R_z-SUPERCONTINUOUS FUNCTIONS

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Abstract. A new class of functions called "R_z-supercontinuous functions" is introduced. Their basic properties are studied and their place in the hierarchy of strong variants of continuity that already exist in the literature is elaborated. The class of R_z-supercontinuous functions properly includes the class of $R_{\rm cl}$-supercontinuous functions, Tyagi, Kohli, Singh (2013), which in its turn contains the class of cl-supercontinuous ($\equiv$ clopen continuous) functions, Singh (2007), Reilly, Vamanamurthy (1983), and is strictly contained in the class of R_δ-supercontinuous, Kohli, Tyagi, Singh, Aggarwal (2014), which in its turn is properly contained in the class of R-supercontinuous functions, Kohli, Singh, Aggarwal (2010).

Keywords: z-supercontinuous function; F-supercontinuous function; cl-supercontinuous function; R_z-supercontinuous function; R-supercontinuous function; r_z-open set; r_z-closed set; z-embedded set; R_z-space; functionally Hausdorff space

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1. Introduction

Strong forms of continuity arise naturally in diverse situations in mathematics and applications of mathematics. For example in many circumstances in geometry, analysis, topology and topologico-analytic situations continuity is not sufficient and a condition stronger than continuity is required to meet the demand of a paricular situation. Hence it is of considerable significance both from intrinsic interest as well as from the applications view point to formulate and study new strong variants of continuity. Several of such strong variants of continuity occur in the lore of mathematical literature. For example, see [15]–[19], [21]–[23], [25], [26], [28], [30], [31]. The purpose of the present paper is to introduce one such strong form of continuity called "R_z-supercontinuity" and study its basic properties. We discuss the interrelations and interconnections of "R_z-supercontinuity" with other strong variants of continuity that already exist in mathematical literature. The class of R_z-supercontinuous functions properly contains the class of R_{cl}-supercontinuous functions [35] which in its turn strictly contains the class of cl-supercontinuous ($\equiv$ clopen continuous) functions [28], [30] and is properly contained in the class of R_δ-supercontinuous functions [21] which is strictly contained in the class of R-supercontinuous functions [18].

The organization of the paper is as follows: Section 2 is devoted to basic definitions and preliminaries. In Section 3 we introduce the notion of an "R_z-supercontinuous function" and discuss its place in the hierarchy of strong variants of continuity that already exist in the literature. Examples are included to reflect upon the distinctiveness of notions so introduced from the existing ones. Basic properties of R_z-supercontinuous functions are studied in Section 4, wherein it is shown that (i) R_z-supercontinuity is stable under the restrictions, shrinking and expansion of range and composition of functions; (ii) a function into a product space is R_z-supercontinuous if and only if its composition with each projection map is R_z-supercontinuous; and (iii) if X is an R_z-space, then f is R_z-supercontinuous if and only if its graph function g is R_z-supercontinuous. The interplay between topological properties and R_z-supercontinuous functions is investigated in Section 5. In Section 6 properties of graphs of R_z-supercontinuous functions are studied. The notion of r_z-quotient topology is introduced in Section 7. In Section 8 we retopologize the domain of an R_z-supercontinuous function in such a way that it is simply a continuous function and conclude with alternative proofs of certain results of the preceding sections.

2. Basic definitions and preliminaries

A subset H of a space X is called a *regular G_δ-set* [24] if H is the intersection of a sequence of closed sets whose interiors contain H, i.e. $H = \bigcap\limits_{n=1}^{\infty} F_n = \bigcap\limits_{n=1}^{\infty} F_n^o$, where each F_n is a closed subset of X. The complement of a regular G_δ-set is called a *regular F_σ-set.* An open set U of a space X is said to be *F-open* [19] (*r-open* [18]) if for each $x \in U$ there exists a zero (closed) set Z in X such that $x \in Z \subset U$, equivalently if U is expressible as a union of zero (closed) sets in X. A subset A of a space X is said to be *regular open* if it is the interior of its closure, i.e. $A = \overline{A}^o$. The complement of a regular open set is referred to as *regular closed.* Any intersection of regular closed (clopen) sets is called a *δ-closed* [37] (cl-*closed* [30]) set and any intersection of zero sets is called a *z-closed set* [29]. An open set U in X is said to be *r_δ-open* [21] (*r_{cl}-open* [35]) if for each $x \in U$ there exists a δ-closed (cl-closed) set A containing x such that $A \subset U$, equivalently U is expressible as a union of δ-closed (cl-closed) sets.

Next we include definitions of those strong variants of continuity which already exist in the literature and are related to the theme of the present paper.

Definitions 2.1. A function $f\colon X \to Y$ from a topological space X into a topological space Y is said to be

(a) *strongly continuous* [22] if $f(\overline{A}) \subset f(A)$ for each subset A of X;
(b) *perfectly continuous* [26] if $f^{-1}(V)$ is clopen in X for every open set $V \subset Y$;
(c) cl-*supercontinuous* [30] ($\equiv$ *clopen continuous* [28]) if for each $x \in X$ and each open set V containing $f(x)$, there is a clopen set U containing x such that $f(U) \subset V$;
(d) *z-supercontinuous* [15] (*D_δ-supercontinuous* [16], *D-supercontinuous* [17]) if for each $x \in X$ and for each open set V containing $f(x)$, there exists a cozero (regular F_σ, open F_σ) set U containing x such that $f(U) \subset V$;
(e) *strongly θ-continuous* [23], [27] if for each $x \in X$ and for each open set V containing $f(x)$, there exists an open set U containing x such that $f(\overline{U}) \subset V$;
(f) *F-supercontinuous* [19], *R-supercontinuous* [18], or R_{cl}-*supercontinuous* [35] if for each $x \in X$ and each open set V containing $f(x)$, there exists respectively an F-open, r-open, or r_{cl}-open set U containing x such that $f(U) \subset V$;
(g) *supercontinuous* [25] if for each $x \in X$ and for each open set V containing $f(x)$, there exists a regular open set U containing x such that $f(U) \subset V$;
(h) R_δ-*supercontinuous* [21] if for each $x \in X$ and for each open set V containing $f(x)$, there exists an r_δ-open set U containing x such that $f(U) \subset V$.

3. R_z-supercontinuous functions

Let X be a topological space. An open subset U of a space X is said to be *r_z-open* if for each $x \in U$ there exists a z-closed set C_x such that $x \in C_x \subset U$, equivalently U is expressible as a union of z-closed sets. Every r_{cl}-open set as well as every F-open set are r_z-open and every r_z-open set is r_δ-open which in its turn is r-open. However, reverse implications are not true in general. For example, if X denotes the real line endowed with the usual topology, then every open set in X is F-open and so r_z-open but not r_{cl}-open. Similarly, if Y denotes the real line with the cofinite topology, then every open set in Y is r-open but not necessarily r_δ-open and so not r_z-open.

Definition 3.1. A function $f\colon X \to Y$ from a topological space X into a topological space Y is said to be *R_z-supercontinuous* at a point $x \in X$, if for each open set V containing $f(x)$ there exists an r_z-open set U containing x such that $f(U) \subset V$.

The function f is said to be *R_z-supercontinuous*, if it is R_z-supercontinuous at each $x \in X$.

We reproduce the following diagram from [35] (with a slight extension) which well illustrates the place of R_z-supercontinuity in the hierarchy of strong variants of continuity that already exist in the literature and are related to the theme of the present paper.

$$
\begin{tikzcd}
\text{perfectly continuous} \arrow[d] & \text{strongly continuous} \arrow[l] & \\
\text{cl-supercontinuous} \arrow[rr] \arrow[d] & & R_{\mathrm{cl}}\text{-supercontinuous} \arrow[d] \\
z\text{-supercontinuous} \arrow[r] \arrow[d] & F\text{-supercontinuous} \arrow[r] & R_z\text{-supercontinuous} \arrow[d] \\
D_\delta\text{-supercontinuous} \arrow[r] \arrow[d] & \text{strongly } \theta\text{-continuous} \arrow[r] \arrow[d] & R_\delta\text{-supercontinuous} \arrow[d] \\
D\text{-supercontinuous} \arrow[dr] & \text{supercontinuous} \arrow[d] & R\text{-supercontinuous} \arrow[dl] \\
& \text{continuous} &
\end{tikzcd}
$$

However, none of the above implications is reversible as is shown by examples in [16], [18], [19], [21], [35] and Remark 3.3 below.

Definitions 3.2. A topological space X is said to be

(i) *functionally regular* [2], [36] if for each closed set A and each $x \notin A$ there exists a continuous real-valued function f defined on X such that $f(x) \notin \overline{f(A)}$;

(ii) an *R_z-space* [32] if for each open set U in X and each $x \in U$ there exists a z-closed set A such that $x \in A \subset U$; equivalently U is expressible as a union of z-closed sets.

Remark 3.3. If X is an R_z-space, then every continuous function $f\colon X \to Y$ is R_z-supercontinuous. In particular, if X is a functionally regular space, then every continuous function f defined on X is F-supercontinuous and so R_z-supercontinuous.

4. Basic properties of R_z-supercontinuous functions

Definition 4.1. Let X be a topological space and let $A \subset X$. A point $x \in X$ is said to be an *r_z-adherent point* of A if every r_z-open set containing x intersects A. Let A_{r_z} denote the set of all r_z-adherent points of the set A. The set A is r_z-closed if and only if $A = A_{r_z}$. Moreover, $A \subset \overline{A} \subset A_{r_z}$.

Theorem 4.2. *For a function $f\colon X \to Y$ from a topological space X into a topological space Y, the following statements are equivalent.*

(i) *f is R_z-supercontinuous.*
(ii) *$f^{-1}(V)$ is r_z-open for every open set $V \subset Y$.*
(iii) *$f^{-1}(B)$ is r_z-closed for every closed set $B \subset Y$.*
(iv) *$f^{-1}(S)$ is r_z-open for every subbasic open set $S \subset Y$.*
(v) *$f(A_{r_z}) \subset \overline{f(A)}$ for every set $A \subset X$.*
(vi) *$(f^{-1}(B))_{r_z} \subset f^{-1}(\overline{B})$ for every set $B \subset Y$.*

Definition 4.3. A filter base $\mathcal{F}$ is said to *R_z-converge* to a point $x \in X$ (written as $\mathcal{F} \xrightarrow{R_z} x$) if every r_z-open set containing x contains a member of $\mathcal{F}$.

Theorem 4.4. *A function $f\colon X \to Y$ is R_z-supercontinuous if and only if $f(\mathcal{F}) \to f(x)$ for each $x \in X$ and each filter base $\mathcal{F}$ in X which R_z-converges to x.*

Proof. Suppose that f is R_z-supercontinuous and that $\mathcal{F}$ is a filter base in X that R_z-converges to $x \in X$. Let W be any open set in Y containing $f(x)$. By Theorem 4.2 (ii), $f^{-1}(W)$ is an r_z-open set containing x. Since the filter base $\mathcal{F}$ R_z-converges to x, there exists an $F \in \mathcal{F}$ such that $F \subset f^{-1}(W)$ and so $f(F) \subset W$. Thus $f(\mathcal{F}) \to f(x)$.

Conversely, let W be an open subset of Y containing $f(x)$. Let $\mathcal{F}_x$ denote the set of all r_z-open subsets of X containing x. Clearly, $\mathcal{F}_x$ is a filter base in X which R_z converges to x. By hypothesis $f(\mathcal{F}_x) \to f(x)$ and so there exists a member $F \in \mathcal{F}_x$ such that $f(F) \subset W$. Since F is an r_z-open set containing x, f is R_z-supercontinuous.

□

Theorem 4.5. *Let $f\colon X \to Y$ be an R_z-supercontinuous function and $g\colon Y \to Z$ a continuous function. Then their composition $g \circ f$ is R_z-supercontinuous. In particular, the composition of two R_z-supercontinuous functions is R_z-supercontinuous.*

Definition 4.6. A function $f\colon X \to Y$ is said to be *R_z-open* (*R_z-closed*) if the image of every r_z-open (r_z-closed) set in X is open (closed) in Y.

Clearly every open (closed) function is R_z-open (R_z-closed). However, the converse is not true in general.

Theorem 4.7. *Let $f\colon X \to Y$ be an R_z-open (R_z-closed), R_z-supercontinuous surjection and let $g\colon Y \to Z$ be any function. Then $g \circ f$ is R_z-supercontinuous if and only if g is continuous. Further, if in addition f maps r_z-open (r_z-closed) sets to r_z-open (r_z-closed) sets, then g is R_z-supercontinuous.*

Definition 4.8 ([1], [6]). A subset S of a space X is said to be *z-embedded* in X if every zero (cozero) set in S is the intersection of a zero (cozero) set in X with S.

Theorem 4.9. *Let $f\colon X \to Y$ be a function. The following statements are true.*

(a) *If f is R_z-supercontinuous and if A is a subspace of X, then the restriction function $f|A\colon A \to Y$ is R_z-supercontinuous.*

(b) *Let $\{U_\alpha\colon \alpha \in \Lambda\}$ be a cover of X by r_z-open sets such that each U_α is z-embedded in X. If $f_\alpha = f|U_\alpha\colon U_\alpha \to Y$ is R_z-supercontinuous for each α, then f is R_z-supercontinuous.*

(c) *Let $X = \bigcup\limits_{i=1}^{n} F_i$, where each F_i is an r_z-closed z-embedded set in X. If for each i, $f_i|F_i$ is R_z-supercontinuous, then f is R_z-supercontinuous.*

Proof. (a) Let W be any open set in Y. Since f is an R_z-supercontinuous function, $f^{-1}(W)$ is an r_z-open set in X. Suppose $f^{-1}(W) = \bigcup W_\alpha$, where each W_α is a z-closed in X and let $W_\alpha = \bigcap W_{\alpha\beta}$, where each $W_{\alpha\beta}$ is a zero set in X. So each $W_{\alpha\beta} \cap A$ is a zero set in A. Now $(f|A)^{-1}(W) = f^{-1}(W) \cap A = \bigcup(W_\alpha \cap A) = \bigcup((\bigcap W_{\alpha\beta}) \cap A) = \bigcup\bigcap(W_{\alpha\beta} \cap A)$. Thus $(f|A)^{-1}(W)$ is an r_z-open set being an open set which is the union of z-closed sets and so $f|A$ is R_z-supercontinuous.

(b) Let W be an open subset of Y. Then $f^{-1}(W) = \bigcup\{f_\alpha^{-1}(W)\colon \alpha \in \Lambda\}$. Since each f_α is R_z-supercontinuous, $f_\alpha^{-1}(W)$ is an r_z-open set in U_α. Let $f_\alpha^{-1}(W) = \bigcup W_{\alpha\beta}$, where each $W_{\alpha\beta}$ is a z-closed set in U_α. Let $W_{\alpha\beta} = \bigcap W_{\alpha\beta\gamma}$, where each $W_{\alpha\beta\gamma}$ is a zero set in U_α. Since U_α is z-embedded in X, there exists a zero set $W^*_{\alpha\beta\gamma}$ in X such that $W_{\alpha\beta\gamma} = W^*_{\alpha\beta\gamma} \cap U_\alpha$. Now $W_{\alpha\beta} = \bigcap(W^*_{\alpha\beta\gamma} \cap U_\alpha) = (\bigcap W^*_{\alpha\beta\gamma}) \cap U_\alpha$. Let $\bigcap W^*_{\alpha\beta\gamma} = W^*_{\alpha\beta}$, which is an r_z-open set in X. Again, $f_\alpha^{-1}(W) = (\bigcup W^*_{\alpha\beta}) \cap U_\alpha$. Since arbitrary unions and finite intersections of r_z-open sets are r_z-open, $f_\alpha^{-1}(W)$ is an r_z-open set in X and so f is R_z-supercontinuous.

(c) Let F be any closed subset of Y. Then $f^{-1}(F) = \bigcup\limits_{i=1}^{n} f_i^{-1}(F)$. Since each f_i is R_z-supercontinuous, each $f_i^{-1}(F)$ is an r_z-closed set in F_i. Again, since each F_i is z-embedded in X, it is routine to verify that $f_i^{-1}(F)$ is an r_z-closed set in X. Since a finite union of r_z-closed sets is r_z-closed, $f^{-1}(F)$ is r_z-closed and hence f is R_z-supercontinuous. □

It is easily verified that R_z-supercontinuity is stable under the shrinking and expansion of range.

Theorem 4.10. *A function into a product space is R_z-supercontinuous if and only if its composition with each projection map is R_z-supercontinuous.*

P r o o f. Suppose that the function $f\colon X \to \prod_{\alpha\in\Lambda} X_\alpha$ is R_z-supercontinuous. Let $f_\alpha = \pi_\alpha \circ f$, where $\pi_\alpha\colon \prod_{\alpha\in\Lambda} X_\alpha \to X_\alpha$ denotes the projection onto the α-coordinate space X_α. Since projection maps are continuous, in view of Theorem 4.5, each f_α is a R_z-supercontinuous.

Conversely, suppose that each $\pi_\alpha \circ f = f_\alpha\colon X \to X_\alpha$ is R_z-supercontinuous. Since arbitrary unions and finite intersections of r_z-open sets are r_z-open, to show that f is R_z-supercontinuous, it suffices to show that the inverse image under f of every subbasic open set in $\prod_{\alpha\in\Lambda} X_\alpha$ is r_z-open in X. Let $V_\beta \times \prod_{\alpha\neq\beta} X_\alpha$ be a subbasic open set in $\prod_{\alpha\in\Lambda} X_\alpha$. Then $f^{-1}\Big(V_\beta \times \prod_{\alpha\neq\beta} X_\alpha\Big) = f^{-1}(\pi_\beta^{-1}(V_\beta)) = f_\beta^{-1}(V_\beta)$ is r_z-open in X. So f is R_z-supercontinuous. $\square$

Theorem 4.11. *Let $f\colon X \to Y$ be any function and let $g\colon X \to X \times Y$ be the graph function defined by $g(x) = (x, f(x))$ for each $x \in X$. Then g is R_z-supercontinuous if and only if f is R_z-supercontinuous and X is an R_z-space.*

P r o o f. Observe that $g = 1_X \times f$, where 1_X denotes the identity function defined on X. Now by Theorem 4.10, g is R_z-supercontinuous if and only if both 1_X and f are R_z-supercontinuous. Again, 1_X is R_z-supercontinuous implies that every open set in X is r_z-open. Hence X is an R_z-space. $\square$

R e m a r k 4.12. The hypothesis of "R_z-space" in Theorem 4.11 cannot be omitted. For let $X = \mathbb{R}$ be the real line with the right ray topology [34] and Y the real line with the indiscrete topology. Let $f\colon X \to Y$ be the identity function. Clearly f is R_z-supercontinuous but the graph function $g\colon X \to X \times Y$ is not R_z-supercontinuous.

Theorem 4.13. *Let $f\colon \prod_{\alpha\in\Lambda} X_\alpha \to \prod_{\alpha\in\Lambda} Y_\alpha$ be a mapping defined by $f((x_\alpha)) = (f_\alpha(x_\alpha))$, where $f_\alpha\colon X_\alpha \to Y_\alpha$ for each $\alpha \in \Lambda$. Then f is R_z-supercontinuous if and only if each f_α is R_z-supercontinuous.*

P r o o f. To prove necessity, let V_β be any open set in Y_β. Then $\pi_\beta^{-1}(V_\beta) = V_\beta \times \prod_{\alpha\neq\beta} Y_\alpha$ is a subbasic open set in $\prod_{\alpha\in\Lambda} Y_\alpha$. Now since f is R_z-supercontinuous,

$f^{-1}(\pi_\beta^{-1}(V_\beta)) = f_\beta^{-1}(V_\beta) \times \Bigl(\prod\limits_{\alpha\neq\beta} X_\alpha\Bigr)$ is an r_z-open set in $\prod\limits_{\alpha\in\Lambda} X_\alpha$. Thus $f_\beta^{-1}(V_\beta)$ is an r_z-open set in X_β and hence f_β is R_z-supercontinuous.

Conversely, let $V = V_\beta \times \prod\limits_{\alpha\neq\beta} Y_\alpha$ be a subbasic open set in the product space $\prod Y_\alpha$. Then $f^{-1}(V) = f^{-1}\Bigl(V_\beta \times \prod\limits_{\alpha\neq\beta} Y_\alpha\Bigr) = f_\beta^{-1}(V_\beta) \times \prod\limits_{\alpha\neq\beta} X_\alpha$. Since each f_β is R_z-supercontinuous, $f_\beta^{-1}(V_\beta)$ is an r_z-open subset of X_β and so $f^{-1}(V)$ is an r_z-open subset of $\prod\limits_{\alpha\in\Lambda} X_\alpha$ and hence f is R_z-supercontinuous. $\square$

Theorem 4.14. *Let $f, g\colon X \to Y$ be R_z-supercontinuous functions from a topological space X into a Hausdorff space Y. Then the equalizer $E = \{x \in X\colon f(x) = g(x)\}$ of the functions f and g is an r_z-closed subset of X.*

P r o o f. To prove that E is r_z-closed, we shall prove that its complement $X \setminus E$ is r_z-open. To this end, let $x \in X \setminus E$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, there exist disjoint open sets U and V containing $f(x)$ and $g(x)$, respectively. Since f and g are R_z-supercontinuous, $f^{-1}(U)$ and $g^{-1}(V)$ are r_z-open sets containing x. Then $W = f^{-1}(U) \cap g^{-1}(V)$ is an r_z-open set containing x and $W \cap E = \emptyset$. Thus E is r_z-closed. $\square$

5. Topological properties and R_z-supercontinuity

Theorem 5.1. *Let $f\colon X \to Y$ be an R_z-supercontinuous open bijection. Then X and Y are homeomorphic R_z-spaces.*

P r o o f. Let U be an open set in X and let $x \in U$. Then $f(U)$ is an open subset of Y containing $f(x)$. Now, since f is R_z-supercontinuous, there exists an r_z-open set G containing x such that $f(G) \subset f(U)$. Now, $x \in f^{-1}(f(G)) \subset f^{-1}(f(U))$. Again, since f is a bijection, $f^{-1}(f(G)) = G$ and $f^{-1}(f(U)) = U$. Thus U being expressible as a union of r_z-open sets is r_z-open and so X is an R_z-space. Since the property of being an R_z-space is a topological property and f is a homeomorphism, Y is also an R_z-space. $\square$

Theorem 5.2. *Let $f\colon X \to Y$ be an R_z-supercontinuous injection into a T_0-space Y. Then X is a functionally Hausdorff space.*

P r o o f. Let $x, y \in X$, $x \neq y$. Then $f(x) \neq f(y)$. Since Y is a T_0-space, there exist an open set W in Y containing one of the points $f(x)$ and $f(y)$ but not both. For definiteness, assume that $f(x) \in W$. Then $f^{-1}(W)$ is an r_z-open set containing x but not y. So there exists a z-closed set C containing x but not y such that

$C \subset f^{-1}(W)$. Let $C = \bigcap_{\alpha \in \Lambda} Z_\alpha$, where each Z_α is a zero set. There exists $\alpha_\circ \in \Lambda$ such that $y \notin Z_{\alpha_\circ}$. Hence there exists a continuous function $h\colon X \to [0,1]$ such that $h(x) = 0$ and $h(y) \neq 0$ and so X is functionally Hausdorff. $\square$

Corollary 5.3 ([15]). *Let $f\colon X \to Y$ be a z-supercontinuous injection into a T_0-space Y. Then X is a functionally Hausdorff space.*

Definitions 5.4. A space X is said to be

(i) *r_z-regular* if for every r_z-closed set A and a point $x \notin A$, there exist disjoint open sets U and V in X containing x and A, respectively;

(ii) *r_z-completely regular* if for every r_z-closed set A and a point $x \notin A$ there exists a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(A) = 1$.

Remark 5.5. For the interested reader we point out that the properties of r_z-regular spaces and r_z-completely regular can be inferred directly by substituting for $P =$ the property of being an r_z-closed set in the relevant results pertaining to P-regular spaces and completely P-regular spaces in [13].

Theorem 5.6. *Let $f\colon X \to Y$ be an R_z-supercontinuous open bijection from an r_z-regular space X onto Y. Then X and Y are homeomorphic regular spaces.*

Proof. Let B be a closed subset of Y and $y \notin B$. Then $f^{-1}(y)$ is a singleton and $f^{-1}(y) \notin f^{-1}(B)$. Since f is R_z-supercontinuous, by Theorem 4.2 (iii) $f^{-1}(B)$ is an r_z-closed subset of X. In view of r_z-regularity of X, there exist disjoint open sets U and V containing $f^{-1}(y)$ and $f^{-1}(B)$, respectively. Since f is an open bijection, $f(U)$ and $f(V)$ are disjoint open sets containing y and B, respectively, and so Y is a regular space. Again, since regularity is a topological property and since f is a homeomorphism, X is also a regular space. $\square$

Definition 5.7. A function $f\colon X \to Y$ is said to be an *R_z-homeomorphism* if f is a bijection such that both f and f^{-1} are R_z-supercontinuous.

Theorem 5.8. *Let $f\colon X \to Y$ be an R_z-homeomorphism from an r_z-completely regular space X onto Y. Then X and Y are homeomorphic completely regular spaces.*

Proof. In view of f being a homeomorphism, it is sufficient to prove that Y is a completely regular space. To this end, let B be a closed set in Y and let y be a point outside B. Then $x = f^{-1}(y)$ is a singleton and x does not belong to the r_z-closed set $f^{-1}(B)$. Since X is an r_z-completely regular space, there exists a continuous function $h\colon X \to [0,1]$ such that $h(x) = 0$ and $h(f^{-1}(B)) = 1$. Let

$g = h \circ f^{-1}$. Since f is an R_z-homeomorphism, g is well defined and is a continuous function from Y into $[0,1]$, since h is continuous. Clearly, $g(y) = 0$ and $g(F) = 1$. Thus Y is a completely regular space. $\square$

6. Properties of graph of an R_z-supercontinuous function

Definition 6.1. The graph $G(f)$ of a function $f\colon X \to Y$ is said to be *r_z-closed with respect to X* if for each $(x,y) \notin G(f)$ there exist open sets U and V containing x and y, respectively, such that U is r_z-open and $(U \times V) \cap G(f) = \emptyset$.

Theorem 6.2. *If $f\colon X \to Y$ is an R_z-supercontinuous function and Y is Hausdorff, then the graph of f is r_z-closed with respect to X.*

P r o o f. Let $x \in X$ and let $y \neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets V and W containing y and $f(x)$, respectively. Again, since f is R_z-supercontinuous, there exists an r_z-open set U containing x such that $f(U) \subset W \subset Y \setminus V$ and so $(U \times V) \cap G(f) = \emptyset$. Consequently, $G(f)$ is r_z-closed with respect to X. $\square$

Theorem 6.3. *Let $f\colon X \to Y$ be an injection such that its graph is r_z-closed with respect to X. Then X is a functionally Hausdorff space.*

P r o o f. Let $x_1, x_2 \in X$, $x_1 \neq x_2$. Since f is an injection, $(x_1, f(x_2)) \notin G(f)$. Since the graph $G(f)$ is r_z-closed with respect to X, there exist open sets U and V containing x_1 and $f(x_2)$, respectively, where U is r_z-open and $(U \times V) \cap G(f) = \emptyset$. Since U is r_z-open, let $U = \bigcup\{C_\alpha\colon \alpha \in \Lambda\}$, where each C_α is a z-closed set in X. Then $x_1 \notin X \setminus U = \bigcap\{X \setminus C_\alpha\colon \alpha \in \Lambda\}$. Hence there exists a $\beta \in \Lambda$ such that $x_1 \notin X \setminus C_\beta$. Let $C_\beta = \bigcap\limits_{\gamma \in \Gamma} C_{\beta\gamma}$, where each $C_{\beta\gamma}$ is a zero set in X. Then $X \setminus C_\beta = X \setminus \bigcap C_{\beta\gamma} = \bigcup\limits_{\gamma\in\Gamma} (X \setminus C_{\beta\gamma})$. So there exists some γ such that $x_1 \in C_{\beta\gamma}$ and $x_2 \notin C_{\beta\gamma}$. Thus there is a continuous function $h\colon X \to [0,1]$ such that $h(x_1) = 0$ and $h(x_2) \neq 0$. So X is a functionally Hausdorff space. $\square$

Theorem 6.4. *Let $f\colon X \to Y$ be a function such that its graph $G(f)$ is r_z-closed with respect to X. Then $f^{-1}(K)$ is r_z-closed in X for every compact subset K of Y.*

P r o o f. Let K be a compact subset of Y. To prove that $f^{-1}(K)$ is r_z-closed, we shall prove that its complement $X \setminus f^{-1}(K)$ is an r_z-open subset of X. To this end, let $x \in X \setminus f^{-1}(K)$. Then $(x,z) \notin G(f)$ for every $z \in K$. Since the graph $G(f)$ is r_z-closed with respect to X, there exist an r_z-open set U_z containing x and an open set V_z containing z such that $(U_z \times V_z) \cap G(f) = \emptyset$. The collection $\{V_z\colon z \in K\}$ is an open cover of the compact set K. So there exist finitely many $z_1, \ldots, z_n \in K$ such that $K \subset \bigcup\{V_{z_i}\colon i = 1, \ldots, n\}$. Let $U = \bigcap_{i=1}^{n} U_{z_i}$. Then U is r_z-open and $f(U) \cap K = \emptyset$. Thus $U \subset X \setminus f^{-1}(K)$. So $X \setminus f^{-1}(K)$ being the union of r_z-open sets is r_z-open and so $f^{-1}(K)$ is r_z-closed. $\square$

7. r_z-quotient topology and r_z-quotient spaces

Several variants of quotient topology occur in the lore of mathematical literature, see [18], [20]. In this section we introduce a new variant of quotient topology which lies strictly between the r_{cl}-quotient topology [35] and the r-quotient topology [18] as well as between the z-quotient topology [15] and the r-quotient topology.

Definitions 7.1. Let $p\colon X \to Y$ be a surjection from a topological space X onto a set Y. The collection of all subsets $A \subset Y$ such that $p^{-1}(A)$ is

(i) r_z-open in X is a topology on Y and is called the *r_z-quotient topology.* The map p is called the *r_z-quotient map* and the set Y with the r_z-quotient topology is called the *r_z-quotient space.*

(ii) r_{cl}-open in X is a topology on Y and is called the *r_{cl}-quotient topology* and the map p is called the *r_{cl}-quotient map.*

(iii) z-open in X is a topology on Y and is called the *z-quotient topology* and the map p is called the *z-quotient map.*

(iv) r-open in X is a topology on Y and is called the *r-quotient topology* and the map p is called the *r-quotient map.*

The following diagram gives a quick comparison among the variants of quotient topologies defined in Definitions 7.1. For a detailed survey of the variants of quotient topologies in the literature and the interrelations among them we refer the interested reader to [18], [20].

$$\begin{array}{ccc}
\text{cl-quotient topology} & \subset & r_{\mathrm{cl}}\text{-quotient topology} \\
\cap & & \cap \\
z\text{-quotient topology} & \subset & r_z\text{-quotient topology} \\
 & & \cap \\
\text{quotient topology} & \supset & r\text{-quotient topology}
\end{array}$$

However, none of the above inclusions is reversible in general as is shown by examples in [18], [20], [35] and the following example.

E x a m p l e 7.2. Let $X = Y$ be the set of natural numbers and let X be endowed with the cofinite topology τ_c. Let f denote the identity function defined on X. Then the r-quotient topology on Y is identical with τ_c, while cl-quotient topology = r_{cl}-quotient topology = r_z-quotient topology = z-quotient topology = indiscrete topology.

Theorem 7.3. *Let $p\colon (X,\tau_1) \to (X,\tau_2)$ be a surjection, where τ_2 is the r_z-quotient topology on Y. Then p is R_z-supercontinuous. Moreover, τ_2 is the largest topology on Y which makes $p\colon (X,\tau_1) \to Y$ R_z-supercontinuous.*

The following result shows that a function out of an r_z-quotient space is continuous if and only if its composition with any r_z-quotient map is R_z-supercontinuous.

Theorem 7.4. *Let $p\colon X \to Y$ be an r_z-quotient map. Then a function $g\colon Y \to Z$ is continuous if and only if $g \circ p$ is R_z-supercontinuous.*

8. Change of topology and R_z-supercontinuous functions

The technique of change of topology of a space is of considerable significance and widely used in topology, functional analysis and several other branches of mathematics. For example, weak and weak* topologies of a Banach space, weak and strong operator topologies on $\mathfrak{B}(H)$, the space of operators on a Hilbert space, the hull kernel topology and the multitude of other topologies on $\mathrm{Id}(A)$, the space of all closed two sided ideals of a Banach algebra A, see [3]–[5], [33]. Furthermore, to taste the flavour of applications of the technique of change in topology see [7]–[9], [14], [18], [30], [38].

Theorem 8.1. *A topological space (X,τ) is an R_z-space if and only if $\tau = \tau_{rz}$.*

In this section we retopologize the domain of an R_z-supercontinuous function such that it transforms into a continuous function with the new topology of the domain. Let (X,τ) be a topological space and let $\mathcal{B}_{r_z}$ denote the collection of all r_z-open subsets of (X,τ). Since arbitrary unions and finite intersections of r_z-open sets are r_z-open, the collection $\mathcal{B}_{r_z}$ is indeed a topology for X, which we denote by τ_{rz}. Clearly $\tau_{rz} \subset \tau$ and the inclusion is proper if (X,τ) is not an R_z-space.

Theorem 8.2. *A function $f\colon (X,\tau) \to (Y,\nu)$ is R_z-supercontinuous if and only if $f\colon (X,\tau_{rz}) \to (Y,\nu)$ is continuous.*

Many of the results of the preceding sections now follow from Theorem 8.2 and the corresponding standard properties of continuous functions.

Theorem 8.3. *For a topological space (X,τ) the following statements are equivalent.*

(i) *(X,τ) is an R_z-space.*

(ii) *Every continuous function $f\colon (X,\tau)\to(Y,\nu)$ from a space (X,τ) into (Y,ν) is R_z-supercontinuous.*

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i). Take $(Y,\nu)=(X,\tau)$. Then the identity function 1_X on X is continuous and so R_z-supercontinuous. Thus by Theorem 8.1, the identity function $1_X\colon (X,\tau_{rz})\to(X,\tau)$ is continuous. Since $U\in\tau$ implies $1_X^{-1}(U)=U\in\tau_{rz}$, we have $\tau\subset\tau_{rz}$. Hence it follows that $\tau=\tau_{rz}$ and so (X,τ) is an R_z-space. □

Definition 8.4. A function $f\colon X\to Y$ from a topological space X into a topological space Y is said to be *R_z-continuous* at $x\in X$ if for each r_z-open set V containing $f(x)$ there exists an open set U containing x such that $f(U)\subset V$. The function f is said to be *R_z-continuous* if it is R_z-continuous at each $x\in X$.

Theorem 8.5. *For a function $f\colon (X,\tau)\to(Y,\nu)$, the following statements are true.*

(i) *f is R_z-continuous if and only if $f\colon (X,\tau)\to(Y,\nu_{rz})$ is continuous.*

(ii) *f is R_z-open if and only if $f\colon (X,\tau_{rz})\to(Y,\nu)$ is open.*

In view of Theorems 8.2 and 8.5, Theorem 4.7 can be restated as follows:

If $f\colon (X,\tau_{rz})\to(Y,\nu)$ is a continuous open surjection and $g\colon (Y,\nu)\to(Z,\omega)$ is a function, then g is continuous if and only if $g\circ f$ is continuous. Further, if f maps open (closed) sets to r_z-open (r_z-closed) sets, then g is R_z-supercontinuous.

Moreover, the r_z-quotient topology on Y determined by the function $f\colon (X,\tau)\to Y$ in Section 7 is identical with the standard quotient topology on Y determined by $f\colon (X,\tau_{rz})\to Y$.

Remark 8.6. For the interested reader we point out that the properties of R_z-continuous functions can be inferred directly by simply substituting $P\equiv$ the property of being an r_z-closed set, in the relevant results pertaining to P-continuous functions and P^*-continuous functions in [10]–[12].

References

[1] *R. A. Alò, H. L. Shapiro*: Normal Topological Spaces. Cambridge Tracts in Mathematics 65, Cambridge University Press, Cambridge, 1974. zbl MR

[2] *C. E. Aull*: Functionally regular spaces. Nederl. Akad. Wet., Proc., Indag. Math. 38, Ser. A *79* (1976), 281–288. zbl MR

[3] *F. Beckhoff*: Topologies on the ideal space of a Banach algebra and spectral synthesis. Proc. Am. Math. Soc. *125* (1997), 2859–2866. zbl MR

[4] *F. Beckhoff*: Topologies of compact families on the ideal space of a Banach algebra. Stud. Math. *118* (1996), 63–75. zbl MR

[5] *F. Beckhoff*: Topologies on the space of ideals of a Banach algebra. Stud. Math. *115* (1995), 189–205. zbl MR

[6] *R. L. Blair, A. W. Hager*: Extensions of zero-sets and of real-valued functions. Math. Z. *136* (1974), 41–52. zbl MR

[7] *D. B. Gauld, M. Mršević, I. L. Reilly, M. K. Vamanamurthy*: Continuity properties of functions. Topology, Theory and Applications (Á. Császár, ed.). 5th Colloq., Eger, Hungary, 1983, Colloq. Math. Soc. János Bolyai 41, North-Holland, Amsterdam; János Bolyai Mathematical Society, Budapest, 1985, pp. 311–322. zbl MR

[8] *A. M. Gleason*: Universal locally connected refinements. Ill. J. Math. *7* (1963), 521–531. zbl MR

[9] *J. K. Kohli*: Change of topology, characterizations and product theorems for semilocally P-spaces. Houston J. Math. *17* (1991), 335–350. zbl MR

[10] *J. K. Kohli*: A framework including the theories of continuous functions and certain noncontinuous functions. Note Mat. *10* (1990), 37–45. zbl MR

[11] *J. K. Kohli*: A unified approach to continuous and certain noncontinuous functions. I. J. Aust. Math. Soc., Ser. A *48* (1990), 347–358. zbl MR

[12] *J. K. Kohli*: A unified approach to continuous and certain noncontinuous functions. II. Bull. Aust. Math. Soc. *41* (1990), 57–74. zbl MR

[13] *J. K. Kohli*: A unified view of (complete) regularity and certain variants of (complete) regularity. Can. J. Math. *36* (1984), 783–794. zbl MR

[14] *J. K. Kohli*: A class of mappings containing all continuous and all semiconnected mappings. Proc. Am. Math. Soc. *72* (1978), 175–181. zbl MR

[15] *J. K. Kohli, R. Kumar*: z-supercontinuous functions. Indian J. Pure Appl. Math. *33* (2002), 1097–1108. zbl MR

[16] *J. K. Kohli, D. Singh*: D_δ-supercontinuous functions. Indian J. Pure Appl. Math. *34* (2003), 1089–1100. zbl MR

[17] *J. K. Kohli, D. Singh*: D-supercontinuous functions. Indian J. Pure Appl. Math. *32* (2001), 227–235. zbl MR

[18] *J. K. Kohli, D. Singh, J. Aggarwal*: R-supercontinuous functions. Demonstr. Math. (electronic only) *43* (2010), 703–723. zbl MR

[19] *J. K. Kohli, D. Singh, J. Aggarwal*: F-supercontinuous functions. Appl. Gen. Topol. (electronic only) *10* (2009), 69–83. zbl MR

[20] *J. K. Kohli, D. Singh, R. Kumar*: Some properties of strongly θ-continuous functions. Bull. Calcutta Math. Soc. *100* (2008), 185–196. MR

[21] *J. K. Kohli, B. K. Tyagi, D. Singh, J. Aggarwal*: R_δ-supercontinuous functions. Demonstr. Math. (electronic only) *47* (2014), 433–448. zbl MR

[22] *N. Levine*: Strong, continuity in topological spaces. Am. Math. Mon. *67* (1960), 269. zbl

[23] *P. E. Long, L. L. Herrington*: Strongly θ-continuous functions. J. Korean Math. Soc. *18* (1981), 21–28. zbl MR

[24] *J. Mack*: Countable paracompactness and weak normality properties. Trans. Am. Math. Soc. *148* (1970), 265–272. zbl MR

[25] *B. M. Munshi, D. S. Bassan*: Super-continuous mappings. Indian J. Pure Appl. Math. *13* (1982), 229–236. zbl MR
[26] *T. Noiri*: Supercontinuity and some strong forms of continuity. Indian J. Pure Appl. Math. *15* (1984), 241–250. zbl MR
[27] *T. Noiri*: On δ-continuous functions. J. Korean Math. Soc. *16* (1980), 161–166. zbl MR
[28] *I. L. Reilly, M. K. Vamanamurthy*: On super-continuous mappings. Indian J. Pure Appl. Math. *14* (1983), 767–772. zbl MR
[29] *M. K. Singal, S. B. Nimse*: z-continuous mappings. Math. Stud. *66* (1997), 193–210. zbl MR
[30] *D. Singh*: cl-supercontinuous functions. Appl. Gen. Topol. *8* (2007), 293–300. zbl MR
[31] *D. Singh*: D^*-supercontinuous functions. Bull. Calcutta Math. Soc. *94* (2002), 67–76. zbl MR
[32] *D. Singh, J. K. Kohli*: Separation axioms between functionally regular spaces and R_0-spaces. Submitted to Sci. Stud. Res., Ser. Math. Inform.
[33] *D. W. B. Somerset*: Ideal spaces of Banach algebras. Proc. Lond. Math. Soc. (3) *78* (1999), 369–400. zbl MR
[34] *L. A. Steen, J. A. Seebach,* Jr.: Counterexamples in Topology. Springer, New York, 1978. zbl MR
[35] *B. K. Tyagi, J. K. Kohli, D. Singh*: R_{cl}-supercontinuous functions. Demonstr. Math. (electronic only) *46* (2013), 229–244. zbl MR
[36] *W. T. van Est, H. Freudenthal*: Trennung durch stetige Funktionen in topologischen Räumen. Nederl. Akad. Wet., Proc., Indagationes Math. 13, Ser. A *54* (1951), 359–368. (In German.) zbl MR
[37] *N. V. Veličko*: H-closed topological spaces. Transl., Ser. 2, Am. Math. Soc. *78* (1968), 103–118; translation from Russian original, Mat. Sb. (N.S.), *70* (1966), 98–112. zbl
[38] *G. S. Young,* Jr.: The introduction of local connectivity by change of topology. Am. J. Math. *68* (1946), 479–494. zbl MR

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