq \mathbb{N};\ u(A) = 0\}$.

An infinite series $\sum\limits_{n=1}^{\infty} a_n$ is said to be I-convergent to $s \in \mathbb{R}$ provided the sequence $(s_n)_{n=1}^{\infty}$ of its partial sums I-converges to s, where $s_n = a_1 + \ldots + a_n$ $(n = 1, 2, \ldots)$ (cf. [4], [13]).

We recall the concept of the dual filter of an ideal. If $I \subseteq 2^{\mathbb{N}}$ is a proper ideal on $\mathbb{N}$ then the class

$$F(I) := \{B \subseteq \mathbb{N};\ \mathbb{N} \setminus B \in I\}$$

is called the *dual filter of the ideal* I.

2. Main results

Theorem 1 of the paper [13], p. 4 claims that "If a series $\sum\limits_{k=1}^{\infty} a_k$ is statistically convergent then there exists a set $M \subseteq \mathbb{N}$, $M = \{m_1 < m_2 < \ldots\}$ with $d(M) = 1$ such that the series $\sum\limits_{k=1}^{\infty} a_{m_k}$ is convergent in the usual sense." Our following considerations show that this result is not correct. Nevertheless, it motivated us to introduce the following definition.

Definition 1. An admissible ideal $I \subseteq 2^{\mathbb{N}}$ is said to have the property (T) if the following assertion holds: If $\sum\limits_{k=1}^{\infty} a_k$ is an arbitrary I-convergent series then there exists a set $M \subseteq \mathbb{N}$, $M = \{m_1 < m_2 < \ldots\}$ belonging to the filter $F(I)$ (i.e., $M = \mathbb{N} \setminus A$ for an $A \in I$) and $\sum\limits_{k=1}^{\infty} a_{m_k}$ converges in the usual sense.

Let us remark that the ideal I_f has the property (T) (for the set M we can take $M = \mathbb{N}$). On the other hand, we will show in what follows that no other ideal among the ideals mentioned in this paper has the property (T). We begin to prove this fact with the ideal $I_c = \Big\{A \subseteq \mathbb{N};\ \sum\limits_{a \in A} a^{-1} < +\infty\Big\}$.

Theorem 1. *The ideal I_c does not have the property* (T).

P r o o f. Put

$$\begin{aligned}(a_k)_{k=1}^{\infty} = \Big(&\frac{1}{2^2}, \frac{1}{2^2}, \Big(-1-\frac{1}{2}\Big), 1, \frac{1}{3^2}, \frac{1}{3^2}, \frac{1}{3^2}, \Big(-1-\frac{1}{3}\Big), 1, \ldots,\\ &\underbrace{\frac{1}{n^2}, \frac{1}{n^2}, \ldots, \frac{1}{n^2}}_{n \text{ terms}}, \Big(-1-\frac{1}{n}\Big), 1, \ldots\Big).\end{aligned}$$

Denote by $(s_n)_{n=1}^{\infty}$ the sequence of the partial sums of the series $\sum_{k=1}^{\infty} a_k$. Then we have

$$(s_n)_{n=1}^{\infty} = \Big(\frac{1}{2^2}, \frac{1}{2}, \underbrace{-1}_{l_2\text{-term}}, 0, \frac{1}{3^2}, \frac{2}{3^2}, \frac{1}{3}, \underbrace{-1}_{l_3\text{-term}}, 0, \dots, \\ \frac{1}{n^2}, \frac{2}{n^2}, \dots, \frac{1}{n}, \underbrace{-1}_{l_n\text{-term}}, 0, \dots\Big).$$

We prove that this sequence $(s_n)_{n=1}^{\infty}$ is I_c-convergent to zero.

Recall that if I is an admissible ideal then a sequence $x = (x_n)$ is said to be I^*-convergent to L provided there is a set $M = \{m_1 < m_2 < \dots\} \in F(I)$ such that $\lim_{k\to\infty} x_{m_k} = L$. It is well-known that if $x = (x_n)$ is I^*-convergent to L then it is also I-convergent to L (cf. [8]).

Hence to prove that $\sum_{k=1}^{\infty} a_k$ is I-convergent to 0 it suffices to show that $(s_n)_{n=1}^{\infty}$ is I^*-convergent to 0.

If we omit from the sequence $(s_n)_{n=1}^{\infty}$ the terms with indices l_n $(n = 2, 3, \dots)$ then we obviously obtain a sequence which is convergent (in the usual sense) to 0 and hence it is I_c^* convergent to 0 and so also I_c-convergent to 0). Hence if we show that $\sum_{n=2}^{\infty} l_n^{-1} < +\infty$ then it will be proved that the series $\sum_{k=1}^{\infty} a_k$ is I_c-convergent to 0.

We are going to prove that

$$\sum_{n=2}^{\infty} l_n^{-1} < +\infty. \tag{2}$$

We have

$$\begin{aligned} l_2 &= 3, \\ l_3 &= l_2 + 5, \\ l_4 &= l_3 + 6, \\ &\ \vdots \\ l_n &= l_{n-1} + (n+2). \end{aligned}$$

By summing these equalities we get

$$\begin{aligned} l_n &= 1 + 2 + 3 + 4 + 5 + \dots + (n+2) - 1 - 2 - 4 = \frac{(n+2)(n+3)}{2} - 7 \\ &= \frac{1}{2}(n^2 + 5n - 8). \end{aligned}$$

This immediately implies (2).

Hence the series $\sum_{k=1}^{\infty} a_k$ is I_c-convergent. However, we will show that there is no set $M = \{m_1 < m_2 < \ldots\} \in F(I_c)$ such that $\sum_{k=1}^{\infty} a_{m_k}$ converges in the usual sense.

We proceed indirectly. Write

$$\sum_{v=1}^{\infty} a_v = \frac{1}{2^2} + \frac{1}{2^2} + \Big(-1-\frac{1}{2}\Big) + 1 + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{3^2} + \Big(-1-\frac{1}{3}\Big) + 1 + \ldots.$$

Suppose that there is a set $M \in F(I_c)$, $M = \{m_1 < m_2 < \ldots\}$, such that the series $\sum_{k=1}^{\infty} a_{m_k}$ converges in the usual sense.

The convergence of $\sum_{k=1}^{\infty} a_{m_k}$ implies that this series contains only a finite number of terms of the form 1, $-1-1/n$ $(n \geqslant 2)$. Hence there is a $v_0 \in \mathbb{N}$ such that for every $v > v_0$ we have $a_v \neq 1$, $a_v \neq -1-1/n$ $(n \geqslant 2)$.

However, then the series $\sum_{v_0<v\in M} a_v$ converges in the usual sense and none of its terms equals 1 or $-1-1/n$ $(n \geqslant 2)$.

Obviously the set $M \cap (v_0, +\infty)$ belongs again to $F(I_c)$ and the series

$$\sum_{v_0<v\in M} a_v \tag{3}$$

is a subseries of a series of the form

$$\underbrace{\frac{1}{m^2} + \ldots + \frac{1}{m^2}}_{m\text{-terms}} + \underbrace{\frac{1}{(m+1)^2} + \ldots + \frac{1}{(m+1)^2}}_{(m+1)\text{-terms}} + \ldots. \tag{4}$$

A term $1/(m+k)^2$ occurs in (3) exactly when it occurs in $\sum_{v=1}^{\infty} a_v$ with an index v belonging to M, $v > v_0$.

Put $H = M \cap (v_0, +\infty)$. Then $H \in F(I_c)$. Hence $H = \mathbb{N} \setminus A$, $\sum_{a\in A} a^{-1} < +\infty$. Then $d(A) = 0$, $d(H) = 1$ (cf. [9], [11, p. 100]).

So we obtain a contradiction if we prove the following statement:

(w) If $\bar{d}(H) > 0$ then the subseries of (4) corresponding to the subscripts from H diverges.

We prove (w). By the assumption of (w) there exists a $\delta > 0$ such that for infinitely many k's (say for $k \in V$, V is an infinite set) we have

$$\frac{H(k)}{k} > \delta > 0, \tag{5}$$

$H(k) = \operatorname{card}(\{1, 2, \ldots, k\} \cap H)$.

The divergence of the subseries $\sum_{v\in H} a_v$ of the series (4) will be established by showing that it fails to satisfy the Cauchy condition of convergence. This condition says:

For every $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that for each $n > n_\varepsilon$ we have

$$\sum_{n_\varepsilon < v \leqslant n, v\in H} a_v < \varepsilon.$$

Hence it suffices to prove that there is an $\varepsilon_0 > 0$ such that for every $n_0 \in \mathbb{N}$ there exists an $n > n_0$ with $\sum_{n_0 < v \leqslant n, v\in H} a_v \geqslant \varepsilon_0$. We show that one can take $\varepsilon_0 = \frac{1}{6}\delta$.

Let n_0 be an arbitrary positive integer. Then a_{n_0} occurs in a block (say in the l-th block consisting of terms of the form $1/l^2$). Choose $k \in V$,

$$k = m + (m+1) + \ldots + (m+r+1) = m(r+2) + \frac{(r+1)(r+2)}{2}.$$

By the definition of the set V there are infinitely many such v's that the corresponding k's belong to V.

We will estimate the sum

$$\sum_{n_0 < v \leqslant k, v\in H} a_v$$

from below. We get

$$\sum_{n_0 < v \leqslant k, v\in H} a_v \geqslant \frac{H(k) - H(n_0)}{(m+r+1)^2} = \frac{H(k)}{(m+r+1)^2} - \frac{H(n_0)}{(m+r+1)^2}.$$

The first summand on the right hand side is by (5) greater than

$$\delta\Big[m(r+2) + \frac{(r+1)(r+2)}{2}\Big].$$

Choosing r sufficiently large the second summand can be made less than $\delta/6$. Hence for r chosen in such a way we get

$$\sum_{n_0 < v \leqslant k, v\in H} a_v \geqslant \delta\frac{m(r+2) + \frac{1}{2}(r+1)(r+2)}{(m+r+1)^2} - \frac{\delta}{6}.$$

The first term on the right hand side converges for $r \to \infty$ to $\delta/2$. Hence by a new suitably enlarged r we get

$$\sum_{n_0 < v \leqslant k, v\in H} a_v \geqslant \frac{\delta}{3} - \frac{\delta}{6} = \frac{\delta}{6}.$$

$\square$

Minor modifications of the proof of Theorem 1 enable us to show that none of the ideals I_d, I_δ, I_u has the property (T).

Theorem 2. *The ideal I_d does not have the property* (T).

Corollary. *The statement in Theorem* 1 *of* [13] *does not hold.*

P r o o f o f T h e o r e m 2. Let $\sum\limits_{k=1}^{\infty} a_k$ have the same meaning as in Theorem 1. This series is I_c-convergent to 0. Since $I_c \subseteq I_d$, it is also I_d-convergent (statistically convergent) to 0. Further, if $M \subseteq \mathbb{N}$ and $d(M) = 1$ (i.e. $M \in F(I_d)$) then by statement (w) (see the proof of Theorem 1) we see that $\sum\limits_{v\in M} a_v$ diverges. Theorem 2 follows. $\square$

Theorem 3. *The ideal I_δ does not have the property* (T).

P r o o f. Modify the proof of Theorem 1. Note that (cf. [6] p. 241)

$$\underline{d}(A) \leqslant \underline{\delta}(A) \leqslant \bar{\delta}(A) \leqslant \bar{d}(A). \tag{6}$$

The series $\sum\limits_{k=1}^{\infty} a_k$ constructed in the proof of Theorem 1 is I_d-convergent to 0 since it is I_c-convergent to 0 and $I_c \subseteq I_d$. But then it is also I_δ-convergent to 0 by (6).

Further, if $M \subseteq \mathbb{N}$, $\delta(M) = 1$ (i.e. $M \in F(I_\delta)$) then by (6) we get $\bar{d}(M) = 1$ and applying the statement (w) (see the proof of Theorem 1) we see that $\sum\limits_{v\in M} a_v$ diverges. $\square$

Theorem 4. *The ideal I_u of the sets of uniform density zero does not have property* (T).

P r o o f. We show that the set

$$A = \{l_2 < l_3 < \ldots\}$$

in the proof of Theorem 1 has the uniform density zero. Let s be fixed, $s \in \mathbb{N}$. A simple calculation shows that

$$l_{n+1} - l_n = \frac{1}{2}[((n+1)^2 + 5(n+1) - 8) - (n^2 + 5n - 8)] = n + 3 \to \infty (n \to \infty).$$

Therefore there is an $t_0(s)$ such that each interval $[t+1, t+s] (t > t_0)$ contains at most one element from the set A. So we get

$$\alpha^s = \limsup_{t\to\infty} A(t+1, t+s) \leqslant 1, \quad \overline{u}(A) = \lim_{s\to\infty} \frac{\alpha^s}{s} = 0, \quad u(A) = 0.$$

If $M \in F(I_u)$, i.e. if $M = \mathbb{N} \setminus K$, $u(K) = 0$, then by (1) we have $d(K) = 0$ and so $M \in F(I_d)$. But then by the proof of Theorem 1 (see (w)) we see that the series $\sum\limits_{v \in M} a_v$ diverges. $\square$

The results we have just obtained about the property (T) suggest to formulate the following

C o n j e c t u r e . The ideal I_f is the only admissible ideal having the property (T).

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