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Asymptotic properties of solutions of ordinary differential equations

Syllabus of the dissertation

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1 Subject, methods and results of the dissertation

The dissertation contains a collection of the published papers [D1]–[D12], which are devoted to the asymptotic theory of ordinary differential equations of the second and higher orders. In particular, we study asymptotic behavior of solutions of the second order nonlinear differential equations with p-Laplacian, oscillatory and asymptotic properties of solutions of the third order differential equations and of the linear differential equations with quasiderivatives.

In the following sections we present employed methods and some main results published in papers [D1]–[D12].

1.1. Second order differential equations with p-Laplacian

Results published in [D9, D10, D11, D12] are concerned with the asymptotic behavior of solutions of the nonlinear differential equation

$$(a(t)\Phi_p(x'))' = b(t)f(x) \tag{1.1}$$

and its special cases — the quasilinear differential equation

$$(a(t)\Phi_p(x'))' = b(t)\Phi_q(x) \tag{1.2}$$

and the half-linear equation

$$(a(t)\Phi_p(x'))' = b(t)\Phi_p(x). \tag{1.3}$$

Throughout this chapter we assume that the functions a, b are continuous for $t \geq 0$,

$$a(t) > 0, \quad b(t) > 0, \quad \Phi_p(u) = |u|^{p-2}u \text{ with } p > 1, \tag{1.4}$$

$q > 1$ and f is a continuous function on $\mathbb{R}$ such that $f(u)u > 0$ for $u \neq 0$.

We start with a short historical and bibliographical survey.

For $p = 2$, equation (1.1) becomes a equation with the Sturm-Liouville differential operator

$$(a(t)x')' = b(t) f(x) \tag{1.5}$$

and (1.3) becomes the linear equation

$$(a(t)x')' = b(t)x. \tag{1.6}$$

It is well known (see, e.g, [39]) that if (1.4) holds, then the linear equation (1.6) is nonoscillatory. In [60], a complete analysis of the asymptotic behavior of solutions of (1.6) has been presented. Later such results have been extended in [12, 13, 14, 15, 19, 58, 59, 65] in many directions to the nonlinear equation (1.5). In some of these quoted papers, the case of b eventually negative has been considered. A survey of the oscillatory and asymptotic theories of equation (1.5) can also be found in the recent monograph [1].

Quasilinear equation (1.2) has been extensively considered in the last years, see, e.g., [19, 64, 76]. Solutions of (1.2) can be examined by interpreting (1.2) as a system of the Emden–Fowler type for the vector $(x, y) = (x, a\Phi_p(x'))$ given by

$$\begin{aligned} x' &= \Phi_{p^*}\left(\frac{1}{a(t)}\right) \Phi_{p^*}(y) \\ y' &= b(t)\, \Phi_q(x), \end{aligned} \tag{1.7}$$

where p^* denotes the conjugate number of p, i.e.

$$p^* = \frac{p}{p-1} \quad \text{or} \quad \frac{1}{p} + \frac{1}{p^*} = 1.$$

For this reason, equation (1.2) is sometimes called a *generalized Emden-Fowler equation*. We refer to the book by Mirzov [61] where the uniqueness, continuability, existence, and asymptotic behavior of solutions of (1.7) have been investigated. Other interesting results concern the special case $p = q$, i.e. the half-linear equation (1.3), see e.g. [26, 27].

Such studies are essentially motivated by the dynamics of positive radial solutions of reaction-diffusion problems modelled by the nonlinear elliptic equation

$$-\operatorname{div}(\alpha(|\nabla u|)\nabla u) + \lambda f(u) = 0 \tag{1.8}$$

where $\alpha : (0, \infty) \to (0, \infty)$ is continuous and such that $\delta(v) := \alpha(|v|)v$ is an odd increasing homeomorphism from $\mathbb{R}$ into $\mathbb{R}$, λ is a positive constant (the so called Thiele modulus), and f represents the ratio of the reaction rate at the concentration u to the reaction rate at concentration unity (see e.g. [20]). If $\alpha(|v|) = |v|^{p-2}$, then the differential operator in (1.8) is the p-Laplacian $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and (1.8) reduces to (1.1).

The aim of papers [D9, D10] is to study the asymptotic behavior of solutions of (1.1) and (1.2). This task is accomplished by introducing an *a-priori* classification of solutions of (1.1) which is a natural extension of the well-known classification stated for the linear case.

As usual, by a solution of (1.1) we always mean a continuously differentiable function x such that $a\Phi_p(x')$ has a continuous derivative satisfying (1.1). Following I. Kiguradze [47, Definitions 12.4 and 12.5], a nontrivial solution x of (1.1) is said to be a *singular solution of the first kind of* (1.1) if there exists $T < \infty$ such that $x(t) \equiv 0 \quad \text{for } t \geq T$, and a solution x of (1.1) is said to be a *singular solution of the second kind of* (1.1) if there exists $T < \infty$ such that $\lim_{t\to T-} |x(t)| = \infty$. For brevity, denote by S_1 (S_2) the set of all singular solutions of the first (second) kind. Observe that the problem whether $S_1 = \emptyset$ is closely related to the uniqueness problem with respect to the initial conditions. A nontrivial and nonsingular solution of (1.1) is said to be a *proper solution.*

We say that a proper solution x of (1.1) defined on (α_x, ∞) is either of class $\mathbb{M}^+$ or of class $\mathbb{M}^-$ according to whether $x(t)x'(t)$ is eventually positive or $x(t)x'(t)$ is negative for every $t > \alpha_x$, i.e.

$$\mathbb{M}^+ = \{x \text{ proper solution of (1.1)} : \exists t_x \geq \alpha_x : \ x(t)x'(t) > 0 \text{ for } t > t_x\},$$
$$\mathbb{M}^- = \{x \text{ proper solution of (1.1)} : \ x(t)x'(t) < 0 \text{ for } t > \alpha_x\}.$$

As for the singular solutions and solutions in classes $\mathbb{M}^-$, $\mathbb{M}^+$ for the quasilinear equation (1.2), the results of [17, 18, 63] yield the following.

Theorem A. *The following holds for equation* (1.2)*:*

(a) *If $p = q$ then $S_1 = S_2 = \emptyset$, $\mathbb{M}^- \neq \emptyset$ and $\mathbb{M}^+ \neq \emptyset$.*
(b) *If $p < q$ then $S_1 = \emptyset$, $S_2 \neq \emptyset$ and $\mathbb{M}^- \neq \emptyset$.*
(c) *If $p > q$ then $S_1 \neq \emptyset$, $S_2 = \emptyset$ and $\mathbb{M}^+ \neq \emptyset$.*

Remark 1.1. When $p > q$, the class $\mathbb{M}^-$ can be empty. For instance, this happens when $p = 2$, $a(t) \equiv 1$, $1 < q < 2$ and $\liminf_{t\to\infty} t^2 b(t) > 0$, see e.g. [47, Corollary 17.3].

When $p < q$, the class $\mathbb{M}^+$ can be empty. For instance, this happens when $p = 2$, $a(t) \equiv 1$, $q > 2$ and $\liminf_{t\to\infty} t^q b(t) > 0$, see e.g. [47, Corollary 17.4]).

In the sequel, we will extend Theorem A to equation (1.1), and we will give conditions which ensure that the classes $\mathbb{M}^-$ and $\mathbb{M}^+$ are not empty when $p > q$ and $p < q$, respectively.

As in the papers [12, 13, 14, 15, 19, 59, 65], both classes $\mathbb{M}^+, \mathbb{M}^-$ can be divided, *a-priori,* into the following four subclasses, which are mutually disjoint:

$$\mathbb{M}_B^- = \{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = \ell_x \neq 0\},$$
$$\mathbb{M}_0^- = \{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = 0\},$$
$$\mathbb{M}_B^+ = \{x \in \mathbb{M}^+ : \lim_{t\to\infty} x(t) = \ell_x < \infty\},$$
$$\mathbb{M}_\infty^+ = \{x \in \mathbb{M}^+ : \lim_{t\to\infty} |x(t)| = \infty\}.$$

In the following sections we are going to characterize these classes in terms of certain integral conditions and, similarly to the linear case, to show that the convergence and/or divergence of the two integrals

$$J_1 = \lim_{T\to\infty} \int_0^T \Phi_{p^*}\Big(\frac{1}{a(t)}\Big)\Phi_{p^*}\Big(\int_0^t b(s)\,ds\Big)\,dt,$$
$$J_2 = \lim_{T\to\infty} \int_0^T \Phi_{p^*}\Big(\frac{1}{a(t)}\Big)\Phi_{p^*}\Big(\int_t^T b(s)\,ds\Big)\,dt,$$

completely characterize the above four classes. We remark that in the linear case $J_1 = I_1$ and $J_2 = I_2$. In the following we will also use the notation

$$J_3 = \int_0^\infty \Phi_{p^*}\Big(\frac{1}{a(t)}\Big)\,dt, \quad J_4 = \int_0^\infty b(t)\,dt.$$

We point out that a different approach and classification of solutions of (1.2) have been used in the recent papers [64] and [76] under the assumptions $a(t) \equiv 1$ and $J_3 < < \infty$, respectively. As we will see, such conditions automatically reduce possible types of solutions of (1.2).

Next results describe solutions in the classes $\mathbb{M}^-$ and $\mathbb{M}^+$ in terms of the integrals J_1, J_2 without any additional conditions on the nonlinearity f. Our approach is based on the Tychonov fixed point theorem as well as on the asymptotic integration of (1.1).

Theorem 1.1 ([D10]). *Eq.* (1.1) *has a solution in the class* $\mathbb{M}_B^-$ *if and only if* $J_2 < \infty$.

Theorem 1.2 ([D10]). *If* $J_1 < \infty$ *and* $J_2 < \infty$*, then equation* (1.1) *has a solution* x *of* (1.1) *such that the limit*

$$\lim_{t\to\infty} \frac{x(t)}{\int_t^\infty \Phi_{p^*}\left(\frac{1}{a(s)}\right)ds} \tag{1.9}$$

exists, is finite and is different from zero.

Theorem 1.3 ([D10]). *Eq.* (1.1) *has a solution in the class* $\mathbb{M}_B^+$ *if and only if* $J_1 < \infty$.

The next two theorems have been proved using a complete asymptotic characterization for the half-linear equations and on the existence of suitable upper and lower solutions for (1.1), see [D10].

Theorem 1.4 ([D10]). *Assume*

$$\limsup_{u\to 0} \frac{f(u)}{\Phi_p(u)} < \infty. \tag{1.10}$$

Then the following holds:

(a) *Eq.* (1.1) *has no solution in the class* S_1.

(b) *Eq.* (1.1) *has solutions in the class* $\mathbb{M}^-$.

(c) *If* $J_1 = \infty$ *and* $J_2 < \infty$, *then every solution* x *of* (1.1) *in the class* $\mathbb{M}^-$ *tends to a non-zero limit as* $t \to \infty$, *i.e.,* $\mathbb{M}^- = \mathbb{M}^-_B$.

(d) *Eq.* (1.1) *has solutions in the classes* $\mathbb{M}^-_0$ *and* $\mathbb{M}^-_B$ *if and only if* $J_1 < \infty$ *and* $J_2 < \infty$.

Theorem 1.5 ([D10]). *Assume*

$$\limsup_{|u|\to\infty} \frac{f(u)}{\Phi_p(u)} < \infty. \tag{1.11}$$

Then the following holds:

(a) *Eq.* (1.1) *has no solution in the class* S_2.

(b) *Eq.* (1.1) *has solutions in the class* $\mathbb{M}^+$.

(c) *If* $J_1 < \infty$, *then every solution* x *of* (1.1) *in the class* $\mathbb{M}^+$ *is bounded, i.e.* $\mathbb{M}^+ = \mathbb{M}^+_B \neq \emptyset$.

(d) *If* $J_1 = \infty$, *then every solution* x *of* (1.1) *in the class* $\mathbb{M}^+$ *is unbounded, i.e.* $\mathbb{M}^+ = \mathbb{M}^+_\infty \neq \emptyset$.

Corollary 1.1 ([D10]). *The following holds for equation* (1.1)*:*

(i) *If* $J_1 = \infty$ *and* $J_2 = \infty$, *then* $\mathbb{M}^-_B = \emptyset$, $\mathbb{M}^+_B = \emptyset$. *In addition, if* (1.10) *is satisfied, then* $\mathbb{M}^- = \mathbb{M}^-_0 \neq \emptyset$, *and if* (1.11) *is satisfied, then* $\mathbb{M}^+ = \mathbb{M}^+_\infty \neq \emptyset$.

(ii) *If* $J_1 = \infty$ *and* $J_2 < \infty$, *then* $\mathbb{M}^+_B = \emptyset$. *In addition, if* (1.10) *is satisfied, then* $\mathbb{M}^- = \mathbb{M}^-_B \neq \emptyset$, $\mathbb{M}^-_0 = \emptyset$, *and if* (1.11) *is satisfied, then* $\mathbb{M}^+ = \mathbb{M}^+_\infty \neq \emptyset$.

(iii) *If* $J_1 < \infty$ *and* $J_2 = \infty$, *then* $\mathbb{M}^-_B = \emptyset$. *If in addition* (1.10) *is satisfied, then* $\mathbb{M}^- = \mathbb{M}^-_0 \neq \emptyset$, *and if* (1.11) *is satisfied, then* $\mathbb{M}^+ = \mathbb{M}^+_B \neq \emptyset$ *and* $\mathbb{M}^+_\infty = \emptyset$.

(iv) *If* $J_1 < \infty$ *and* $J_2 < \infty$, *then both classes* $\mathbb{M}^-_0$ *and* $\mathbb{M}^-_B$ *are nonempty. In addition, if* (1.11) *is satisfied, then* $\mathbb{M}^+ = \mathbb{M}^+_B \neq \emptyset$ *and* $\mathbb{M}^+_\infty = \emptyset$.

Remark 1.2. For the quasilinear equation (1.2), i.e. for $f(u) = \Phi_q(u)$, (1.10) reduces to the condition $p \le q$ and (1.11) reduces to the condition $p \ge q$.

For the linear equation (1.6), Theorem 1.5-(c),(d) reduces to Theorem 3 in [60], and Corollary 1.1 to Theorem 2 in [12]. For equation (1.5), Theorem 1.4-(b), (d) gives Theorems 6,8 in [12].

Our main results for equation (1.2) concern the following problems:

(1) Reciprocity principle and asymptotic behavior of quasiderivatives of solutions.

(2) Uniqueness problem in the class $\mathbb{M}^-$.

The asymptotic behavior of quasiderivatives of solutions of (1.2) can be investigated by the so called *reciprocity principle*. This principle extends to (1.3) a classical result of Potter [71] on the oscillation of the second order self-adjoint linear equation (1.6), and was later used in [12, 13] for the asymptotics of nonoscillatory linear equation (1.6). It links solutions of (1.2) to those of a suitable associated equation and enables, by a simple way, to describe the qualitative behavior of solutions of (1.2) and their quasiderivatives .

Let x be a solution of (1.2) and let us denote the quasiderivative $x^{[1]}$ of x by

$$x^{[1]}(t) = a(t)\Phi_p(x'(t)).$$

Then $z = x^{[1]}$ is a solution of the *reciprocal equation* of (1.2)

$$\Big(\frac{1}{\Phi_{q^*}(b(t))}\Phi_{q^*}(z')\Big)' = \frac{1}{\Phi_{p^*}(a(t))}\Phi_{p^*}(z). \tag{1.12}$$

This equation follows from (1.2) when a is replaced by $1/\Phi_{q^*}(b)$ and b by $1/\Phi_{p^*}(a)$. Consequently, the integral J_3 for (1.2) plays the same role as J_4 for (1.12) and vice versa; analogously J_4 for (1.2) plays the same role as J_3 for (1.12). Similarly, for (1.12) the integrals J_1, J_2 become

$$J_5 = \lim_{T\to\infty}\int_0^T b(t)\,\Phi_q\Big(\int_0^t \Phi_{p^*}\Big(\frac{1}{a(s)}\Big)\,ds\Big)dt,$$
$$J_6 = \lim_{T\to\infty}\int_0^T b(t)\,\Phi_q\Big(\int_t^T \Phi_{p^*}\big(\frac{1}{a(s)}\big)\,ds\Big)dt,$$

respectively.

Following the classification used in [64, 76], we distinguish these types of solutions x of (1.2):

Type (1)	$\lim_{t\to\infty} x(t) = 0,$	$\lim_{t\to\infty} x^{[1]}(t) = 0;$
Type (2)	$\lim_{t\to\infty} x(t) = 0,$	$\lim_{t\to\infty} x^{[1]}(t) = c_1 < 0;$
Type (3)	$\lim_{t\to\infty} x(t) = c > 0,$	$\lim_{t\to\infty} x^{[1]}(t) = c_1 \le 0;$
Type (4)	$\lim_{t\to\infty} x(t) = c > 0,$	$\lim_{t\to\infty} x^{[1]}(t) = c_1 > 0;$
Type (5)	$\lim_{t\to\infty} x(t) = c > 0,$	$\lim_{t\to\infty} x^{[1]}(t) = \infty;$
Type (6)	$\lim_{t\to\infty} x(t) = \infty,$	$\lim_{t\to\infty} x^{[1]}(t) = c_1;$
Type (7)	$\lim_{t\to\infty} x(t) = \infty,$	$\lim_{t\to\infty} x^{[1]}(t) = \infty.$

Theorem 1.6 ([D10]). *The following holds for equation* (1.2)*:*

(a) *Let $p \le q$. Every solution in $\mathbb{M}^-$ is of Type (1) if and only if $J_2 = \infty$ and $J_6 = \infty$.*
(b) *Eq.* (1.2) *has solutions of Type (2) if and only if $J_6 < \infty$.*
(c) *Eq.* (1.2) *has solutions of Type (3) if and only if $J_2 < \infty$.*
(d) *Eq.* (1.2) *has solutions of Type (4) if and only if $J_1 < \infty$ and $J_5 < \infty$.*

(e) *Eq.* (1.2) *has solutions of Type (5) if and only if* $J_1 < \infty$ *and* $J_5 = \infty$.

(f) *Eq.* (1.2) *has solutions of Type (6) if and only if* $J_1 = \infty$ *and* $J_5 < \infty$.

(g) *Let* $p \geq q$. *Every solution in* $\mathbb{M}^+$ *is of Type (7) if and only if* $J_1 = \infty$ *and* $J_5 = \infty$.

Remark 1.3. In [76], the same problem has been studied under the assumption $J_3 < \infty$. In [64], the case $a(t) \equiv 1$ is considered which represents the case $J_3 = \infty$.

Theorem 1.6 gives

- in case $J_3 < \infty$ Theorems 3.1, 3.2, 3.3, 3.4, 4.2, 4.5 and 4.6 of [76];
- in case $a(t) \equiv 1$ Theorems 2.2, 2.3, 2.4, 2.5, 3.6, 3.7 and 3.8 of [64].

Next result concerns the uniqueness in $\mathbb{M}^-$ and has been proved in [D9].

Theorem 1.7 (Uniqueness in M$^-$). *Consider equation* (1.2) *with* $p \leq q$.

For any $(t_0, x_0) \in [0,\infty) \times \mathbb{R} \setminus \{0\}$, *there exists a unique solution* x *of* (1.1) *in the class* $\mathbb{M}^-$ *such that* $x(t_0) = x_0$ *if and only if*

$$\int_0^{\infty} \Big(\frac{1}{\Phi_{p^*}(a(t))} + b(t) \Big)\, dt = \infty \tag{1.13}$$

is satisfied.

This result has been later used in [D12] to prove the limit characterization of principal solutions of half-linear equations.

A particular attention among quasilinear equations is paid to the half-linear equation (1.3). The half-linear case is characterized by the fact that every solution of (1.3) is proper and defined on the whole interval $[0,\infty)$. It is straightforward to verify that solutions x in the class $\mathbb{M}^-$ are positive decreasing or negative increasing in the whole interval $[0,\infty)$. In addition, the *homogeneity property* holds, that is, if x is a solution of (1.3), then so is λx for any constant λ.

In the nonoscillation theory of the half-linear equations, an important role is played by the *principal solution*. This concept has been introduced independently by Elbert–Kusano [27] and by Mirzov [62]. In papers [D12], [11] we have proved the limit and integral characterizations of the principal solutions of (1.3) in the case when b does not change its sign.

It is well known that if the linear equation (1.6) is nonoscillatory, then there exists a solution u of (1.6), called the *principal solution at* ∞, that is uniquely determined up to a constant factor by one of the following conditions (in which x denotes an arbitrary solution of (1.6) linearly independent of u):

$$\lim_{t\to\infty} \frac{u(t)}{x(t)} = 0; \tag{π_1}$$

$$\frac{u'(t)}{u(t)} < \frac{x'(t)}{x(t)} \quad \text{for large } t\,; \tag{π_2}$$

$$\int^{\infty} \frac{dt}{a(t)u^2(t)} = \infty. \tag{π_3}$$

Roughly speaking, property (π_1) means that the principal solution of (1.6) is the smallest solution in a neighbourhood of infinity; property (π_2) is connected with the associated Riccati equation, and property (π_3) is related to the Wronskian identity and to the so called *reduction of order formula*. The notion of the principal solution was introduced by W. Leighton and M. Morse [57] in studying positiveness of certain quadratic functionals associated with (1.6). It was later characterized by means of the properties (π_1)-(π_3) by P. Hartman and A. Wintner, see, e.g., Chapter 11 in [39].

Recently, a considerable effort has been devoted to extend the notion of the principal solution to half-linear equation (1.3) (see, e.g., [25, 27, 62, 63]). In [27, 62], the principal solution of (1.3) has been defined by the Riccati approach. It is shown that, when (1.3) is nonoscillatory, among all eventually nonvanishing solutions

$$w_x(t) = a(t)\Phi_p\Big(\frac{x'(t)}{x(t)}\Big)$$

of the Riccati equation associated with (1.3), there exists one, say w_u, that is continuable to infinity and minimal in the sense that any other solution w_x of the Riccati equation that is continuable to infinity satisfies $w_u(t) < w_x(t)$ for large t. The corresponding solution u of (1.3) is then called principal. This definition can be formulated in the following way.

Definition 1.1. [27, 62] A nontrivial solution u of (1.3) is said to be the *principal solution* of (1.3) if for every solution x of (1.3) such that $x \neq \lambda u$, $\lambda \in \mathbb{R}$,

$$\frac{u'(t)}{u(t)} < \frac{x'(t)}{x(t)} \quad \text{for large } t. \tag{1.14}$$

The set of principal solutions is nonempty and, obviously, principal solutions are determined up to a constant factor. Since the sets $\mathbb{M}^-, \mathbb{M}^+$ of (1.3) are nonempty, any principal solution necessarily belongs to $\mathbb{M}^-$. As we will show later, in general, the converse does not hold, i.e. there exist nonprincipal solutions in the class $\mathbb{M}^-$.

As pointed out in [25], the simplest and most characteristic property (π_1) of the principal solutions to be the “smallest solutions in a neighbourhood of infinity” has been until now a open problem for (1.3) . Our following result gives the answer to the question posed in [25], by showing the equivalence between properties (π_1) and (π_2) in the half-linear case.

Theorem 1.8 ([D12]). *A solution u of* (1.3) *is principal if and only if*

$$\lim_{t\to\infty} \frac{u(t)}{x(t)} = 0 \tag{1.15}$$

for any solution x of (1.3) *such that $x \neq \lambda u$, $\lambda \in \mathbb{R}$.*

Principal solutions of (1.3) can be completely characterized by the following integral criterion which yields in the linear case the property (π_3).

Theorem 1.9 ([D12]). *A solution u of* (1.3) *is principal if and only if*

$$\int^{\infty} \frac{dt}{\Phi_{p^*}(a(t))u^2(t)} = \infty. \tag{1.16}$$

In [25], a different integral characterization of principal solutions for (1.3) has been introduced, as the following theorem shows.

Theorem B. *The following holds for equation* (1.3)*:*

(a) *If $p \geq 2$ and u is a principal solution of* (1.3)*, then*

$$\int^{\infty} \frac{u'(t)\,dt}{a(t)u^2(t)\Phi_p(u'(t))} = \infty. \tag{1.17}$$

(b) *If $p \in (1, 2]$ and there exists a solution u of* (1.3) *such that* (1.17) *holds, then u is a principal solution of* (1.3)*.*

Remark 1.4. The above result has been stated in [25] for a more general case by assuming, instead of $b(t) > 0$ on $[0, \infty)$, the condition "(1.3) is nonoscillatory and u is its solution such that $u'(t) \neq 0$ eventually". The question whether the divergence of (1.17) equivalently characterizes the principal solutions when $b(t) > 0$, is also posed in [25]. In [D12], we have given examples illustrating that this conjecture is not true.

When $b(t) < 0$, the integral characterization of the principal solution is more complicated. This is due to the fact that no of properties (1.16), (1.17) characterizes the principal solution in general, see [11].

Some problems solved for equation (1.1) have been investigated in [D11] for nonlinear differential equations with deviating argument

$$\big(a(t)\Phi_p(x')\big)' = b(t) f\big(x(g(t))\big) \tag{1.18}$$

where (1.4) is assumed, f is a continuous function on $\mathbb{R}$ satisfying $f(u)u > 0$ for $u \neq 0$, and $g : [0, \infty) \to \mathbb{R}$ is a continuous function satisfying $\lim_{t\to\infty} g(t) = \infty$.

Equation (1.18) exhibits phenomena which are different in comparison with equation (1.1) without a deviating argument. For instance, every solution of (1.1) is nonoscillatory, but the deviating argument can produce the oscillation of some or of all solutions of certain equations of type (1.18) (see, e.g., [54], [33]).

The aim of [D11] is to establish sufficient conditions for the existence of monotone solutions of (1.18) approaching zero as $t \to \infty$. Such solutions are often called *decaying nonoscillatory solutions* and have been deeply studied in the literature. We refer to [70] and [54], where the particular case of (1.18) with $a(t) \equiv 1$ has been considered. Other interesting contributions can be found in the recent book [33] and in the references contained therein.

Some of our results hold without any additional conditions on g: we have proved that Theorems 1.1, 1.2 and 1.3 remain to hold for (1.18), see Theorems 3, 5 and 6 in [D11]. However, the delayed argument sometimes generates a different situation than it occurs for the corresponding equation without delay, as the following results illustrate.

Theorem 1.10 ([D11]). *Assume that g is eventually nondecreasing and $g(t) < t$ eventually. If conditions*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} \Phi_{p^*}\Big(\frac{1}{a(s)}\Big)\Phi_{p^*}\Big(\int_s^t b(\tau)\,d\tau\Big)\,ds > 1 \tag{1.19}$$

and

$$\exists\, \varepsilon > 0 \text{ such that } \frac{\Phi_p(u)}{f(u)} \le 1 \text{ for } |u| < \varepsilon \tag{1.20}$$

are satisfied, then (1.18) has no solution in the class $\mathbb{M}^-$.

Corollary 1.2 ([D11]). *Assume that g is eventually nondecreasing and $g(t) < t$ eventually. If the conditions $J_1 = \infty$, (1.19) and (1.20) are verified, then all bounded solutions of (1.18) (if any) are oscillatory.*

Theorem 1.11 ([D11]). *Eq. (1.18) has decaying nonoscillatory solutions, i.e. $\mathbb{M}_0^- \neq \emptyset$, if any of the following condition is satisfied:*

(a) $J_2 < \infty$, $g(t) < t$ *eventually;*

(b) f *is nondecreasing,* $J_3 < \infty$ *and*

$$\lim_{T\to\infty} \int_0^T b(t)\, f\left(\int_{g(t)}^{T} \Phi_{p^*}\big(\frac{1}{a(s)}\big)\,ds\right) dt < \infty.$$

Notes. In [D12], the notion of the principal solution and its characterizations have been extended to the nonlinear differential equation (1.1) by defining the concept of the *minimal set*. The limit and integral characterizations for the half-linear equation (1.3) with $b(t) < 0$ have been proved in [11].

The relationships between integrals J_1 and J_6 (and similarly between J_2 and J_5) has been proved in [24] where applications to equations (1.3) and (1.2) with $b(t) < 0$ have also been made.

1.2. Third order differential equations

Results published in [D1]–[D7] are concerned the asymptotic and oscillatory behavior of solutions of the third order linear and nonlinear differential equations.

Many authors have studied the oscillatory, nonoscillatory and asymptotic behavior of the third order linear differential equation in the normal form

$$x'''(t) + p(t)x'(t) + q(t)x(t) = 0 \tag{1.21}$$

where p, q are real continuous functions for $t \ge 0$. Among the numerous results dealing with this subject, we refer to the books by Greguš [36], Kiguradze-Chanturia [47], Swanson [75] and the references contained therein.

We say that x is an *oscillatory solution* of (1.21) if it has arbitrary large zeros. Otherwise this solution is said to be *nonoscillatory*. Equation (1.21) is said to be *oscillatory* if it has at least one nontrivial oscillatory solution, and *nonoscillatory* if all its solutions are nonoscillatory.

Some authors (see, e. g., [34, 38]) consider third order differential equations of the form

$$(r(t)x'(t))'' + q(t)x(t) = 0,$$
$$(r(t)x''(t))' + q(t)x(t) = 0.$$

A prototype of these equations is the binomial equation

$$x''' \pm q(t)x = 0 \tag{E$\pm$}$$

where q is a positive continuous function for $t \geq 0$. It is well known that there is an analogy between the space of solutions of $(E+)$ and $(E-)$. For instance, by using the notion of an equation of the class I and II introduced by Hanan in [38], it is easy to show that $(E+)$ is nonoscillatory if and only if $(E-)$ is nonoscillatory. Another result in this direction is given in [78] (see also [74]) where it is proved that if there exists $\lambda > 0$ such that

$$\int^{\infty} t^{2-\lambda} q(t)dt = \infty,$$

then $(E\pm)$ have both oscillatory and nonoscillatory solutions. In addition, every nonoscillatory solution x of $(E+)$ tends to zero as $t \to \infty$ and satisfies, for all large t, either the inequalities $x(t) > 0$, $x'(t) < 0$, $x''(t) > 0$ or the inequalities $x(t) < 0$, $x'(t) > 0$, $x''(t) < 0$, while every nonoscillatory solution of $(E-)$ tends to infinity as $t \to \infty$ and satisfies, for large t, either $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$ or $x(t) < 0$, $x'(t) < 0$, $x''(t) < 0$. Some authors refer to this property of $(E+)$ as the *property A* and of $(E-)$ as the *property B*. Both properties have been extended in several directions to linear and nonlinear equations of $n-$th order, which will be treated later.

Another basic classification concerning third order linear differential equaitons has been employed, in an implicit form, by Sansone [73] and later on has been formalized by Hanan [38].

Definition 1.2. Equation (1.21) is said to be of *Class I* if every solution x for which $x(a) = x'(a) = 0$, $x''(a) > 0$ $(a > 0)$ satisfies $x(t) > 0$ on $(0, a)$.

Equation (1.21) is said to be of *Class II* if every solution x for which $x(a) = x'(a) = 0$, $x''(a) > 0$ satisfies $x(t) > 0$ on (a, ∞).

The terminology (1.21) being of *Class I* or *Class II* plays an important role in the study of the conjugate points (zeros) for the solution of the linear equation (1.21). In the literature, there are numerous papers dealing with this classification: we refer to the books [36, 75] for an extensive bibliography.

In [D1, D7], we consider the differential equation

$$y''' + 2q(t)y' + (q'(t) + r(t))y = 0 \tag{1.22}$$

where

$$q \in C^1([0,\infty),\mathbb{R}), \quad r \in C([0,\infty),\mathbb{R})$$

(functions q, r may change its sign).

If $r(t) = 0$ we have the self-adjoint equation (known as *Appel's equation*)

$$x'''(t) + 2q(t)x'(t) + q'(t)x(t) = 0 \tag{1.23}$$

and all its solutions are given by

$$x = c_1 z_1^2 + c_2 z_1 z_2 + c_3 z_2^2, \tag{1.24}$$

where z_1, z_2 are linearly independent solutions of the second order linear equation

$$z''(t) + \frac{1}{2}q(t)z(t) = 0. \tag{1.25}$$

In [41] Jones described the types of possible bases for the solution space of (1.22) with respect to the possible number of oscillatory solutions in a given basis.

In view of (1.24), the self-adjoint equation (1.23) has the following properties:

- If (1.25) has an oscillatory solution, then (1.23) has bases consisting of 0, 1, 2 or 3 oscillatory solutions, (see e.g. [36, Theorem 2.52]);
- If all solutions of (1.25) are bounded, then equation (1.25) has all solutions in L^2 if and only if equation (1.23) has all solutions in L^2.

We seek for the possibility of perturbing (1.23) to (1.22) in such a way that these properties are preserved. Fro this purpose, in [D1] we have studied the problem when equations (1.22) and (1.23) are asymptotically equivalent.

Let X and Y be the space of all solutions of (1.22) and (1.23) on $[0,\infty)$, respectively. The continuity of coefficients of equations (1.22), (1.23) ensures $X \neq \emptyset$, $Y \neq \emptyset$ and thus, X, Y are linear spaces of the dimension three.

Theorem 1.12 ([D1]). *Assume that*

$$\int_0^\infty |r(t)|\,dt < \infty \tag{1.26}$$

holds and that z_1, z_2 are bounded solutions of (1.25) *with $z_1(t)z_2'(t) - z_1'(t)z_2(t) \equiv 1$. Then the mapping $V : Y \to X$ defined by*

$$(Vy)(t) = y(t) - \int_t^\infty K(t,s)r(s)y(s)\,ds, \tag{1.27}$$

where

$$K(t,s) = \begin{vmatrix} z_1(t) & z_2(t) \\ z_1(s) & z_2(s) \end{vmatrix} \tag{1.28}$$

is a one-to-one mapping and

$$\lim_{t\to\infty} |x(t) - y(t)| = 0 \tag{1.29}$$

for every $x(t) \in X$ and $y(t) \in Y$ such that $x(t) = Vy(t)$.

Remark 1.5. Equations (1.22) and (1.23) satisfying (1.29) are said to be *asymptotically equivalent*. Using Theorem 1.12 and applying the asymptotic properties of (1.25), we can obtain asymptotic properties of solutions of (1.22). For example, if all solutions of (1.25) converge to zero as $t \to \infty$ and (1.26) holds, then all solutions of (1.22) converge to zero as $t \to \infty$.

Under a stronger assumption than (1.26) the following result holds.

Theorem 1.13. *Suppose that every solution of* (1.25) *is bounded on* $[0, \infty)$ *and*

$$\int_0^\infty \Big(\int_t^\infty |r(s)|\, ds \Big)^2 dt < \infty. \tag{1.30}$$

Then

$$\int_0^\infty (x(t) - y(t))^2\, dt < \infty \tag{1.31}$$

holds for every $x \in X$ *and* $y \in Y$ *such that* $x = Vy$, *where* $V : Y \to X$ *is determined by* (1.27). *In particular, every solution* x *of* (1.23) *is in* L^2 *if and only if every solution* y *of* (1.22) *is in* L^2.

Now we show some applications of Theorems 1.12 and 1.13. Our first result concerns the types of bases which are possible for the solution space of (1.22) with respect to the number of oscilatory solutions in a given basis.

Lemma 1.1 ([D1]). *Assume* (1.26). *Let* $q(t) > 0$ *be such that* q, q^{-1} *are bounded and there exists* $\gamma \neq 0$ *such that* q^γ *is either convex or concave. Then* (1.22) *has a nonoscillatory solution* $y(t)$ *such that* $\liminf_{t\to\infty} y(t) > 0$. *Furthermore, every solution of* (1.22) *is bounded.*

Theorem 1.14 ([D1]). *Let equation* (1.22) *be of Class I or Class II, oscillatory and let the assumptions of Lemma 1.1 be fulfilled. Then the solution space of* (1.22) *has bases consisting of exactly* 0, 1, 2 *or* 3 *oscillatory solutions.*

Our second application concerns the problem of the existence of the square integrable solutions of (1.22). This problem has been initiated by H. Weyl (1910) for the second order equation (1.25), and later it has been deeply developed for the self-adjoint $2n$th order linear differential equations in the connection with the deficiency index of the associated differential operators.

As we mentioned above, the self-adjoint equation (1.23) has the following property: *Let all solutions of* (1.25) *be bounded. Equation* (1.25) *has all solutions in* L^2 *if and only if the Appel's equation* (1.23) *has all solutions in* L^2.

The next theorem extends this property for the perturbed equation (1.22).

Theorem 1.15 ([D7]). *Let all solutions of* (1.25) *be bounded.*

(a) *If* (1.26) *holds and all solutions of* (1.25) *belong to* L^2, *then all solutions of* (1.22) *belong to* L^2.

(b) *Let* (1.30) *hold. Equation* (1.25) *has all solutions in* L^2 *if and only if equation* (1.22) *has all solutions in* L^2.

Remark 1.6. If all solutions of (1.25) belong to L^2, then equation (1.25) is oscillatory (see e.g. [8, Theorem 5.1]). Hence, equation (1.23) has an oscillatory solution. Conditions ensuring that all solutions of (1.25) are bounded and belong to L^2 can be found in e.g.[8].

In [D2]—[D5], we study the oscillatory and asymptotic properties of solutions of the third order linear equation in the form

$$\Big(\frac{1}{p(t)}\Big(\frac{1}{r(t)}\,x'(t)\Big)'\Big)' + q(t)x(t) = 0 \tag{L}$$

and of its adjoint equation

$$\Big(\frac{1}{r(t)}\Big(\frac{1}{p(t)}\,x'(t)\Big)'\Big)' - q(t)x(t) = 0 \tag{$L^{\mathcal{A}}$}$$

where

$$r, p, q \in C^0([0,\infty),\mathbb{R}), \quad r(t) > 0,\ p(t) > 0,\ q(t) > 0 \ \text{ on } \ [0,\infty). \tag{1.32}$$

When the functions p and/or r do not have a continuous first and/or second derivative, then (L) may be interpreted as a first order differential system for the vector $(x^{[0]}, x^{[1]}, x^{[2]})$ given by

$$x^{[0]} = x, \quad x^{[1]} = \frac{1}{r}x', \quad x^{[2]} = \frac{1}{p}\Big(\frac{1}{r}x'\Big)' = \frac{1}{p}\big(x^{[1]}\big)',$$

where x is a solution of (L). The functions $x^{[i]}$ are called the *quasiderivatives* of x.

Following Kiguradze-Chanturia [47], we introduce the following definitions.

Definition 1.3. A solution x of (L) is said to be a *Kneser solution* if for $i = 0, 1$

$$x^{[i]}(t)x^{[i+1]}(t) < 0 \quad \text{for } t \ge 0. \tag{1.33}$$

A solution u of ($L^{\mathcal{A}}$) is said to be a *strongly increasing solution* if for $i = 0, 1$

$$u^{[i]}(t)u^{[i+1]}(t) > 0 \quad \text{for large } t. \tag{1.34}$$

Eq. (L) is said to have the *property A* if every solution x of this equation is either oscillatory or satisfies for $i = 0, 1, 2$

$$|x^{[i]}(t)| \downarrow 0 \ \text{ as } t \to \infty. \tag{1.35}$$

Eq. ($L^{\mathcal{A}}$) is said to have the *property B* if every solution u of this equation is either oscillatory or satisfies for $i = 0, 1, 2$

$$|u^{[i]}(t)| \uparrow \infty \ \text{ as } t \to \infty, \tag{1.36}$$

where the notation $y(t) \downarrow 0$ and $y(t) \uparrow \infty$ mean that y is monotone decreasing to zero as $t \to \infty$ or monotone increasesing to infinity as $t \to \infty$, respectively.

It is well-known (see e.g. the book by Elias [31]) that (L) is oscillatory if and only if $(L^{\mathcal{A}})$ is oscillatory. It follows from the results of Hartman/Wintner (see [39], p.506) and of Kusano et al.[50, Lemma 2] that equation (L) has Kneser solutions and ($L^{\mathcal{A}}$) has strongly monotone solutions.

If (L) has property A, then every nonoscillatory solution is a Kneser solution and if $(L^{\mathcal{A}})$ has property B, then every nonoscillatory solution is a strongly monotone solution. The following questions arise for equation (L):

(i) Do the converse statements hold?

(ii) Does there exist a relationship between the property A and property B?

(iii) Does there exist a relationship between the property A [property B] and oscillation?

In the sequel we give an affirmative answer to these questions, and we state the oscillation and nonoscillation criteria for (L).

We start with the basic property of linear equations with quasiderivatives, which is often used in our later investigation.

Theorem 1.16 ([D2]). *Equation (L) is of Class I and $(L^{\mathcal{A}})$ is of Class II.*

Using this property we can describe the structure of the solution space of equations (L) and $(L^{\mathcal{A}})$. From a result in [21], this space always contains a two-dimensional subspace either of oscillatory solutions or of nonoscillatory ones. The following holds:

Theorem 1.17 ([D2]). *If equation (L) has oscillatory solution, then the solution space of (L) has a basis consisting of exactly two or three oscillatory solutions.*

If equation $(L^{\mathcal{A}})$ has oscillatory solution, then the solution space of $(L^{\mathcal{A}})$ has a basis consisting of exactly two oscillatory solutions.

Using Theorem 1.16 we also get the following oscillation and nonoscillation criteria. For this purpose, the following notation is used:

$$I(u_i) = \int_0^\infty u_i(t)\,dt, \quad I(u_i, u_j) = \int_0^\infty u_i(t) \int_0^t u_j(s)\,ds\,dt, \ \ i, j = 1, 2$$
$$I(u_i, u_j, u_k) = \int_0^\infty u_i(t) \int_0^t u_j(s) \int_0^s u_k(\tau)\,d\tau\,ds\,dt, \ \ i, j, k = 1, 2, 3,$$

where u_i, $i = 1, 2, 3$, are continuous positive functions on $[0, \infty)$.

Theorem 1.18 ([D2]). *Let one of the following integrals*

$$I(q, r, p), \quad I(p, q, r) \quad I(r, p, q)$$

be convergent. Then (L) is nonoscillatory.

Theorem 1.19 ([D2]). *Let one of the following conditions be satisfied:*

(i) $I(p) = I(r) = I(q, r) = \infty$;

(ii) $I(q) = I(p) = I(r, p) = \infty$;

(iii) $I(r) = I(q) = I(p, q) = \infty$.

Then (L) is oscillatory,

Our main result is the following theorem which has been proved in [D4].

Theorem 1.20 (Equivalence Theorem). *Eq.* (L) *has property A if and only if* $(L^{\mathcal{A}})$ *has property B.*

The above results yield the following result.

Corollary 1.3 ([D5]). *The following assertions are equivalent:*

(a) *Eq.* (L) *has property A.*

(b) *Eq.* $(L^{\mathcal{A}})$ *has property B.*

(c) *Eq.* (L) *is oscillatory and we have*

$$I(q, p, r) = I(r, q, p) = I(p, r, q) = \infty. \tag{1.37}$$

(d) *Eq.* $(L^{\mathcal{A}})$ *is oscillatory and* (1.37) *holds.*

In [D6], we consider the third order linear differential equation

$$x''' + q(t)x' \pm r(t)x = 0 \tag{e$\pm$}$$

and the corresponding nonlinear one

$$x''' + q(t)x' \pm r(t)f(x) = 0 \tag{n$\pm$}$$

where

$$q, r \ \text{ are continuous functions for } t \geq 0,\ q(t) \leq 0, r(t) > 0 \tag{1.38}$$

and

$$f \ \text{ is a continuous function in } \mathbb{R} \text{ such that } f(u)u > 0 \text{ for } u \neq 0. \tag{1.39}$$

By a *solution* of $(n\pm)$ we mean a three times differentiable function x satisfying $(n\pm)$ for large t and $\sup\{|x(t)| : t > T\} > 0$ for every T sufficiently large. For the results concerning continuability to infinity of solutions of $(n\pm)$, we refer the reader to [37, 47]. A nontrivial solution of (e$\pm$) [$(n\pm)$] is said to be *oscillatory* or *nonoscillatory* according to whether it does or does not have arbitrarily large zeros. Equation (e$\pm$) [$n\pm$] is called *oscillatory* if it has at least one oscillatory solution and *nonoscillatory* otherwise, i.e. if all its solutions are nonoscillatory.

Equation (e$+$) is said to have *property A* if every solution of (e$+$) either is oscillatory or satisfies the conditions

$$x(t)x'(t) < 0, \quad x(t)x''(t) > 0 \ \text{ for } t \geq 0 \tag{1.40}$$

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0.$$

Equation (e−) is said to have *property B* if every solution of (e−) either is oscillatory or satisfies the conditions

$$x(t)x'(t) > 0, \quad x(t)x''(t) > 0 \text{ for all large } t \tag{1.41}$$

$$\lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |x'(t)| = \lim_{t\to\infty} |x''(t)| = \infty.$$

We recall that a solution x of (e+) which satisfies condition (1.40) is a *Kneser solution* of (e+), and similarly a solution x of (e−) which satisfies (1.41) is a *strongly monotone solution* of (e−). We also recall that (e+) always has Kneser solutions and (e−) always has strongly monotone solutions (see, e.g, [39] and [56]). Similar definitions hold for the nonlinear equations $(n\pm)$. In this case condition (1.40) holds for large t.

For the linear equations, the relationship between the oscillatory solutions and asymptotic behavior of nonoscillatory ones is often considered. In particular, in the quoted paper [56], Lazer proved the following:

Lazer Theorem. *Assume* $q(t) \le 0$ *for* $t \ge 0$.

(a) *Equation* (e+) *is oscillatory if and only if every nonoscillatory solution* x *of* (e+) *is a Kneser solution and* $\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0$.

(b) *If equation* (e−) *is oscillatory, then every nonoscillatory solution* x *of* (e−) *is a strongly monotone solution and* $\lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |x'(t)| = \infty$.

Motivated by equations with constant coefficients, Lazer posed the following question (see [56], p.444): *Does every nonoscillatory solution of* (e+) *tend to zero as* t *tends to infinity when* (e+) *is oscillatory?*

In the special case when $q \equiv 0$, this conjecture was proved by Gaet. Villari in [78] and later by Lazer ([56, Th.1.5]) and Greguš ([36, Th.3.12]) under some additional assumptions on q, r. In particular, Greguš proved the equivalence between the oscillation and property A for (e+) provided $q \in C^1([0,\infty))$. Gera [35] has proved the converse of the Lazer Theorem (b), i.e., if every nonoscillatory solution of (e−) is a strongly monotone solution, then (e−) is oscillatory.

Our next result gives a complete positive answer to the Lazer conjecture, by showing that it is true without any additional condition.

Theorem 1.21 ([D6]). *Assume* (1.38). *Then,*

(a) *Equation* (e+) *is oscillatory if and only if* (e+) *has property A.*

(b) *Equation* (e−) *is oscillatory if and only if* (e−) *has property B.*

The obtained results in the linear case are interesting themselves by virtue of their necessary and sufficient character, but they are useful also in the study of the nonlinear equations.

Theorem 1.22 ([D6]). *Assume* (1.38), (1.39) *and* $\liminf_{|u|\to\infty} f(u)/u > 0$.

(a) *If the linear equation*

$$y''' + q(t)y' + \mu\, r(t)y = 0$$

is oscillatory for every $\mu > 0$*, then* $(n+)$ *has property A.*

(b) *If the linear equation*

$$y''' + q(t)y' - \mu\, r(t)y = 0$$

is oscillatory for every $\mu > 0$*, then* $(n-)$ *has property B.*

Notes. Applications of the Equivalence Theorem to the nonlinear third order differential equations with quasiderivatives has been given in [D5] and to the nonlinear differential equations with a delay in [9]. Further contribution about the asymptotic behavior of solutions of the nonlinear third order differential equations can be found in author's papers [3, 4, 5, 6].

1.3. Linear differential equations of *n*-th order

Consider the two-term linear differential equation

$$L_n x + p(t)x = 0 \tag{1.42}$$

where $p(t) \neq 0$ is continuous on $t \in [0, \infty)$ and L_n is the disconjugate differential operator

$$L_n x \equiv \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{a_{n-1}(t)} \cdots \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{a_1(t)}\frac{\mathrm{d}}{\mathrm{d}t}x,$$

with $a_i(t) > 0$ $(i = 1, \ldots, n)$, $a_i \in C^{n-i}([0, \infty))$. These equations, sometimes called *equations with quasiderivatives*, enjoy a very rich structure of solutions and they are the natural generalization of the binomial equations

$$x^{(n)} + p(t)x = 0.$$

By the result of Trench [77], the disconjugate differential operator L_n can be written in a *canonical form*, that is,

$$L_n x \equiv \frac{1}{b_n}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{b_{n-1}(t)} \cdots \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{b_1(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{b_0(t)}x$$

such that the functions $b_i(t) > 0$ $(i = 0, 1, 2, \ldots, n)$ are continuous, $\int^{\infty} b_i = \infty$, $(i = = 0, \ldots, n-1)$ and determined up to positive multiplicative constants with the product 1. The explicit formulas for the functions b_i depend on the convergence or divergence of the integrals $\int^{\infty} a_i$. In [D3], we have investigated the third order differential operator L_3, the canonical representation of L_3, and the relations between the corresponding linear equations.

Oscillatory and asymptotic properties of solutions of (1.42) are usually described by the property A and property B. We refer to the recent papers [16, 23, 28, 29, 30, 48, 50, 53, 68] and the books [31, 47], where an extensive bibliography on this topic can be found. Usually, it has been assumed that L_n is in the canonical form. This fact implies the special structure of the set of nonoscillatory solutions, see e.g. [47, 53, 50].

Our main contribution in this field is the generalization of the Equivalence Theorem 1.20 to higher order equations with quasiderivatives. It is worth noting that our result holds without assuming that the operator L_n is in the canonical form.

Consider the differential equation

$$\left(\frac{1}{a_{n-1}(t)}\left(\cdots\left(\frac{1}{a_1(t)}x'\right)'\cdots\right)'\right)' + \lambda\, a_0(t)x = 0 \tag{1.43}$$

where the coefficients a_i, $i = 0, 1, \dots, n-1$, are continuous positive functions on the interval $I = [0, \infty)$ and λ is a real parameter different from zero. Equation (1.43) can be interpreted as a first order differential system for the vector $(x^{[0]}, x^{[1]}, \dots, x^{[n-1]})$ given by

$$x^{[0]}(t) = x(t), \quad x^{[1]}(t) = \frac{1}{a_1(t)}x'(t), \dots, \; x^{[n-1]}(t) = \frac{1}{a_{n-1}(t)}\left(x^{[n-2]}(t)\right)'.$$

The functions $x^{[i]}$, $i = 0, 1, \dots, n-1$ are called the *quasiderivatives* of x. By a solution of (1.43) we mean a continuously differentiable function x such that its quasiderivatives $x^{[i]}$ exist and are continuous on I, and for $t \in I$ it satisfies (1.43). As usual, a nontrivial solution of (1.43) is said to be *oscillatory* or *nonoscillatory* according to whether it does or does not have arbitrarily large zeros.

Jointly with (1.43), consider the adjoint equation

$$\left(\frac{1}{a_1(t)}\left(\cdots\left(\frac{1}{a_{n-1}(t)}u'\right)'\cdots\right)'\right)' + (-1)^n\lambda\, a_0(t)u = 0. \tag{1.44}$$

The spaces of solutions of (1.43), (1.44) are mutually related. For instance, (1.43) has at least one oscillatory solution if and only if the same happens for (1.44) (see, e.g., [31, Th.8.33] or [28, Cor.2]). Other related contributions are in [42] and, when $n = 3$, in [36].

Following Kiguradze and Kondratiev, (see, e.g., [47]), we use the following definition.

Definition 1.4. Equation (1.43) is said to have *property A* if, for n even, all its solutions are oscillatory and, for n odd, every solution x is either oscillatory or satisfies (1.35) for $i = 0, 1, \dots, n-1$.

Equation (1.43) is said to have *property B* if, for n even, every solution x is either oscillatory or satisfies (1.35) or (1.36) for $i = 0, 1, \dots, n-1$, and, for n odd, every solution is either oscillatory or satisfies (1.36) for $i = 0, 1, \dots, n-1$.

Remark 1.7. Property A (property B) ensures the existence of all types of solutions occuring in its definition, i.e. the existence of Kneser (and/or strongly monotone) solutions as well as the existence of oscillatory solutions (see [31, Theorems 8.5 and 8.8]). Note that solution of (1.43) is said to be a Kneser solution [strongly monotone solution] if it satisfies (1.33) [(1.34)] for $i = 0, 1, \dots, n-2$.

The following relationship between the properties A and B holds for the binomial equations, see [47, Theorem 1.3]:

Theorem C. *Let n be odd and p be a continuous positive function on $[0, \infty)$. Equation $x^{(n)} + p(t)x = 0$ has property A if and only if its adjoint equation $x^{(n)} - p(t)x = 0$ has property B.*

Our main result, proved in [D8], extends this theorem and Theorem 1.20 to equations (1.43) and reads as follows.

Theorem 1.23 (Equivalence Theorem). *The following holds:*

(a) *Let $\lambda > 0$ and n be odd. Equation* (1.43) *has property A if and only if equation* (1.44) *has property B.*

(b) *Let $\lambda > 0$ and n be even. Equation* (1.43) *has property A if and only if equation* (1.44) *has property A.*

(c) *Let $\lambda < 0$ and n be odd. Equation* (1.43) *has property B if and only if equation* (1.44) *has property A.*

(d) *Let $\lambda < 0$ and n be even. Equation* (1.43) *has property B if and only if equation* (1.44) *has property B.*

There are many papers in the literature devoted to the property A or B independently. The Equivalence Theorem enables us to apply criteria on property A to obtain criteria on property B and vice versa.

Notes. The extension of property A and property B to the nonlinear differential equations associated with the disconjugate operators have been given in [10].

2 List of presented papers

[D1] Z. Došlá, *On oscillatory solutions of third-order linear differential equations*, Čas. Pěst. Mat. **114** (1989), 28-34.

[D2] M. Cecchi, Z. Došlá, M. Marini, Gab. Villari, *On the qualitative behavior of solutions of third order differential equations*, J. Math. Anal. Appl. **197**, 2 (1996), 749–766.

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[D4] M. Cecchi, Z. Došlá, M. Marini, *An equivalence theorem on properties A, B for third order differential equations*, Ann. Mat. Pura Appl. IV. Ser. **173** (1997), 373–389.

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[D7] Z. Došlá, *On square integrable solutions of third order linear differential equations*, Hascak, A. (ed.) et al., International scientific conference on mathematics. Proceedings of the conference, Herlany, Slovakia, Oct. 21-23, 1999. Košice: University of Technology Košice (2000), 68-72.

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4 Résumé

The aim of the dissertation is to present main results published in the papers [D1]–[D12]. These results contribute to the asymptotic theory of the following ordinary differential equations:

(1) Second order differential equations with p-Laplacian

Consider the nonlinear equation

$$(a(t)\Phi_p(x'))' = b(t)f(x) \tag{4.1}$$

and the functional differential equation

$$\big(a(t)\Phi_p(x')\big)' = b(t)f\big(x(g(t))\big),$$

where the functions a, b are continuous and positive for $t \geq 0$, $\Phi_p(u) = |u|^{p-2}u$ with $p > 1$, f is continuous on $\mathbb{R}$ such that $f(u)u > 0$ for $u \neq 0$, and $g : [0, \infty) \to \mathbb{R}$ is a continuous function satisfying $\lim_{t\to\infty} g(t) = \infty$.

These equations arise in the study of radially symmetric solutions of the nonlinear partial differential equation with p-Laplacian. They are natural generalizations of nonlinear equations with the Sturm-Liouville operator. When $f(x) = \Phi_q(x)$ ($q > 1$) and $f(x) = \Phi_p(x)$, (4.1) is called a *quasilinear equation* and a *half-linear equation*, respectively. The integral conditions describing the asymptotic behavior of all nonoscillatory solutions of (4.1) are given. Moreover, applications to the quasilinear equations and the half-linear equations are given as well.

Our results in this direction generalize or complete the results of Cecchi-Marini-Villari, Elbert, Elbert-Kusano, Mirzov, Mizukami-Naito-Usami, Philos, Potter, Tanigawa, and others.

(2) Third order linear and nonlinear differential equations

We establish new results in the oscillatory and asymptotic theories of the third order differential equation

$$x''' + p(t)x' + q(t)f(x) = 0.$$

Our results in this direction generalize or complete the results of Dolan, Erbe, Gregus, Gera, Hanan, Jones, Lazer, Gaet.Villari, and others.

(3) Linear differential equations with the disconjugate operators

Consider the two-term linear differential equation

$$L_n x + p(t)x = 0$$

where $n \geq 3$, $p \neq 0$, and L_n is the n-th order disconjugate operator

$$L_n x \equiv \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{a_{n-1}(t)} \cdots \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{a_1(t)} \frac{\mathrm{d}}{\mathrm{d}t} x$$

with continuous and positive real functions a_i $(i = 1, \ldots, n)$ on $[0, \infty)$. These equations, sometimes called *equations with quasiderivatives,* enjoy a very rich structure of solutions. A particular attention is devoted to the oscillatory and asymptotic properties of these equations described in terms of the so-called property A and property B. In particular, the equivalence theorem between both properties is proved.

Our results in this direction generalize or complete the results of Chanturia, Chanturia-Kiguradze, Dzurina, Elias, Kiguradze, Kusano, Kusano-Naito, Kusano-Naito-Tanaka, Ohriska, Švec, Trench, and others.

The dissertation is organized into three chapters:

Chapter 1: Differential equations with p-Laplacian – papers [D9, D10, D11, D12]

Chapter 2: Third order differential equations – papers [D1, D2, D3, D4, D5, D6, D7]

Chapter 3: Linear differential equations of n-th order – paper [D8].

The main contribution of the presented papers consists of the following topics:

(1) Possible types of nonoscillatory solutions of the second and third order differential equations.

(2) Necessary and sufficient conditions ensuring the existence of zero-convergent solutions and the existence of bounded/unbounded solutions.

(3) A description of the asymptotic behavior of all nonoscillatory solutions of (4.1) and their quasiderivatives.

(4) Limit and integral characterizations of the principal solution of the half-linear differential equations.

(5) Theorems on the equivalence between the properties A and B for the third and higher order linear differential equations with quasiderivatives.

(6) Oscillatory properties for the third order linear differential equations.

(7) Sufficiency theorems for property A and property B for the third order nonlinear differential equations.