ON THE EQUALITY BETWEEN SOME CLASSES OF OPERATORS ON BANACH LATTICES

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(Received November 12, 2010)

Abstract. We establish some sufficient conditions under which the subspaces of Dunford-Pettis operators, of M-weakly compact operators, of L-weakly compact operators, of weakly compact operators, of semi-compact operators and of compact operators coincide and we give some consequences.

Keywords: M-weakly compact operator, L-weakly compact operator, Dunford-Pettis operator, weakly compact operator, semi-compact operator, compact operator, order continuous norm, discrete Banach lattice, positive Schur property

MSC 2010: 46A40, 46B40, 46B42

1. Introduction and notation

In [2] and [6] ([5], [7]) the compactness (weak compactness, semi-compactness) of positive Dunford-Pettis operators was studied, but as a compact (weakly compact, semi-compact) operator is not necessarily L-weakly compact (M-weakly compact), we cannot deduce anything on the L-weak compactness (M-weak compactness, respectively) of positive Dunford-Pettis operators. Also, a M-weakly compact (L-weakly compact) operator is not necessarily Dunford-Pettis. In fact, the inclusion map $i\colon L^2[0,1] \to L^1[0,1]$ is both L-weakly compact and M-weakly compact but it is not Dunford-Pettis. Finally, note that Chen and Wickstead [9] used the Schur property to study the L-weak compactness and the M-weak compactness of weakly compact operators.

Recall that an operator T from a Banach space E into another F is said to be Dunford-Pettis if it carries weakly compact subsets of E onto compact subsets of F. It is well known that each compact operator is Dunford-Pettis but a Dunford-Pettis operator is not necessarily compact. However, they coincide if the Banach space E is reflexive.

On the other hand, an operator T from a Banach lattice E into a Banach space F is M-weakly compact if for each disjoint bounded sequence (x_n) of E, we have $\lim_n \|T(x_n)\| = 0$. An operator T from a Banach space E into a Banach lattice F is called L-weakly compact if for each disjoint bounded sequence (y_n) in the solid hull of $T(B_E)$, we have $\lim_n \|y_n\| = 0$.

Meyer-Nieberg ([12], Proposition 3.6.11) proved that between two Banach lattices, an operator T is L-weakly compact (M-weakly compact) if and only if its adjoint T' is M-weakly compact (L-weakly compact). He also proved that the class of Dunford-Pettis operators does not satisfy the duality problem. Some results on this problem were given in [8].

Finally, unlike Dunford-Pettis operators [2], [11], [13], the class of L-weakly compact (M-weakly compact) operators satisfies the domination problem. Indeed, if S and T are operators from a Banach lattice E into another F such that $0 \leqslant S \leqslant T$ and T is L-weakly compact (respectively M-weakly compact), then S is L-weakly compact (respectively M-weakly compact) (Theorem 3.6.16 of Meyer-Nieberg [12]).

Our goal in this paper is to give some sufficient conditions under which the class of Dunford-Pettis (compact, weakly compact, semi-compact) operators coincides with the class of M-weakly compact (respectively L-weakly compact) operators. Also, we will give some interesting consequences.

To state our results, we need to fix some notation and recall some definitions. A vector lattice E is an ordered vector space in which $\sup(x, y)$ and $\inf(x, y)$ exist for every $x, y \in E$. A subspace F of a vector lattice E is said to be a sublattice if for every pair of elements a, b of F the supremum and the infimum of a and b taken in E belong to F. A subset B of a vector lattice E is said to be solid if it follows from $|y| \leqslant |x|$ with $x \in B$ and $y \in E$ that $y \in B$. An order ideal of E is a solid subspace. Let E be a vector lattice, then for each $x, y \in E$ with $x \leqslant y$, the set $[x, y] = \{z \in E \colon x \leqslant z \leqslant y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm possesses the following property: for each $x, y \in E$ such that $|x| \leqslant |y|$, we have $\|x\| \leqslant \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice. Recall that a norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. Finally, a nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the lattice subspace generated by x. The vector lattice E is discrete, if it admits a complete disjoint system of discrete elements. We refer the reader to Zaanen [15] for unexplained terminology on Banach lattice theory.

2. Main results

We will use the term operator $T\colon E \to F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geqslant 0$ in F whenever $x \geqslant 0$ in E. The operator T is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F.

Let us recall that if an operator $T\colon E \to F$ between two Banach lattices is positive, then its adjoint operator $T'\colon F' \to E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. For more information on positive operators see the book of Aliprantis-Burkinshaw [3].

In [6] it is proved that if E' is discrete and its norm is order continuous, then the class of positive Dunford-Pettis operators coincides with that of positive compact operators. In the following we show that these two classes coincide also with the subspace of M-weakly compact operators not necessarily positive.

Theorem 2.1. *Let $T\colon E \to F$ be an operator from a Banach lattice E into a Banach space F. If E' is discrete and its norm is order continuous, then the following assertions are equivalent:*
(i) *T is Dunford-Pettis.*
(ii) *T is M-weakly compact.*
(iii) *T is compact.*

Proof. (i) $\Longrightarrow$ (ii) Since the norm of E' is order continuous, it follows from Corollary 2.9 of Dodds-Fremlin [10] that each bounded disjoint sequence (x_n) of E is convergent to 0 in the weak topology $\sigma(E, E')$. Since the operator $T\colon E \to F$ is Dunford-Pettis, we obtain $\|T(x_n)\| \to 0$. Hence T is M-weakly compact.

(ii) $\Longrightarrow$ (iii) Let $T\colon E \to F$ be an M-weakly compact operator, its adjoint $T'\colon F' \to E'$ is L-weakly compact ([12], Proposition 3.6.11). We have to prove that T' is compact. Let A be the solid hull of $T'(B_{F'})$ where $B_{F'}$ is the closed ball of F'. Since T' is L-weakly compact, each disjoint sequence of $T'(B_{F'})$ converges to 0 in the norm. Now, as E' is discrete, it follows from Theorem 21.15 of Aliprantis and Burkinshaw [1] that the solid and bounded subset A of E' is relatively compact in the norm if and only if each disjoint sequence of A converges to 0 in the norm. Hence $T'(B_{F'})$ is relatively compact in the norm. And this proves that T' is compact.

(iii) $\Longrightarrow$ (i) Obvious.

A non-empty bounded subset A of a Banach lattice E is L-weakly compact if for every disjoint sequence (x_n) in the solid hull of A, we have $\|x_n\| \to 0$.

Recall that a Banach space E has the Dunford-Pettis property if each weakly compact operator on E into another Banach space F is Dunford-Pettis. If we replace the class of compact operators by the class of weakly compact operators, we obtain

Theorem 2.2. *Let $T\colon E \to F$ be an operator from a Banach lattice E into a Banach space F. If E has the Dunford-Pettis property and the norm of E' is order continuous, then the following assertions are equivalent:*
(i) *T is Dunford-Pettis.*
(ii) *T is M-weakly compact.*
(iii) *T is weakly compact.*

Proof. (i) $\Longrightarrow$ (ii) It is just the implication $1 \Longrightarrow 2$ of Theorem 2.1.

(ii) $\Longrightarrow$ (iii) If T is an M-weakly compact operator then its adjoint T' is L-weakly compact. We have just to prove that T' is weakly compact. In fact, since $T'(B_{F'})$ is L-weakly compact in E', where $B_{F'}$ denotes the closed unit ball in F', hence $T'(B_{F'})$ is relatively weakly compact. In fact, let $S = \mathrm{sol}(T'(B_{F'}))$ be the solid hull of $T'(B_{F'})$, then for every disjoint sequence (x_n) in S we have $\|x_n\| \to 0$. It follows from Theorem 21.8 of Aliprantis-Burkinshaw [1] that S is relatively weakly compact. Hence $T'(B_{F'})$ is relatively weakly compact (because $T'(B_{F'}) \subset S$). Then the adjoint T' is weakly compact. Hence T is weakly compact.

(iii) $\Longrightarrow$ (i) Obvious since E has the Dunford-Pettis property.

Now, as a consequence of Theorem 2.1 and Theorem 2.2, we obtain a sufficient condition for the four classes of operators to coincide.

Corollary 2.3. *Let $T\colon E \to F$ be an operator from a Banach lattice E into a Banach space F. If E has the Dunford-Pettis property and E' is discrete with an order continuous norm, then the following assertions are equivalent:*
(1) *T is Dunford-Pettis.*
(2) *T is M-weakly compact.*
(3) *T is weakly compact.*
(4) *T is compact.*

Proof. Clearly (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) by Theorem 2.2.

(1) $\Longrightarrow$ (2) $\Longrightarrow$ (4) It is just Theorem 2.2.

Let us recall that a subset S of a Banach lattice E is called almost order bounded if for each $\varepsilon > 0$ there exists $u \in E^+$ such that $S \subset [-u, u] + \varepsilon B_E$ where B_E is the closed unit ball of E.

Recall from [4] that an operator T from a Banach space E into a Banach lattice F is said to be semi-compact if $T(B_E)$ is almost order bounded, i.e., for each $\varepsilon > 0$ there exists $u \in F^+$ such that $T(B_E) \subset [-u, u] + \varepsilon B_F$ where $F^+ = \{y \in F\colon 0 \leqslant y\}$.

Each L-weakly compact subset of a Banach lattice E is almost order bounded. In fact, let A be a subset of E which is L-weakly compact, i.e., for every disjoint sequence (x_n) in the solid hull of A we have $\|x_n\| \to 0$. It follows from Corollary 2.10 of Dodds-Fremlin [10] that for each $\varepsilon > 0$ there exists $u \in E^+$ such that $\|(|x| - u)^+\| \leqslant \varepsilon$ for every $x \in A$. Now, Theorem 122.1 of Zaanen [15] implies that A is almost order bounded. Hence, each L-weakly compact operator $T\colon E \to F$ is semi-compact.

A semi-compact operator is not necessarily L-weakly compact (M-weakly compact). In fact, the identity operator $\mathrm{Id}_c\colon c \to c$ is semi-compact but it is not L-weakly compact (M-weakly compact) where c is the Banach lattice of all convergent sequences. If not, Id_c would be weakly compact and this is false.

Now, we give a sufficient condition under which the two classes of L-weakly and M-weakly compact operators coincide with the class of semi-compact operators.

Theorem 2.4. *Let $T\colon E \to F$ be a regular operator between two Banach lattices. If E' and F have order continuous norms, then the following assertions are equivalent:*
(1) *T is semi-compact.*
(2) *T is L-weakly compact.*
(3) *T is M-weakly compact.*

Proof. (1) $\Longrightarrow$ (2) Follows from Theorem 1 of [6].

(2) $\Longleftrightarrow$ (3) It is just Theorem 5.2 of Dodds-Fremlin [10].

(2) $\Longrightarrow$ (1) We will prove that each L-weakly compact operator $T\colon E \to F$ is semi-compact, i.e., if $T(B_E)$ is an L-weakly compact subset of F, then $T(B_E)$ is an almost order bounded subset of F. Since for every disjoint sequence (x_n) in the solid hull of $T(B_E)$ we have $\|x_n\| \to 0$, it follows from Corollary 2.10 of Dodds-Fremlin [10] that for each $\varepsilon > 0$ there exists $u \in F^+$ such that $\|(|x| - u)^+\| \leqslant \varepsilon$ for every $x \in T(B_E)$. Now, Theorem 122.1 of Zaanen [15] implies that $T(B_E)$ is almost order bounded.

As a consequence of Proposition 3.7.10 of Meyer-Nieberg [12], Theorem 2.4 and Theorem 2.1, we obtain the following corollary:

Corollary 2.5. *Let $T\colon E \to F$ be a regular operator between two Banach lattices. If E' is discrete with an order continuous norm and the norm of F is order continuous, then the following assertions are equivalent:*
(1) *T is Dunford-Pettis.*
(2) *T is M-weakly compact.*
(3) *T is L-weakly compact.*
(4) *T is semi-compact.*
(5) *T is compact.*

Proof. In fact, since the norm of E' is order continuous, it follows from Proposition 3.7.10 of Meyer-Nieberg [12] that T is M-weakly compact.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (2) It is just Theorem 2.4.

(2) $\Longrightarrow$ (5) It is just the implication (ii) $\Longrightarrow$ (iii) of Theorem 2.1.

(5) $\Longrightarrow$ (1) Obvious.

As a consequence of Corollary 2.5, Theorem 2.4 and Theorem 2.1, we obtain the following result:

Corollary 2.6. *Let $T\colon\ E \to F$ be a regular operator between two Banach lattices. If the norm of $E'$is order continuous and F is discrete and its norm is order continuous, then the following assertions are equivalent:*

(1) *T is Dunford-Pettis.*
(2) *T is M-weakly compact.*
(3) *T is L-weakly compact.*
(4) *T is semi-compact.*
(5) *T is compact.*

Proof. (1) $\Longrightarrow$ (2) Follows from Proposition 3.7.10 of Meyer-Nieberg [12].

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) It is just Theorem 2.4.

(4) $\Longrightarrow$ (5) If $T\colon\ E \to F$ is semi-compact then for each $\varepsilon > 0$ there exists $u \in F^+$ such that $T(B_E) \subset [-u, u] + \varepsilon B_F$. Now, since F is discrete and its norm is order continuous, the order intervall $[-u, u]$ is compact (see Corollary 21.13 of [1]). Then $T(B_E)$ is precompact and hence T is compact.

(3) $\Longrightarrow$ (5) It is just the implication (ii) $\Longrightarrow$ (iii) of Theorem 2.1.

(5) $\Longrightarrow$ (1) Obvious.

We also have the following consequence:

Corollary 2.7. *Let $T\colon\ E \to F$ be a regular operator between two Banach lattices. If E has the Dunford-Pettis property and the norm of E' is order continuous and F is discrete and its norm is order continuous, then the following assertions are equivalent:*

(1) *T is Dunford-Pettis.*
(2) *T is M-weakly compact.*
(3) *T is L-weakly compact.*
(4) *T is semi-compact.*
(5) *T is compact.*
(6) *T is weakly compact.*

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$(4) $\Longrightarrow$ (5) It is just Corollary 2.6.

(5) $\Longrightarrow$ (6) Obvious.

(6) $\Longrightarrow$ (1) Obvious (because E has the Dunford-Pettis property).

To give the next result, we need to recall the following notions. A Banach space E is said to have the Schur property if every sequence weakly convergent to zero is norm convergent to zero in E. For example, the Banach space l^1 has the Schur property.

The Banach lattice E has the positive Schur property if weakly null sequences with positive terms are norm null. For example, the Banach lattice $L^1([0,1])$ has the positive Schur property but does not have the Schur property. Fore more information about this notion see [14].

Theorem 2.8. *Let E and F be two Banach lattices. If E' has the positive Schur property and F is discrete with an order continuous norm, then for every regular operator $T\colon E \to F$ the following assertions are equivalent:*

(1) *T is Dunford-Pettis.*
(2) *T is M-weakly compact.*
(3) *T is L-weakly compact.*
(4) *T is semi-compact.*
(5) *T is compact.*
(6) *T is weakly compact.*

Proof. Note that if E' has the positive Schur property, then the norm of E' is order continuous.

(1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (1) It is just Corollary 2.6.

(5) $\Longrightarrow$ (6) Obvious.

(6) $\Longrightarrow$ (2) If T is weakly compact, then its adjoint $T'\colon F' \to E'$ is weakly compact. Put $A = T'(B_{F'})$. Then A is relatively weakly compact in E'. Since E' has the positive Schur property, it follows from Theorem 3.1 (3) of Chen-Wickstead [9] that A is an L-weakly compact subset of E. And hence T' is L-weakly compact. This proves that T is M-weakly compact.

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