SEVERAL REFINEMENTS AND COUNTERPARTS OF RADON'S INEQUALITY

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Abstract. We establish that the inequality of Radon is a particular case of Jensen's inequality. Starting from several refinements and counterparts of Jensen's inequality by Dragomir and Ionescu, we obtain a counterpart of Radon's inequality. In this way, using a result of Simić we find another counterpart of Radon's inequality. We obtain several applications using Mortici's inequality to improve Hölder's inequality and Liapunov's inequality. To determine the best bounds for some inequalities, we used Matlab program for different cases.

Keywords: Radon's inequality; Jensen's inequality; Hölder's inequality; Liapunov's inequality

MSC 2010: 26D15

1. Introduction

Most recent theoretical results related to inequalities include: generalized vector variational-type inequalities [9], Jensen type inequalities [1], Hölder's inequality [17] or weighted norm inequalities [13]. The reader can find in the literature many applications of these inequalities. In the sequel, we focus on inequalities that employ Jensen's inequality.

Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ be n-tuples, where $\mathbf{ab} = (a_1b_1, a_2b_2, \dots, a_nb_n)$ and $\mathbf{a}^m = (a_1^m, a_2^m, \dots, a_n^m)$ for any real number m. We write $\mathbf{a} \geqslant 0$ and $\mathbf{b} > 0$, if $a_i \geqslant 0$ and $b_i > 0$ for every $1 \leqslant i \leqslant n$.

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We consider the expression

$$\Delta_n^{[p]}(\mathbf{a};\mathbf{b}) := \sum_{i=1}^{n} \frac{a_i^p}{b_i^{p-1}} - \frac{\left(\sum_{i=1}^{n} a_i\right)^p}{\left(\sum_{i=1}^{n} b_i\right)^{p-1}} \tag{1.1}$$

for a real number $p > 1$ and for n-tuples $\mathbf{a} \geqslant 0$ and $\mathbf{b} > 0$.

Radon proved in [18] the inequality

$$\Delta_n^{[p]}(\mathbf{a};\mathbf{b}) \geqslant 0, \tag{1.2}$$

known in the literature as Radon's inequality. Other solutions of this inequality can be found in [8], [10]. A particular case of this inequality is the well known inequality of Bergström (see [2]) which is equivalent to Cauchy-Buniakovski-Schwarz's inequality. Refinements of Bergström's inequality were established by many authors, in particular, by Mărghidanu-Barrero-Rădulescu (see [11]) and Pop (see [16]). In [4] we can find interesting applications of these inequalities developed by Ciurdariu. In [12], Mortici studies another refinement of Radon's inequality, namely

$$\Delta_n^{[p]}(\mathbf{a};\mathbf{b}) \geqslant T_n^{[p]}(\mathbf{a};\mathbf{b}) := (p-1) \max_{1 \leqslant i < j \leqslant n} \frac{(a_i + a_j)^{p-2}(a_i b_j - a_j b_i)^2}{b_i b_j (b_i + b_j)^{p-1}} \tag{1.3}$$

for every $n \geqslant 2$, $p > 1$, $a_i \geqslant 0$, $b_i > 0$, $1 \leqslant i \leqslant n$. We remark that Radon's inequality implies Hölder's inequality (see [3], [14]), by a simple change of variables $a_i = x_i y_i$ and $b_i = y_i^{p/(p-1)}$, where $p > 1$, $x_i \geqslant 0$, $y_i > 0$ and $1 \leqslant i \leqslant n$, so that we have

$$\left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n} y_i^q\right)^{1/q} \geqslant \sum_{i=1}^{n} x_i y_i, \tag{1.4}$$

where $1/p + 1/q = 1$. If we replace in Radon's inequality $a_i = x_i^s$, $b_i = x_i^r$, and $p = (r-t)/(r-s)$ for the real numbers $r > s > t > 0$, $1 \leqslant i \leqslant n$, we obtain Liapunov's inequality (see [3], [14]):

$$\left(\sum_{i=1}^{n} x_i^t\right)^{r-s} \left(\sum_{i=1}^{n} x_i^r\right)^{s-t} \geqslant \left(\sum_{i=1}^{n} x_i^s\right)^{r-t}, \tag{1.5}$$

where $x_i > 0$ for all $1 \leqslant i \leqslant n$.

2. Main results

In [6], Dragomir and Ionescu proved a reverse of Jensen's inequality:

Theorem 2.1. *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex mapping on $\mathring{I}$ ($\mathring{I}$ is the interior of I), $x_i \in \mathring{I}$, $p_i \geqslant 0$ $(i = 1, \dots, n)$ and $\sum_{i=1}^n p_i = 1$. Then we have the inequality*

$$0 \leqslant \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \leqslant \sum_{i=1}^n p_i x_i f'(x_i) - \sum_{i=1}^n p_i x_i \sum_{i=1}^n p_i f'(x_i), \tag{2.1}$$

where f' is the derivative of f on $\mathring{I}$.

In [5], Dragomir obtained another counterpart of Jensen's inequality:

Theorem 2.2. *With the above assumptions for f and if m, $M \in \mathring{I}$, $m \leqslant x_i \leqslant M$ $(i = 1, \dots, n)$ we have*

$$0 \leqslant \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \leqslant \frac{1}{4}(M-m)(f'(M) - f'(m)). \tag{2.2}$$

For $b_i > 0$, $i = 1, \dots, n$, if we take $p_i = b_i / \sum_{i=1}^n b_i$, then we have the condition $\sum_{i=1}^n p_i = 1$. Therefore, inequalities (2.1) and (2.2) can be written in the following form:

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^n b_i f(x_i) - \left(\sum_{i=1}^n b_i\right) f\left(\frac{\sum_{i=1}^n b_i x_i}{\sum_{i=1}^n b_i}\right) \\ &\leqslant \sum_{i=1}^n b_i x_i f'(x_i) - \frac{\left(\sum_{i=1}^n b_i x_i\right)\left(\sum_{i=1}^n b_i f'(x_i)\right)}{\sum_{i=1}^n b_i}, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^n b_i f(x_i) - \left(\sum_{i=1}^n b_i\right) f\left(\frac{\sum_{i=1}^n b_i x_i}{\sum_{i=1}^n b_i}\right) \\ &\leqslant \frac{1}{4}(M-m)(f'(M) - f'(m)) \sum_{i=1}^n b_i, \end{aligned} \tag{2.4}$$

Theorem 2.3. *For every $n \geqslant 2$, $p \geqslant 1$, $a_k \geqslant 0$, $b_k > 0$, $1 \leqslant k \leqslant n$, the following inequalities hold:*

$$0 \leqslant \Delta_n^{[p]}(\mathbf{a};\mathbf{b}) \leqslant p\left(\Delta_n^{[p]}(\mathbf{a};\mathbf{b}) - \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}\Delta_n^{[p-1]}(\mathbf{a};\mathbf{b})\right), \tag{2.5}$$

and

$$0 \leqslant \Delta_n^{[p]}(\mathbf{a};\mathbf{b}) \leqslant \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})\sum_{i=1}^n b_i, \tag{2.6}$$

where $m \leqslant a_i/b_i \leqslant M$, for $i = 1, \ldots, n$.

Proof. First, we consider the function $f\colon I \to \mathbb{R}$, defined by $f(x) = x^p$, where $p \geqslant 1$. Since $f''(x) = p(p-1)x^{p-2} \geqslant 0$, we conclude that the function f is convex.

By (2.3), (2.4) and taking $x_i = a_i/b_i$, $i = 1, \ldots, n$, we have

$$0 \leqslant \sum_{i=1}^n \frac{a_i^p}{b_i^{p-1}} - \frac{\left(\sum_{i=1}^n a_i\right)^p}{\left(\sum_{i=1}^n b_i\right)^{p-1}} \leqslant p\left(\sum_{i=1}^n \frac{a_i^p}{b_i^{p-1}} - \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}\sum_{i=1}^n \frac{a_i^{p-1}}{b_i^{p-2}}\right), \tag{2.7}$$

and

$$0 \leqslant \sum_{i=1}^n \frac{a_i^p}{b_i^{p-1}} - \frac{\left(\sum_{i=1}^n a_i\right)^p}{\left(\sum_{i=1}^n b_i\right)^{p-1}} \leqslant \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})\sum_{i=1}^n b_i. \tag{2.8}$$

Using the notation (1.1), i.e.,

$$\Delta_n^{[p]}(\mathbf{a};\mathbf{b}) = \sum_{i=1}^n \frac{a_i^p}{b_i^{p-1}} - \frac{\left(\sum_{i=1}^n a_i\right)^p}{\left(\sum_{i=1}^n b_i\right)^{p-1}},$$

in relations (2.7) and (2.8), we deduce the required inequalities. □

Remark 2.1. We observe that Radon's inequality is a particular case of Jensen's inequality. Therefore, we obtained another proof for Radon's inequality.

Remark 2.2. Inequality (2.5) can be written in the following form:

$$p\left(\sum_{i=1}^n a_i\right)\Delta_n^{[p-1]}(\mathbf{a};\mathbf{b}) \leqslant (p-1)\left(\sum_{i=1}^n b_i\right)\Delta_n^{[p]}(\mathbf{a};\mathbf{b}). \tag{2.9}$$

Writing relation (2.9) for $p = 2, 3, \ldots$, and multiplying the relations obtained, we have

$$0 \leqslant \frac{p}{2}\left(\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}\right)^{p-2}\Delta_n^{[2]}(\mathbf{a};\mathbf{b}) \leqslant \Delta_n^{[p]}(\mathbf{a};\mathbf{b}), \tag{2.10}$$

which is another refinement of Radon's inequality.

In [11], we have

$$\Delta_n^{[2]}(\mathbf{a};\mathbf{b}) \geqslant \max_{1\leqslant i<j\leqslant n} \frac{(a_ib_j - a_jb_i)^2}{b_ib_j(b_i+b_j)},$$

so, according to inequality (2.10), we deduce

$$\Delta_n^{[p]}(\mathbf{a};\mathbf{b}) \geqslant \frac{p}{2}\left(\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}\right)^{p-2} \max_{1\leqslant i<j\leqslant n} \frac{(a_ib_j - a_jb_i)^2}{b_ib_j(b_i+b_j)} \geqslant 0, \tag{2.11}$$

where p is an integer number, $p \geqslant 2$.

Another refinement of Radon's inequality can be found in [10].

In [7], we find the following results:

Theorem 2.4. *Let $f\colon I \to \mathbb{R}$ be a twice differentiable function such that there exist real constants α and β, $0 \leqslant \alpha \leqslant f''(x) \leqslant \beta$, for any $x \in I$. Then we have*

$$\begin{aligned} \frac{\alpha}{2}\sum_{1\leqslant i<j\leqslant n} p_ip_j(x_j-x_i)^2 &\leqslant \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_ix_i\right) \\ &\leqslant \frac{\beta}{2}\sum_{1\leqslant i<j\leqslant n} p_ip_j(x_j-x_i)^2, \end{aligned} \tag{2.12}$$

where $p_i > 0$ with $\sum_{i=1}^n p_i = 1$, and $x_i \in I$ for all $i = 1, \ldots, n$.

Theorem 2.5. *If $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ are n-tuples, then we have the inequality*

$$\begin{aligned} \frac{p(p-1)m^{p-2}}{\sum_{i=1}^n b_i} \sum_{1\leqslant i<j\leqslant n} \frac{(a_ib_j-a_jb_i)^2}{b_ib_j} &\leqslant \Delta_n^{[p]}(\mathbf{a};\mathbf{b}) \\ &\leqslant \frac{p(p-1)M^{p-2}}{\sum_{i=1}^n b_i} \sum_{1\leqslant i<j\leqslant n} \frac{(a_ib_j-a_jb_i)^2}{b_ib_j}, \end{aligned} \tag{2.13}$$

where $m \leqslant a_i/b_i \leqslant M$, $p > 1$, $a_i \geqslant 0$, $b_i > 0$, for $i = 1, \ldots, n$.

P r o o f. If we take $p_i = b_i/\sum_{i=1}^n b_i$, $x_i = a_i/b_i$ for $1 \leqslant i \leqslant n$, and $f(x) = x^p$ in relation (2.12), then we obtain the inequality of the statement. $\square$

R e m a r k 2.3. Relation (2.13) of Theorem 2.5 is a new refinement of Radon's inequality.

In [19], Simić proved:

Theorem 2.6. *If f is convex on I and $\sum_{i=1}^{n} p_i = 1$, then*

$$\max_{1\leqslant i<j\leqslant n}\left[p_i f(x_i)+p_j f(x_j)-(p_i+p_j) f\Big(\frac{p_i x_i+p_j x_j}{p_i+p_j}\Big)\right] \leqslant \sum_{i=1}^{n} p_i f(x_i)-f\Big(\sum_{i=1}^{n} p_i x_i\Big), \tag{2.14}$$

and

$$\sum_{i=1}^{n} p_i f(x_i)-f\Big(\sum_{i=1}^{n} p_i x_i\Big) \leqslant f(a)+f(b)-2f\Big(\frac{a+b}{2}\Big) \tag{2.15}$$

for any $a \leqslant x_i \leqslant b$, $i=1,\ldots,n$.

We apply this theorem and we will obtain another characterization of Radon's inequality.

Theorem 2.7. *For $n \geqslant 2$, $p \geqslant 1$ we have the following inequalities:*

$$\Delta_n^{[p]}(\mathbf{a};\mathbf{b}) \geqslant \max_{1\leqslant i<j\leqslant n}\left[\frac{a_i^p}{b_i^{p-1}}+\frac{a_j^p}{b_j^{p-1}}-\frac{(a_i+a_j)^p}{(b_i+b_j)^{p-1}}\right], \tag{2.16}$$

and

$$0 \leqslant \Delta_n^{[p]}(\mathbf{a};\mathbf{b}) \leqslant \left[M^p+m^p-\frac{(M+m)^p}{2^{p-1}}\right]\Big(\sum_{i=1}^{n} b_i\Big), \tag{2.17}$$

where $m \leqslant a_i/b_i \leqslant M$, $a_i \geqslant 0$, $b_i > 0$, $1 \leqslant i \leqslant n$.

P r o o f. We take in Theorem 2.6 $p_i = b_i/\sum_{i=1}^{n} b_i$, $x_i = a_i/b_i$ for $1 \leqslant i \leqslant n$, and $f(x) = x^p$, which is convex for $p \geqslant 1$.

Therefore, by simple calculations, we obtain the required inequalities. □

R e m a r k 2.4. Inequality (2.16) can be found in [10], but with another proof.

In [15], Pečarić and Perić refined the relation (2.15) as follows:

Theorem 2.8. *If $f\colon [a,b] \to \mathbb{R}$ is a convex function, $x_i \in [a,b]$, $i=1,\ldots,n$ and $p_i > 0$ with $\sum_{i=1}^{n} p_i = 1$, then*

$$\sum_{i=1}^{n} p_i f(x_i)-f\Big(\sum_{i=1}^{n} p_i x_i\Big) \leqslant f\Big(a+b-\sum_{i=1}^{n} p_i x_i\Big) - 2f\Big(\frac{a+b}{2}\Big)+\sum_{i=1}^{n} p_i f(x_i) \leqslant f(a)+f(b)-2f\Big(\frac{a+b}{2}\Big). \tag{2.18}$$

Therefore, we will obtain an improvement of inequality (2.17):

Theorem 2.9. *We have the inequality*

$$0 \leqslant \Delta_n^{[p]}(\mathbf{a};\mathbf{b}) \leqslant \frac{\left[(M+m)\sum_{i=1}^n b_i - \sum_{i=1}^n a_i\right]^p}{\left(\sum_{i=1}^n b_i\right)^{p-1}} - \frac{(M+m)^p}{2^{p-1}}\left(\sum_{i=1}^n b_i\right) + \left(\sum_{i=1}^n \frac{a_i^p}{b_i^{p-1}}\right), \tag{2.19}$$

where $m \leqslant a_i/b_i \leqslant M$, $a_i \geqslant 0$, $b_i > 0$, $1 \leqslant i \leqslant n$, $p \geqslant 1$, $n \geqslant 2$.

Proof. Similarly to the proof of Theorem 2.7, we deduce the statement. □

The inequalities (2.5), (2.6), (2.13) and (2.17) give us upper bounds for the term $\Delta_n^{[p]}(\mathbf{a};\mathbf{b})$. We want to study which of these bounds is closer to $\Delta_n^{[p]}(\mathbf{a};\mathbf{b})$.

We say that inequality $T \leqslant T_1$ is stronger than inequality $T \leqslant T_2$ if $T_1 \leqslant T_2$. To verify the inequalities (2.5), (2.6), (2.13) and (2.17) that are stronger, we use Matlab program for the cases: $n = 4$, $\mathbf{a} = (1,1,1,2)$, $\mathbf{b} = (2,2,2,1)$, $p \in \{3,4,5,7,10,15\}$.

We observe that there are situations in which each of the inequalities (2.5), (2.6), (2.13) and (2.17) are stronger.

Now, for the lower bound of the term $\Delta_n^{[p]}(\mathbf{a};\mathbf{b})$, we will compare inequalities (2.11) and (2.13). Using Matlab program and the cases presented above, we deduce that there are situations in which any of the inequalities (2.11) and (2.13) are stronger.

3. Several applications

We will use Mortici's inequality to improve Hölder's inequality and Liapunov's inequality.

Lemma 3.1. *For every real numbers* $a > 0$, $b \geqslant 0$ *and* $p > 1$, *the following inequality holds:*

$$(a^p + b)^{1/p} \geqslant a + \frac{b}{p(a^p+b)^{(p-1)/p}}. \tag{3.1}$$

Proof. For $b = 0$, we obtain the equality in relation (3.1).

For $b > 0$, we consider the function $f\colon [a^p, a^p+b] \to \mathbb{R}$, defined by $f(x) = x^{1/p}$. It follows that

$$f'(x) = \frac{1}{px^{(p-1)/p}}.$$

Applying Lagrange's Theorem, we have that there is $c \in (a^p, a^p+b)$ such that

$$(a^p+b)^{1/p} - a = \frac{b}{pc^{(p-1)/p}} > \frac{b}{p(a^p+b)^{(p-1)/p}}.$$

Thus, the inequality of the statement is true. □

We note that

$$F(a,b,p) := \frac{b}{p(a^p+b)^{(p-1)/p}}.$$

Theorem 3.1. *We have the inequalities*

$$\tag{3.2} \begin{aligned}&\left(\sum_{k=1}^n x_k^p\right)^{1/p}\left(\sum_{k=1}^n y_k^q\right)^{1/q}\\ &\qquad \geqslant \sum_{k=1}^n x_k y_k + \left(\sum_{k=1}^n y_k^q\right)^{1/q} F\left(\frac{\sum_{k=1}^n x_k y_k}{\left(\sum_{k=1}^n y_k^q\right)^{1/q}}, T_n^{[p]}(\mathbf{xy};\mathbf{y}^q), p\right),\end{aligned}$$

where $p>1$, $x_k \geqslant 0$, $y_k>0$, $1\leqslant k\leqslant n$, $1/p+1/q=1$, and

$$\tag{3.3} \begin{aligned}&\left(\sum_{k=1}^n x_k^t\right)^{r-s}\left(\sum_{k=1}^n x_k^r\right)^{s-t}\\ &\qquad \geqslant \left(\sum_{k=1}^n x_k^s\right)^{r-t} + \left(\sum_{k=1}^n x_k^r\right)^{s-t} F\left(\frac{\left(\sum_{k=1}^n x_k^s\right)^{r-t}}{\left(\sum_{k=1}^n x_k^r\right)^{s-t}}, T_n^{[p]}(\mathbf{x}^s;\mathbf{x}^r), \frac{1}{r-s}\right),\end{aligned}$$

where $s+1>r>s>t>0$ are real numbers and $x_k\geqslant 0$ for any $1\leqslant k\leqslant n$.

Proof. Using inequality (1.3) for $\mathbf{a}=\mathbf{xy}$ and $\mathbf{b}=\mathbf{y}^{p/(p-1)}$, where $p>1$ and $\mathbf{x}\geqslant 0$, $\mathbf{y}>0$, we obtain

$$\Delta_n^{[p]}\big(\mathbf{xy};\mathbf{y}^{p/(p-1)}\big) \geqslant T_n^{[p]}\big(\mathbf{xy};\mathbf{y}^{p/(p-1)}\big) \geqslant 0,$$

which is equivalent to the inequality

$$\sum_{k=1}^n x_k^p \geqslant \left(\frac{\sum_{k=1}^n x_k y_k}{\left(\sum_{k=1}^n y_k^q\right)^{1/q}}\right)^p + T_n^{[p]}(\mathbf{xy};\mathbf{y}^q),$$

where $1/p+1/q=1$.

We apply Lemma 3.1 for $a=\sum_{k=1}^n x_k y_k/\big(\sum_{k=1}^n y_k^q\big)^{1/q}$ and $b=T_n^{[p]}(\mathbf{xy};\mathbf{y}^q)$, which implies the inequality

$$\left(\sum_{k=1}^n x_k^p\right)^{1/p} \geqslant \frac{\sum_{k=1}^n x_k y_k}{\left(\sum_{k=1}^n y_k^q\right)^{1/q}} + F\left(\frac{\sum_{k=1}^n x_k y_k}{\left(\sum_{k=1}^n y_k^q\right)^{1/q}}, T_n^{[p]}(\mathbf{xy};\mathbf{y}^q), p\right).$$

This proves the inequality of the statement. In inequality (1.3), if we take $\mathbf{a}=\mathbf{x}^s$, $\mathbf{b}=\mathbf{x}^r$, and $p=(r-t)/(r-s)$, then it follows that $\Delta_n^{[p]}(\mathbf{x}^s;\mathbf{x}^r)\geqslant T_n^{[p]}(\mathbf{x}^s;\mathbf{x}^r)\geqslant 0$, which implies the inequality

$$\sum_{k=1}^n x_k^t \geqslant \left(\frac{\left(\sum_{k=1}^n x_k^s\right)^{r-t}}{\left(\sum_{k=1}^n x_k^r\right)^{s-t}}\right)^{1/(r-s)} + T_n^{[p]}(\mathbf{x}^s;\mathbf{x}^r).$$

Now, by raising to the power $r-s$ and by simple calculations, we deduce the required inequality. □

R e m a r k 3.1. Inequality (3.2) is an improvement of Hölder's inequality and inequality (3.3) is an improvement of Liapunov's inequality.

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