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A Pressure Associated with a Weak Solution to the Navier–Stokes Equations with Navier's Boundary Conditions

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Abstract

We show that if $\mathbf{u}$ is a weak solution to the Navier–Stokes initial–boundary value problem with Navier's slip boundary conditions in $Q_T := \Omega \times (0,T)$, where Ω is a domain in $\mathbb{R}^3$, then an associated pressure p exists as a distribution with a certain structure. Furthermore, we also show that if Ω is a "smooth" domain in $\mathbb{R}^3$ then the pressure is represented by a function in Q_T with a certain rate of integrability. Finally, we study the regularity of the pressure in sub-domains of Q_T, where $\mathbf{u}$ satisfies Serrin's integrability conditions.

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1 Introduction

1.1. The Navier–Stokes initial–boundary value problem with Navier's boundary conditions. Let $T > 0$ and Ω be a locally Lipschitz domain in $\mathbb{R}^3$, satisfying the condition

(i) *there exists a sequence of bounded Lipschitz domains* $\Omega_1 \subseteq \Omega_2 \subseteq \dots$ *such that* $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ *and* $(\partial\Omega_n \cap \Omega) \subset \{\mathbf{x} \in \mathbb{R}^3;\ |\mathbf{x}| \geq n\}$ *for all* $n \in \mathbb{N}$.

Note that condition (i) is automatically satisfied e.g. if $\Omega = \mathbb{R}^3$ or Ω is a half-space in $\mathbb{R}^3$ or Ω is a bounded or exterior Lipschitz domain in $\mathbb{R}^3$. Put $Q_T := \Omega \times (0,T)$ and $\Gamma_T := \partial\Omega \times (0,T)$. We deal with the Navier–Stokes system

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p \ = \ \nu \Delta \mathbf{u} + \mathbf{f} \qquad \text{in } Q_T, \tag{1.1}$$

$$\operatorname{div} \mathbf{u} \ = \ 0 \qquad \text{in } Q_T \tag{1.2}$$

with the slip boundary conditions

$$\text{a)} \quad \mathbf{u} \cdot \mathbf{n} = 0, \qquad \text{b)} \quad [\mathbb{T}_{\mathrm{d}}(\mathbf{u}) \cdot \mathbf{n}]_\tau + \gamma \mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_T \tag{1.3}$$

and the initial condition

$$\mathbf{u}\big|_{t=0} \ = \ \mathbf{u}_0. \tag{1.4}$$

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Equations (1.1), (1.2) describe the motion of a viscous incompressible fluid in domain Ω in the time interval $(0,T)$. The unknowns are $\mathbf{u}$ (the velocity) and p (the pressure). Factor ν in equation (1.1) denotes the kinematic coefficient of viscosity (it is supposed to be a positive constant) and $\mathbf{f}$ denotes an external body force. The outer normal vector field on Ω is denoted by $\mathbf{n}$, $\mathbb{T}_{\rm d}(\mathbf{u})$ denotes the dynamic stress tensor, $-\mathbb{T}_{\rm d}(\mathbf{u}) \cdot \mathbf{n}$ is the force with which the fluid acts on the boundary of Ω (we put the minus sign in front of $\mathbb{T}_{\rm d}(\mathbf{u}) \cdot \mathbf{n}$ because $\mathbf{n}$ is the outer normal vector and we express the force acting on $\partial\Omega$ from the interior of Ω), subscript τ denotes the tangential component and γ (which is supposed to be a nonnegative constant) is the coefficient of friction between the fluid and the boundary of Ω. The density of the fluid is supposed to be constant and equal to one. In an incompressible Newtonian fluid, the dynamic stress tensor satisfies $\mathbb{T}_{\rm d}(\mathbf{u}) = 2\nu\mathbb{D}(\mathbf{u})$, where the rate of deformation tensor $\mathbb{D}(\mathbf{u})$ equals $(\nabla\mathbf{u})_s$ (the symmetric part of $\nabla\mathbf{u}$).

Equations (1.1), (1.2) are mostly studied together with the no–slip boundary condition

$$\mathbf{u} \ = \ \mathbf{0} \tag{1.5}$$

on Γ_T. However, an increasing attention in recent years has also been given to boundary conditions (1.3), which have a good physical sense. While condition (1.3a) expresses the impermeability of $\partial\Omega$, condition (1.4b) expresses the requirement that the tangential component of the force with which the fluid acts on the boundary be proportional to the tangential velocity. Conditions (1.3) are mostly called Navier's boundary conditions, because they were proposed by H. Navier in the first half of the 19th century.

1.2. Briefly on the qualitative theory of the problem (1.1)–(1.4). As to the qualitative theory for the problem (1.1)–(1.4), it is necessary to note that it is not at the moment so elaborated as in the case of the no-slip boundary condition (1.5). Nevertheless, the readers can find the definition of a weak solution to the problem (1.1)–(1.4) and the proof of the global in time existence of a weak solution e.g. in the papers [6] (with $\mathbf{f} = \mathbf{0}$), [20] (in a time-varying domain Ω) and [25] (in a half-space). We repeat the definition in section 3. Theorems on the local in time existence of a strong solution are proven e.g. in [6] (for $\mathbf{f} = \mathbf{0}$) and [15] (in a smooth bounded domain Ω). Steady problems are studied in [2] and [3].

1.3. On the contents and results of this paper. We shall see in section 3 that the definition of a weak solution to the problem (1.1)–(1.4) does not explicitly contain the pressure. (This situation is well known from the theory of the Navier–Stokes equations with the no–slip boundary condition (1.5).) This is also why we usually understand, under a "weak solution", only the velocity $\mathbf{u}$ and not the pair $(\mathbf{u},p)$. There arises a question whether one can naturally assign some pressure p to a weak solution $\mathbf{u}$. It is known from the theory of the Navier–Stokes equations with the no–slip boundary condition (1.5) that the pressure, associated with a weak solution, generally exists only as a distribution in Q_T. (See [16], [34], [29], [11], [32], [35] and [22].) The distribution is regular (i.e. it can be identified with a function with some rate of integrability in Q_T) if domain Ω is "smooth", see [31], [13] and [22]. In section 4 of this paper, we show that one can naturally assign a pressure, as a distribution, to a weak solution to the Navier–Stokes equations with Navier's boundary conditions (1.3), too. Moreover, we show in section 4 that the associated pressure is not just a distribution, satisfying together with the weak solution $\mathbf{u}$ equations (1.1), (1.2) in the sense of distributions in Q_T (where the distributions are applied to test functions from $\mathbf{C}_0^\infty(Q_T)$), but that it is a distribution with a certain structure, which can be applied to functions from $\mathbf{C}^\infty(\overline{Q_T})$ with a compact support in $\overline{\Omega} \times (0,T)$ and with the normal component equal to zero on Γ_T. In section 5, we show that if domain Ω is smooth and bounded then the associated pressure is a function with a certain rate of integrability in Q_T. Finally, in section 6, we study the regularity of the associated pressure in a sub-domain $\Omega' \times (t_1, t_2)$ of Q_T, where $\mathbf{u}$ satisfies Serrin's integrability conditions. We shall see that the regularity depends on boundary conditions, satisfied by the velocity on Γ_T.

2 Notation and auxiliary results

2.1. Notation. We use this notation of functions, function spaces, dual spaces, etc.:

- $\Omega_0 \subset\subset \Omega$ means that Ω_0 is a bounded domain in $\mathbb{R}^3$ such that $\overline{\Omega_0} \subset \Omega$.
- Vector functions and spaces of vector functions are denoted by boldface letters.
- $\mathbf{C}^\infty_{0,\sigma}(\Omega)$ denotes the linear space of infinitely differentiable divergence-free vector functions in Ω, with a compact support in Ω.
- Let $1 < q < \infty$. We denote by $\mathbf{L}^q_{\tau,\sigma}(\Omega)$ the closure of $\mathbf{C}^\infty_{0,\sigma}(\Omega)$ in $\mathbf{L}^q(\Omega)$. The subscript τ means that functions from $\mathbf{L}^q_{\tau,\sigma}(\Omega)$ have the normal component on $\partial\Omega$ equal to zero in a certain weak sense of traces and they are therefore tangential on $\partial\Omega$. The subscript σ expresses the fact that functions from $\mathbf{L}^q_{\tau,\sigma}(\Omega)$ are divergence–free in Ω in the sense of distributions. (See e.g. [10] for more information.)
- Put $\mathbf{G}_q(\Omega) := \{\nabla\psi \in \mathbf{L}^q(\Omega);\ \psi \in W^{1,q}_{\mathrm{loc}}(\Omega)\}$. $\mathbf{G}_q(\Omega)$ is a closed subspace of $\mathbf{L}^q(\Omega)$, see [10, Exercise III.1.2].
- $\mathbf{W}^{1,q}_\tau(\Omega) := \{\mathbf{v} \in \mathbf{W}^{1,q}(\Omega);\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ a.e. on } \partial\Omega\}$,
  $\mathbf{W}^{1,q}_{\tau,c}(\Omega) := \big\{\boldsymbol{\varphi} \in \mathbf{W}^{1,q}_\tau(\Omega), \operatorname{supp}\boldsymbol{\varphi} \text{ is a compact set in } \mathbb{R}^3\big\}$,
  $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega) := \mathbf{W}^{1,q}(\Omega) \cap \mathbf{L}^q_{\tau,\sigma}(\Omega) \equiv \mathbf{W}^{1,q}_\tau(\Omega) \cap \mathbf{L}^q_{\tau,\sigma}(\Omega)$,
  $\mathbf{W}^{1,q}_{\tau,\sigma,c}(\Omega) := \mathbf{W}^{1,q}_{\tau,\sigma}(\Omega) \cap \mathbf{W}^{1,q}_{\tau,c}(\Omega)$.
- The norms in $L^q(\Omega)$ and in $\mathbf{L}^q(\Omega)$ are denoted by $\|\,.\,\|_q$. The norms in $W^{k,q}(\Omega)$ and in $\mathbf{W}^{k,q}(\Omega)$ (for $k \in \mathbb{N}$) are denoted by $\|\,.\,\|_{k,q}$. If the considered domain differs from Ω then we use e.g. the notation $\|\,.\,\|_{q;\,\Omega'}$ or $\|\,.\,\|_{k,q;\,\Omega'}$, etc. The scalar products in $L^2(\Omega)$ and in $\mathbf{L}^2(\Omega)$ are denoted by $(\,.\,,\,.\,)_2$ and the scalar products in $W^{1,2}(\Omega)$ and in $\mathbf{W}^{1,2}(\Omega)$ are denoted by $(\,.\,,\,.\,)_{1,2}$.
- The conjugate exponent is denoted by prime, so that e.g. $q' = q/(q-1)$. $\mathbf{W}^{-1,q'}_\tau(\Omega)$ denotes the dual space to $\mathbf{W}^{1,q}_\tau(\Omega)$ and $\mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$ denotes the dual space to $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$. The norm in $\mathbf{W}^{-1,q'}_\tau(\Omega)$, respectively $\mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$, is denoted by $\|\,.\,\|_{-1,q'}$, respectively by $\|\,.\,\|_{-1,q';\,\sigma}$.
- The duality between elements of $\mathbf{W}^{-1,q'}_\tau(\Omega)$ and $\mathbf{W}^{1,q}_\tau(\Omega)$ is denoted by $\langle\,.\,,\,.\,\rangle_\tau$ and the duality between elements of $\mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$ and $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$ is denoted by $\langle\,.\,,\,.\,\rangle_{\tau,\sigma}$.
- $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)^\perp$ denotes the space of annihilators of $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$ in $\mathbf{W}^{-1,q'}_\tau(\Omega)$. i.e. the space $\big\{\mathbf{g} \in \mathbf{W}^{-1,q'}_\tau(\Omega);\ \forall\boldsymbol{\varphi} \in \mathbf{W}^{1,q}_{\tau,\sigma}(\Omega) : \langle\mathbf{g},\boldsymbol{\varphi}\rangle_\tau = 0\big\}$.

2.2. $\mathbf{L}^{q'}(\Omega)$ and $\mathbf{L}^{q'}_{\tau,\sigma}(\Omega)$ as subspaces of $\mathbf{W}^{-1,q'}_\tau(\Omega)$ and $\mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$, respectively. The Lebesgue space $\mathbf{L}^{q'}(\Omega)$ can be identified with a subspace of $\mathbf{W}^{-1,q'}_\tau(\Omega)$ so that if $\mathbf{g} \in \mathbf{L}^{q'}(\Omega)$ then

$$\langle\mathbf{g},\boldsymbol{\varphi}\rangle_\tau \ := \int_\Omega \mathbf{g}\cdot\boldsymbol{\varphi}\;\mathrm{d}\mathbf{x} \tag{2.1}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,q}_{\tau}(\Omega)$. Similarly, $\mathbf{L}^{q'}_{\tau,\sigma}(\Omega)$ can be identified with a subspace of $\mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$ so that if $\mathbf{g} \in \mathbf{L}^{q'}_{\tau,\sigma}(\Omega)$ then

$$\langle \mathbf{g}, \boldsymbol{\varphi} \rangle_{\tau,\sigma} \ := \ \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \tag{2.2}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$. Thus, if $\mathbf{g} \in \mathbf{L}^{q'}_{\tau,\sigma}(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$ then the dualities $\langle \mathbf{g}, \boldsymbol{\varphi} \rangle_{\tau}$ and $\langle \mathbf{g}, \boldsymbol{\varphi} \rangle_{\tau,\sigma}$ coincide.

Note that if $\mathbf{g} \in \mathbf{L}^{q'}(\Omega)$ then the integral on the right hand side of (2.1) also defines a bounded linear functional on $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$. This, however, does not mean that $\mathbf{L}^{q'}(\Omega)$ can be identified with a subspace of $\mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$. The reason is, for instance, that the spaces $\mathbf{L}^{q'}(\Omega)$ and $\mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$ do not have the same zero element. (If ψ is a non-constant function in $C^{\infty}_0(\Omega)$ then $\nabla\psi$ is a non-zero element of $\mathbf{L}^{q'}(\Omega)$, but it induces the zero element of $\mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$.)

2.3. Definition and some properties of operator $\mathcal{P}_{q'}$. $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$ is a closed subspace of $\mathbf{W}^{1,q}_{\tau}(\Omega)$. If $\mathbf{g} \in \mathbf{W}^{-1,q'}_{\tau}(\Omega)$ (i.e. $\mathbf{f}$ is a bounded linear functional on $\mathbf{W}^{1,q}_{\tau}(\Omega)$) then we denote by $\mathcal{P}_{q'}\mathbf{f}$ the element of $\mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$, defined by the equation

$$\langle \mathcal{P}_{q'}\mathbf{g}, \boldsymbol{\varphi} \rangle_{\tau,\sigma} \ := \ \langle \mathbf{g}, \boldsymbol{\varphi} \rangle_{\tau} \qquad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,q}_{\tau,\sigma}(\Omega).$$

Obviously, $\mathcal{P}_{q'}$ is a linear operator from $\mathbf{W}^{-1,q'}_{\tau}(\Omega)$ to $\mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$, whose domain is the whole space $\mathbf{W}^{-1,q'}_{\tau}(\Omega)$.

Lemma 2.1. *The operator $\mathcal{P}_{q'}$ is bounded, its range is $\mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$ and $\mathcal{P}_{q'}$ is not one-to-one.*

Proof. The boundedness of operator $\mathcal{P}_{q'}$ directly follows from the definition of the norms in the spaces $\mathbf{W}^{-1,q'}_{\tau}(\Omega)$, $\mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$ and the definition of $\mathcal{P}_{q'}$.

Let $\mathbf{g} \in \mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$. There exists (by the Hahn-Banach theorem) an extension of $\mathbf{g}$ from $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$ to $\mathbf{W}^{1,q}_{\tau}(\Omega)$, which we denote by $\widetilde{\mathbf{g}}$. The extension is an element of $\mathbf{W}^{-1,q'}_{\tau}(\Omega)$, satisfying $\|\widetilde{\mathbf{g}}\|_{-1,q'} = \|\mathbf{g}\|_{-1,q';\,\sigma}$ and

$$\langle \widetilde{\mathbf{g}}, \boldsymbol{\varphi} \rangle_{\tau} \ = \ \langle \mathbf{g}, \boldsymbol{\varphi} \rangle_{\tau,\sigma}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$. This shows that $\mathbf{g} = \mathcal{P}_{q'}\widetilde{\mathbf{g}}$. Consequently, the range of $\mathcal{P}_{q'}$ is the whole space $\mathbf{W}^{-1,q'}_{0,\sigma}(\Omega)$.

Finally, considering $\mathbf{g} = \nabla\psi$ for $\psi \in C^{\infty}_0(\Omega)$, we get

$$\langle \mathcal{P}_{q'}\mathbf{g}, \boldsymbol{\varphi} \rangle_{\tau,\sigma} \ = \ \langle \mathbf{g}, \boldsymbol{\varphi} \rangle_{\tau} \ = \ \int_{\Omega} \nabla\psi \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \ = \ 0$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$. This shows that the operator $\mathcal{P}_{q'}$ is not one-to-one. $\square$

2.4. The relation between operator $\mathcal{P}_{q'}$ and the Helmholtz projection. If each function $\mathbf{g} \in \mathbf{L}^{q'}(\Omega)$ can be uniquely expressed in the form $\mathbf{g} = \mathbf{v} + \nabla\psi$ for some $\mathbf{v} \in \mathbf{L}^{q'}_{\tau,\sigma}(\Omega)$ and $\nabla\psi \in \mathbf{G}_{q'}(\Omega)$, which is equivalent to the validity of the decomposition

$$\mathbf{L}^{q'}(\Omega) \ = \ \mathbf{L}^{q'}_{\tau,\sigma}(\Omega) \oplus \mathbf{G}_{q'}(\Omega), \tag{2.3}$$

then we write $\mathbf{v} = P_{q'}\mathbf{g}$. Decomposition (2.3) is called the *Helmholtz decomposition* and the operator $P_{q'}$ is called the *Helmholtz projection.* The existence of the Helmholtz decomposition depends on exponent q' and the shape of domain Ω. If $q' = 2$ then the Helmholtz decomposition exists on an arbitrary domain Ω and P_2, respectively $I - P_2$, is an orthogonal projection of $\mathbf{L}^2(\Omega)$ onto $\mathbf{L}^2_{\tau,\sigma}(\Omega)$, respectively onto $\mathbf{G}_2(\Omega)$. (See e.g. [10].) If $q' \neq 2$ then various sufficient conditions for the existence of the Helmholtz decomposition can be found e.g. in [7], [9], [10], [12], [14] and [28].

Further on in this paragraph, we assume that the Helmholtz decomposition of $\mathbf{L}^{q'}(\Omega)$ exists. Let $\mathbf{g} \in \mathbf{L}^{q'}(\Omega)$. Treating $\mathbf{g}$ as an element of $\mathbf{W}^{-1,q'}_{\tau}(\Omega)$ in the sense of paragraph 2.2, we have $\langle \mathcal{P}_{q'}\mathbf{g}, \boldsymbol{\varphi}\rangle_{\tau,\sigma} = \langle \mathbf{g}, \boldsymbol{\varphi}\rangle_{\tau}$ for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$. Writing $\mathbf{g} = P_{q'}\mathbf{g} + (I - P_{q'})\mathbf{g}$, we also have

$$\langle \mathbf{g}, \boldsymbol{\varphi}\rangle_{\tau} \;=\; \big\langle P_{q'}\mathbf{g} + (I - P_{q'})\mathbf{g}, \boldsymbol{\varphi}\big\rangle_{\tau} \;=\; \big\langle P_{q'}\mathbf{g}, \boldsymbol{\varphi}\big\rangle_{\tau}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$, because $(I - P_{q'})\mathbf{g} \in \mathbf{G}_{q'}(\Omega)$. Furthermore,

$$\big\langle P_{q'}\mathbf{g}, \boldsymbol{\varphi}\big\rangle_{\tau} \;=\; \big\langle P_{q'}\mathbf{g}, \boldsymbol{\varphi}\big\rangle_{\tau,\sigma},$$

because $P_{q'}\mathbf{g} \in \mathbf{L}^{q'}_{\tau,\sigma}(\Omega)$, $\boldsymbol{\varphi} \in \mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$ and the formulas (2.1) and (2.2) show that the dualities $\langle P_{q'}\mathbf{g}, \boldsymbol{\varphi}\rangle_{\tau}$ and $\langle P_{q'}\mathbf{g}, \boldsymbol{\varphi}\rangle_{\tau,\sigma}$ are expressed by the same integrals. Hence $\langle \mathcal{P}_{q'}\mathbf{g}, \boldsymbol{\varphi}\rangle_{\tau,\sigma}$ coincides with $\langle P_{q'}\mathbf{g}, \boldsymbol{\varphi}\rangle_{\tau,\sigma}$ for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$. Consequently, $\mathcal{P}_{q'}\mathbf{g}$ and $P_{q'}\mathbf{g}$ represent the same element of $\mathbf{W}^{-1,q'}_{\tau,\sigma}(\Omega)$. As $P_{q'}\mathbf{g} \in \mathbf{L}^{q'}_{\tau,\sigma}(\Omega)$, $\mathcal{P}_{q'}\mathbf{g}$ can also be considered to be an element of $\mathbf{L}^{q'}_{\tau,\sigma}(\Omega)$, which induces a functional in $\mathbf{W}^{-1,q}_{\tau,\sigma}(\Omega)$ in the sense of paragraph 2.2. Thus, *the Helmholtz projection $P_{q'}$ coincides with the restriction of $\mathcal{P}_{q'}$ to $\mathbf{L}^{q'}(\Omega)$.*

2.5. More on the space $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)^{\perp}$. Identifying $\mathbf{G}_{q'}(\Omega)$ with a subspace of $\mathbf{W}^{-1,q'}_{\tau}(\Omega)$ in the sense of paragraph 2.2, *we denote by ${}^{\perp}\mathbf{G}_{q'}(\Omega)$ the linear space $\big\{\boldsymbol{\varphi} \in \mathbf{W}^{1,q}_{\tau}(\Omega);\ \forall \mathbf{g} \in \mathbf{G}_{q'}(\Omega) : \langle \mathbf{g}, \boldsymbol{\varphi}\rangle_{\tau} = 0\big\}$.* Using [10, Lemma III.2.1], we deduce that $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega) = {}^{\perp}\mathbf{G}_{q'}(\Omega)$. Hence $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)^{\perp} = ({}^{\perp}\mathbf{G}_{q'}(\Omega))^{\perp}$ and applying Theorem 4.7 in [24], we observe that $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)^{\perp}$ is a closure of $\mathbf{G}_{q'}(\Omega)$ in the weak-$*$ topology of $\mathbf{W}^{-1,q'}_{\tau}(\Omega)$. The next lemma tells us more on elements of $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)^{\perp}$.

Lemma 2.2. *Let $\mathbf{F} \in \mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)^{\perp}$ and $\Omega_0 \subset\subset \Omega$ be a nonempty sub-domain of Ω. Then there exists a unique $p \in L^{q'}_{loc}(\Omega)$ such that $p \in L^{q'}(\Omega_R)$ for all $R > 0$, $\int_{\Omega_0} p \, \mathrm{d}\mathbf{x} = 0$ and*

$$\|p\|_{q';\Omega_R} \;\leq\; c(R)\, \|\mathbf{F}\|_{-1,q} \qquad \textit{for all } R > 0, \tag{2.4}$$

$$\big\langle \mathbf{F}, \boldsymbol{\psi}\big\rangle_{\tau} \;=\; -\int_{\Omega} p \, \mathrm{div}\, \boldsymbol{\psi} \, \mathrm{d}\mathbf{x} \quad \textit{for all } \boldsymbol{\psi} \in \mathbf{W}^{1,q}_{\tau,c}(\Omega). \tag{2.5}$$

Proof. Let $\{\Omega_n\}$ be the sequence of domains from condition (i). We can assume without the loss of generality that $\Omega_0 \subseteq \Omega_1$. Let $n \in \mathbb{N}$. Denote by $L^q_{\mathrm{mv}=0}(\Omega_n)$ the space of all functions from $L^q(\Omega_n)$, whose mean value in Ω_n is zero. There exists a bounded linear operator $\mathfrak{B} : L^q_{\mathrm{mv}=0}(\Omega_n) \to \mathbf{W}^{1,q}_0(\Omega_n)$, such that

$$\mathrm{div}\, \mathfrak{B}(g) \;=\; g$$

for all $g \in L^q_{\mathrm{mv}=0}(\Omega_n)$. Operator $\mathfrak{B}$ is often called the *Bogovskij* or *Bogovskij–Pileckas* operator. More information on operator $\mathfrak{B}$, including its construction, can be found e.g. in [10, Sec. III.3] or in [5].

Denote by $\mathbf{W}^{1,q}_{\tau}(\Omega)_n$, respectively $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)_n$, the space of all functions from $\mathbf{W}^{1,q}_{\tau}(\Omega)$, respectively from $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$, that have a support in $\overline{\Omega_n}$. Let $\boldsymbol{\psi} \in \mathbf{W}^{1,q}_{\tau}(\Omega)_n$. Then the restriction of $\operatorname{div}\boldsymbol{\psi}$ to Ω_n (which we again denote by $\operatorname{div}\boldsymbol{\psi}$ in order to keep a simple notation) belongs to $L^q_{\mathrm{mv}=0}(\Omega_n)$ and $\mathfrak{B}(\operatorname{div}\boldsymbol{\psi}_n) \in \mathbf{W}^{1,q}_0(\Omega_n)$. Identifying $\mathfrak{B}(\operatorname{div}\boldsymbol{\psi})$ with a function from $\mathbf{W}^{1,q}_0(\Omega)$ that equals zero in $\Omega \smallsetminus \Omega_n$, we have

$$\boldsymbol{\psi} \ = \ \mathfrak{B}(\operatorname{div}\boldsymbol{\psi}) + \mathbf{w},$$

where $\mathbf{w}$ is an element of $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$, satisfying $\mathbf{w} = \boldsymbol{\psi} = \mathbf{0}$ in $\Omega \smallsetminus \Omega_n$. Hence

$$\big\langle \mathbf{F}, \boldsymbol{\psi} \big\rangle_{\tau} \ = \ \big\langle \mathbf{F}, \mathfrak{B}(\operatorname{div}\boldsymbol{\psi} \big\rangle_{\tau}. \tag{2.6}$$

As $\mathbf{F}$ is a bounded linear functional on $\mathbf{W}^{1,q}_{\tau}(\Omega)$, vanishing on the subspace $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)$, its restriction to $\mathbf{W}^{1,q}_{\tau}(\Omega)_n$ is an element of $\mathbf{W}^{-1,q'}_{\tau}(\Omega)_n$, vanishing on $\mathbf{W}^{1,q}_{\tau,\sigma}(\Omega)_n$. Furthermore, identifying functions from $\mathbf{W}^{1,q}_{\tau}(\Omega)_n$ with their restrictions to Ω_n, we can also consider $\mathbf{F}$ to be an element of $\mathbf{W}^{-1,q'}_0(\Omega_n)$, vanishing on $\mathbf{W}^{1,q}_{0,\sigma}(\Omega_n)$. Thus, due to Lemma 1.4 in [22], there exists $c(n) > 0$ and a unique function $p_n \in L^{q'}(\Omega_n)$ such that $\int_{\Omega_0} p_n \,\mathrm{d}\mathbf{x} = 0$ and

$$\|p_n\|_{q';\,\Omega_n} \ \leq \ c(n)\, \|\mathbf{F}\|_{-1,q;\,\Omega_n} \ \leq \ c(n)\, \|\mathbf{F}\|_{-1,q}, \tag{2.7}$$

$$\big\langle \mathbf{F}, \boldsymbol{\zeta} \big\rangle_{\Omega_n} \ = \ -\int_{\Omega_n} p_n \operatorname{div}\boldsymbol{\zeta} \,\mathrm{d}\mathbf{x} \tag{2.8}$$

for all $\boldsymbol{\zeta} \in \mathbf{W}^{1,q}_0(\Omega_n)$. Using identity (2.8) with $\boldsymbol{\zeta} = \mathfrak{B}(\operatorname{div}\boldsymbol{\psi})$, we obtain

$$\big\langle \mathbf{F}, \mathfrak{B}(\operatorname{div}\boldsymbol{\psi}) \big\rangle_{\tau} \ \equiv \ \big\langle \mathbf{F}, \mathfrak{B}(\operatorname{div}\boldsymbol{\psi}) \big\rangle_{\Omega_n} \ = \ -\int_{\Omega_n} p_n \operatorname{div}\mathfrak{B}(\operatorname{div}\boldsymbol{\psi}) \,\mathrm{d}\mathbf{x} \ = \ -\int_{\Omega_n} p_n \operatorname{div}\boldsymbol{\psi} \,\mathrm{d}\mathbf{x}.$$

As the same identities also hold for $n+1$ instead of n, we deduce that $p_{n+1} = p_n$ in Ω_n. Hence we may define function p in Ω by the formula $p := p_n$ in Ω_n and we have

$$\big\langle \mathbf{F}, \mathfrak{B}(\operatorname{div}\boldsymbol{\psi}) \big\rangle_{\tau} \ = \ -\int_{\Omega} p \operatorname{div}\boldsymbol{\psi} \,\mathrm{d}\mathbf{x}. \tag{2.9}$$

If $\boldsymbol{\psi} \in \mathbf{W}^{1,q}_{\tau,c}(\Omega)$ then $\boldsymbol{\psi} \in \mathbf{W}^{1,q}_{\tau}(\Omega)_n$ for sufficiently large n and (2.9) holds as well. Inequality (2.4) now follows from (2.7). Identities (2.6) and (2.9) imply (2.5). $\square$

Note that if Ω is a bounded Lipschitz domain then the choice $\Omega_0 = \Omega$ is also possible in Lemma 2.2.

3 Three equivalent weak formulations of the Navier–Stokes initial-boundary value problem (1.1)–(1.4)

Recall that Ω is supposed to be a locally Lipschitz domain in $\mathbb{R}^3$.

3.1. The 1st weak formulation of the Navier–Stokes IBVP (1.1)–(1.4). *Given* $\mathbf{u}_0 \in \mathbf{L}^2_{\tau,\sigma}(\Omega)$ *and* $\mathbf{f} \in L^2(0,T;\, \mathbf{W}^{-1,2}_{\tau}(\Omega))$. *A function* $\mathbf{u} \in L^\infty(0,T;\ \mathbf{L}^2_{\tau,\sigma}(\Omega)) \cap L^2(0,T;\ \mathbf{W}^{1,2}_{\tau,\sigma}(\Omega))$ *is said to be a weak solution to the problem (1.1)–(1.4) if the trace of* $\mathbf{u}$ *on* Γ_T *is in* $L^2(0,T;\, \mathbf{L}^2(\partial\Omega))$ *and* $\mathbf{u}$ *satisfies*

$$\int_0^T \int_{\Omega} \big[-\partial_t \boldsymbol{\phi} \cdot \mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} \cdot \boldsymbol{\phi} + 2\nu (\nabla\mathbf{u})_s : (\nabla\boldsymbol{\phi})_s \big] \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

$$+\int_0^T\int_{\partial\Omega}\gamma\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}S\,\mathrm{d}t\ =\ \int_0^T\big\langle\mathbf{f},\boldsymbol{\phi}\big\rangle_\tau\,\mathrm{d}t+\int_\Omega\mathbf{u}_0\cdot\boldsymbol{\phi}(\,.\,,0)\,\mathrm{d}\mathbf{x} \tag{3.1}$$

for all vector–functions $\boldsymbol{\phi}\in C_0^\infty\big([0,T);\ \mathbf{W}^{1,2}_{\tau,\sigma,c}(\Omega)\big)$.

Equation (3.1) follows from (1.1), (1.2) if one formally multiplies equation (1.1) by the test function $\boldsymbol{\phi}\in C_0^\infty\big([0,T);\ \mathbf{W}^{1,2}_{\tau,\sigma,c}(\Omega)\big)$, applies the integration by parts and uses the boundary conditions (1.3) and the initial condition (1.4). As the integral of $\nabla p\cdot\boldsymbol{\phi}$ vanishes, the pressure p does not explicitly appear in (3.1).

On the other hand, if $\mathbf{f}\in\mathbf{L}^2(Q_T)$ and $\mathbf{u}$ is a weak solution with the additional properties $\partial_t\mathbf{u}\in\mathbf{L}^2(Q_T)$ and $\mathbf{u}\in L^2(0,T;\,\mathbf{W}^{2,2}(\Omega))$ then, considering the test functions $\boldsymbol{\phi}$ in (3.1) of the form $\boldsymbol{\phi}(\mathbf{x},t)=\boldsymbol{\varphi}(\mathbf{x})\,\vartheta(t)$ where $\boldsymbol{\varphi}\in\mathbf{W}^{1,2}_{\tau,\sigma,c}(\Omega)$ and $\vartheta\in C_0^\infty((0,T))$, and applying the backward integration by parts, one obtains the equation

$$\int_\Omega\big(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\nu\Delta\mathbf{u}-\mathbf{f}\big)\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\ =\ 0$$

for a.a. $t\in(0,T)$. As $\mathbf{W}^{1,2}_{\tau,\sigma,c}(\Omega)$ is dense in $\mathbf{L}^2_{\tau,\sigma}(\Omega)$, this equation shows that $P_2[\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\nu\Delta\mathbf{u}-\mathbf{f}]=\mathbf{0}$ at a.a. time instants $t\in(0,T)$. Consequently, to a.a. $t\in(0,T)$, there exists $p\in W^{1,2}_{\mathrm{loc}}(\Omega)$ such that $\nabla p=(I-P_2)[\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\nu\Delta\mathbf{u}-\mathbf{f}]$ and the functions $\mathbf{u}$ and p satisfy equation (1.1) (as an equation in $\mathbf{L}^2(\Omega)$) at a.a. time instants $t\in(0,T)$. It follows from the boundedness of projection P_2 in $\mathbf{L}^2(\Omega)$ and the assumed properties of functions $\mathbf{u}$ and $\mathbf{f}$ that $\nabla p\in\mathbf{L}^2(Q_T)$. Considering afterwards the test functions $\boldsymbol{\phi}$ as in (3.1), and integrating by parts in (3.1), we get

$$\int_0^T\int_\Omega\big(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\nu\Delta\mathbf{u}-\mathbf{f}\big)\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}+\int_0^T\int_{\partial\Omega}\big([\mathbb{T}_{\mathrm{d}}(\mathbf{u})\cdot\mathbf{n}]+\gamma\mathbf{u}\big)\cdot\boldsymbol{\phi}\,\mathrm{d}S\,\mathrm{d}t\ =\ 0$$

The first integral is equal to zero, because the expression in the parentheses equals $-\nabla p$ a.e. in Q_T and the integral $\nabla p\cdot\boldsymbol{\phi}$ in Ω equals zero for a.a. $t\in(0,T)$. In the second integral, since both $\mathbf{u}(\,.\,,t)$ and $\boldsymbol{\phi}(\,.\,,t)$ are tangent on $\partial\Omega$, we can replace $[\mathbb{T}_{\mathrm{d}}(\mathbf{u})\cdot\mathbf{n}]+\gamma\mathbf{u}$ by $[\mathbb{T}_{\mathrm{d}}(\mathbf{u})\cdot\mathbf{n}]_\tau+\gamma\mathbf{u}$ and we thus obtain

$$\int_0^T\int_{\partial\Omega}\big([\mathbb{T}_{\mathrm{d}}(\mathbf{u})\cdot\mathbf{n}]_\tau+\gamma\mathbf{u}\big)\cdot\boldsymbol{\phi}\,\mathrm{d}S\,\mathrm{d}t\ =\ 0.$$

As this equation holds for all test functions $\boldsymbol{\phi}\in C_0^\infty\big([0,T);\ \mathbf{W}^{1,2}_{\tau,\sigma,c}(\Omega)\big)$, we deduce that $\mathbf{u}$ satisfies the boundary condition (1.3b). Recall that this procedure works only under additional assumptions on smoothness of the weak solution $\mathbf{u}$ and function $\mathbf{f}$. On a general level, however, it is not known whether the existing weak solution is smooth. Nevertheless, we show in subsection 4.4 that there exists a certain pressure, which can be naturally associated with the weak solution to (1.1)–(1.4). The pressure generally exists only as a distribution, see Theorem 4.2.

3.2. The 2nd weak formulation of the Navier-Stokes IBVP (1.1)–(1.4). We define the operators $\mathcal{A}:\mathbf{W}^{1,2}_\tau(\Omega)\to\mathbf{W}^{-1,2}_\tau(\Omega)$ and $\mathcal{B}:\big[\mathbf{W}^{1,2}_\tau(\Omega)\big]^2\to\mathbf{W}^{-1,2}_\tau(\Omega)$ by the equations

$$\begin{aligned}\big\langle\mathcal{A}\mathbf{v},\boldsymbol{\varphi}\big\rangle_\tau\ &:=\ \int_\Omega 2\nu(\nabla\mathbf{v})_s:(\nabla\boldsymbol{\varphi})_s\,\mathrm{d}\mathbf{x}+\int_{\partial\Omega}\gamma\mathbf{v}\cdot\boldsymbol{\varphi}\,\mathrm{d}S &&\text{for }\mathbf{v},\boldsymbol{\varphi}\in\mathbf{W}^{1,2}_\tau(\Omega),\\ \big\langle\mathcal{B}(\mathbf{v},\mathbf{w}),\boldsymbol{\varphi}\big\rangle_\tau\ &:=\ \int_\Omega\mathbf{v}\cdot\nabla\mathbf{w}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} &&\text{for }\mathbf{v},\mathbf{w},\boldsymbol{\varphi}\in\mathbf{W}^{1,2}_\tau(\Omega).\end{aligned}$$

By Korn's inequality (see e.g. [33, Lemma 4]) and inequality [10, (II.4.5), p. 63], we have

$$\big\langle \mathcal{A}\mathbf{v},\mathbf{v}\big\rangle_\tau \;=\; \int_\Omega \nu\,|(\nabla\mathbf{v})_s|^2\;\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \gamma\,|\mathbf{v}|^2\;\mathrm{d}S \;\geq\; c_1\nu\,\|\nabla\mathbf{v}\|_2^2. \tag{3.2}$$

Furthermore, using the boundedness of the operator of traces from $\mathbf{W}^{1,2}_\tau(\Omega)$ to $\mathbf{L}^2(\partial\Omega)$, we can also deduce that there exists $c_2>0$ such that

$$\|\mathcal{A}\mathbf{v}\|_{-1,2} \;\leq\; c_2\,\|\nabla\mathbf{v}\|_2 \tag{3.3}$$

for all $\mathbf{v}\in\mathbf{W}^{1,2}_\tau(\Omega)$. Thus, $\mathcal{A}$ is a bounded one–to–one operator, mapping $\mathbf{W}^{1,2}_\tau(\Omega)$ into $\mathbf{W}^{-1,2}_\tau(\Omega)$. If $k>0$ then the range of $\mathcal{A}+kI$ is the whole space $\mathbf{W}^{-1,2}_\tau(\Omega)$ (by the Lax–Milgram theorem) and $(\mathcal{A}+kI)^{-1}$ is a bounded operator from $\mathbf{W}^{-1,2}_\tau(\Omega)$ onto $\mathbf{W}^{1,2}_\tau(\Omega)$. If Ω is bounded then the same statements also hold for $k=0$. The bilinear operator $\mathcal{B}$ satisfies

$$\begin{aligned}
\|\mathcal{B}(\mathbf{v},\mathbf{w})\|_{-1,2} \;&=\; \sup_{\boldsymbol{\varphi}\in\mathbf{W}^{1,2}_\tau(\Omega),\ \boldsymbol{\varphi}\neq\mathbf{0}} \frac{|\langle\mathcal{B}(\mathbf{v},\mathbf{w}),\boldsymbol{\varphi}\rangle_\tau|}{\|\boldsymbol{\varphi}\|_{1,2}} \\
&=\; \sup_{\boldsymbol{\varphi}\in\mathbf{W}^{1,2}_\tau(\Omega),\ \boldsymbol{\varphi}\neq\mathbf{0}} \frac{|(\mathbf{v}\cdot\nabla\mathbf{w},\,\boldsymbol{\varphi})_2|}{\|\boldsymbol{\varphi}\|_{1,2}} \;\leq\; \sup_{\boldsymbol{\varphi}\in\mathbf{W}^{1,2}_\tau(\Omega),\ \boldsymbol{\varphi}\neq\mathbf{0}} \frac{\|\mathbf{v}\|_2^{1/2}\,\|\mathbf{v}\|_6^{1/2}\,\|\nabla\mathbf{w}\|_2\,\|\boldsymbol{\varphi}\|_6}{\|\boldsymbol{\varphi}\|_{1,2}} \\
&\leq\; c\,\|\mathbf{v}\|_2^{1/2}\,\|\nabla\mathbf{v}\|_2^{1/2}\,\|\nabla\mathbf{w}\|_2.
\end{aligned} \tag{3.4}$$

(We have used the imbedding inequality $\|\mathbf{v}\|_6\leq c\,\|\mathbf{v}\|_{1,2}$. Here and further on, c denotes the generic constant.)

Let $\mathbf{u}$ be a weak solution of the IBVP (1.1)–(1.4) in the sense of paragraph 3.1. It follows from the estimates (3.3) and (3.4) that

$$\mathcal{A}\mathbf{u}\in L^2(0,T;\ \mathbf{W}^{-1,2}_\tau(\Omega)) \quad\text{and}\quad \mathcal{B}(\mathbf{u},\mathbf{u})\in L^{4/3}(0,T;\ \mathbf{W}^{-1,2}_\tau(\Omega)). \tag{3.5}$$

Considering $\boldsymbol{\phi}$ in (3.1) in the form $\boldsymbol{\phi}(\mathbf{x},t)=\boldsymbol{\varphi}(\mathbf{x})\,\vartheta(t)$, where $\boldsymbol{\varphi}\in\mathbf{W}^{1,2}_{\tau,\sigma,c}(\Omega)$ and $\vartheta\in C_0^\infty((0,T))$, we deduce that $\mathbf{u}$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{u},\boldsymbol{\varphi})_2 + \big\langle\mathcal{A}\mathbf{u},\boldsymbol{\varphi}\big\rangle_\tau + \big\langle\mathcal{B}(\mathbf{u},\mathbf{u}),\boldsymbol{\varphi}\big\rangle_\tau \;=\; \langle\mathbf{f},\boldsymbol{\varphi}\rangle_\tau \tag{3.6}$$

a.e. in $(0,T)$, where the derivative of $(\mathbf{u},\boldsymbol{\varphi})_2$ means the derivative in the sense of distributions. As the space $\mathbf{W}^{1,2}_{\tau,\sigma,c}(\Omega)$ is dense in $\mathbf{W}^{1,2}_{\tau,\sigma}(\Omega)$, (3.6) holds for all $\boldsymbol{\varphi}\in\mathbf{W}^{1,2}_{\tau,\sigma}(\Omega)$. It follows from (3.5) that $\langle\mathcal{A}\mathbf{u},\boldsymbol{\varphi}\rangle_\tau\in L^2(0,T)$ and $\langle\mathcal{B}(\mathbf{u},\mathbf{u}),\boldsymbol{\varphi}\rangle_\tau\in L^{4/3}(0,T)$. Since $\langle\mathbf{f},\boldsymbol{\varphi}\rangle_\tau\in L^2(0,T)$, we obtain from (3.6) that the distributional derivative of $(\mathbf{u},\boldsymbol{\varphi})_2$ with respect to t is in $L^{4/3}(0,T)$. Hence $(\mathbf{u},\boldsymbol{\varphi})_2$ is a.e. in $[0,T)$ equal to a continuous function and the weak solution $\mathbf{u}$ is (after a possible redefinition on a set of measure zero) a weakly continuous function from $[0,T)$ to $\mathbf{L}^2_{\tau,\sigma}(\Omega)$. Now, one can easily deduce from (3.1) that $\mathbf{u}$ satisfies the initial condition (1.4) in the sense that

$$(\mathbf{u},\boldsymbol{\varphi})_2\big|_{t=0} = \;(\mathbf{u}_0,\boldsymbol{\varphi})_2 \tag{3.7}$$

for all $\boldsymbol{\varphi}\in\mathbf{W}^{1,2}_{\tau,\sigma}(\Omega)$. Thus, we come to the 2nd weak formulation of the IBVP (1.1)–(1.4):

Given $\mathbf{u}_0\in\mathbf{L}^2_{\tau,\sigma}(\Omega)$ and $\mathbf{f}\in L^2(0,T;\ \mathbf{W}^{-1,2}_\tau(\Omega))$. Find $\mathbf{u}\in L^\infty(0,T;\ \mathbf{L}^2_{\tau,\sigma}(\Omega))\cap L^2(0,T;\ \mathbf{W}^{1,2}_{\tau,\sigma}(\Omega))$ (called the weak solution) such that $\mathbf{u}$ satisfies equation (3.6) a.e. in $(0,T)$ and the initial condition (3.7) for all $\boldsymbol{\varphi}\in\mathbf{W}^{1,2}_{\tau,\sigma}(\Omega)$.

We have shown that if $\mathbf{u}$ is a weak solution of the IBVP (1.1)–(1.4) in the sense of the 1st definition (see paragraph 3.1) then it also satisfies the 2nd definition. Applying standard arguments, one can also show the opposite, i.e. if $\mathbf{u}$ satisfies the 2nd definition then it also satisfies the 1st definition.

3.3. The 3rd weak formulation of the Navier-Stokes IBVP (1.1)–(1.4). Equation (3.6) can also be written in the equivalent form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{u},\boldsymbol{\varphi})_2 + \big\langle \mathcal{P}_2\mathcal{A}\mathbf{u},\boldsymbol{\varphi}\big\rangle_{\tau,\sigma} + \big\langle \mathcal{P}_2\mathcal{B}(\mathbf{u},\mathbf{u}),\boldsymbol{\varphi}\big\rangle_{\Omega,\sigma} \;=\; \big\langle \mathcal{P}_2\mathbf{f},\boldsymbol{\varphi}\big\rangle_{\tau,\sigma}. \tag{3.8}$$

Let us denote by $(\mathbf{u}')_\sigma$ the distributional derivative with respect to t of $\mathbf{u}$, as a function from $(0,T)$ to $\mathbf{W}^{-1,2}_{\tau,\sigma}(\Omega)$. (We explain later why we use the notation $(\mathbf{u}')_\sigma$ and not just $\mathbf{u}'$.) Equation (3.8) can also be written in the form

$$(\mathbf{u}')_\sigma + \mathcal{P}_2\mathcal{A}\mathbf{u} + \mathcal{P}_2\mathcal{B}(\mathbf{u},\mathbf{u}) \;=\; \mathcal{P}_2\mathbf{f}, \tag{3.9}$$

which is an equation in $\mathbf{W}^{-1,2}_{\tau,\sigma}(\Omega)$, satisfied a.e. in the time interval $(0,T)$. (This can be deduced by means of Lemma III.1.1 in [34].) Due to (3.5) and (3.6), $(\mathbf{u}')_\sigma \in L^{4/3}(0,T;\,\mathbf{W}^{-1,2}_{\tau,\sigma}(\Omega))$. Hence $\mathbf{u}$ coincides a.e. in $(0,T)$ with a continuous function from $[0,T)$ to $\mathbf{W}^{-1,2}_{\tau,\sigma}(\Omega)$ and it is therefore meaningful to prescribe an initial condition for $\mathbf{u}$ at time $t=0$. Thus, we obtain the 3rd equivalent definition of a weak solution to the IBVP (1.1)–(1.4):

Given $\mathbf{u}_0 \in \mathbf{L}^2_{\tau,\sigma}(\Omega)$ *and* $\mathbf{f} \in L^2(0,T;\,\mathbf{W}^{-1,2}_\tau(\Omega))$. *Function* $\mathbf{u} \in L^\infty(0,T;\,\mathbf{L}^2_{\tau,\sigma}(\Omega)) \cap L^2(0,T;\,\mathbf{W}^{1,2}_{\tau,\sigma}(\Omega))$ *is called a weak solution to the IBVP (1.1)–(1.4) if* $\mathbf{u}$ *satisfies equation (3.9) a.e. in the interval* $(0,T)$ *and the initial condition (1.4).*

We have explained that if $\mathbf{u}$ is a weak solution in the sense of the 2nd definition then it satisfies the 3rd definition. The validity of the opposite implication can be again verified by means of Lemma III.1.1 in [34].

3.4. Remark. Recall that $(\mathbf{u}')_\sigma$ is the distributional derivative with respect to t of $\mathbf{u}$, as a function from $(0,T)$ to $\mathbf{W}^{-1,2}_{\tau,\sigma}(\Omega)$. It is not the same as the distributional derivative with respect to t of $\mathbf{u}$, as a function from $(0,T)$ to $\mathbf{W}^{-1,2}_\tau(\Omega)$, which can be naturally denoted by $\mathbf{u}'$. As it is important to distinguish between these two derivatives, we use the different notation. We can formally write $(\mathbf{u}')_\sigma = \mathcal{P}_2\mathbf{u}'$.

Since $(\mathbf{u}')_\sigma \in L^{4/3}(0,T;\,\mathbf{W}^{-1,2}_{\tau,\sigma}(\Omega))$, $\mathbf{u}$ coincides a.e. in $(0,T)$ with a continuous function from $[0,T)$ to $\mathbf{W}^{-1,2}_{\tau,\sigma}(\Omega)$. According to what is said in the first part of this remark, this, however, does not imply that $\mathbf{u}$ coincides a.e. in $(0,T)$ with a continuous function from $[0,T)$ to $\mathbf{W}^{-1,2}_\tau(\Omega)$.

4 An associated pressure, its uniqueness and existence

4.1. An associated pressure. *Let* $\mathbf{u}$ *be a weak solution to the IBVP (1.1)–(1.4). A distribution* p *in* Q_T *is called an associated pressure if the pair* $(\mathbf{u},p)$ *satisfies the equations (1.1), (1.2) in the sense of distributions in* Q_T.

4.2. On uniqueness of the associated pressure. Let $\mathbf{u}$ be a weak solution to the IBVP (1.1)–(1.4) and p be an associated pressure.

If G is a distribution in $(0,T)$ and $\psi \in C^\infty_0(Q_T)$ then we define a distribution g in Q_T by the formula

$$\langle\!\langle g, \psi \rangle\!\rangle_{Q_T} \ := \ \Big\langle G,\ \int_\Omega \psi \ \mathrm{d}\mathbf{x} \Big\rangle_{(0,T)}, \tag{4.1}$$

where $\langle\!\langle\, .\,,\, .\,\rangle\!\rangle_{Q_T}$, respectively $\langle\, .\,,\, .\,\rangle_{(0,T)}$, denotes the action of a distribution in Q_T on a function from $C_0^\infty(Q_T)$ or $\mathbf{C}_0^\infty(Q_T)$, respectively the action of a distribution in $(0,T)$ on a function from $C_0^\infty((0,T))$. Obviously, if $\boldsymbol{\phi} \in C_0^\infty\big((0,T);\ \mathbf{W}^{1,2}_{\tau,c}(\Omega)\big)$ then

$$\langle\!\langle \nabla g, \boldsymbol{\phi} \rangle\!\rangle_{Q_T} \ = \ -\langle\!\langle g, \operatorname{div} \boldsymbol{\phi} \rangle\!\rangle_{Q_T} \ = \ -\Big\langle G,\ \int_\Omega \operatorname{div} \boldsymbol{\phi} \ \mathrm{d}\mathbf{x} \Big\rangle_{(0,T)} \ = \ 0, \tag{4.2}$$

because $\int_\Omega \operatorname{div} \boldsymbol{\phi}(\,.\,,t)\ \mathrm{d}\mathbf{x} = 0$ for all $t \in (0,T)$. Thus, $p + g$ is a pressure, associated with the weak solution $\mathbf{u}$ to the IBVP (1.1)–(1.4), too.

For $h \in C_0^\infty((0,T))$, define

$$\big\langle G, h \big\rangle_{(0,T)} \ := \ \langle\!\langle g, \psi \rangle\!\rangle_{Q_T}, \tag{4.3}$$

where $\psi \in C_0^\infty(Q_T)$ is chosen so that $h(t) = \int_\Omega \psi(\mathbf{x},t)\ \mathrm{d}\mathbf{x}$ for all $t \in (0,T)$. The definition of the distribution G is independent of the concrete choice of function ψ due to these reasons: let ψ_1 and ψ_2 be two functions from $C_0^\infty(Q_T)$ such that $h(t) = \int_\Omega \psi_1(\mathbf{x},t)\ \mathrm{d}\mathbf{x} = \int_\Omega \psi_2(\mathbf{x},t)\ \mathrm{d}\mathbf{x}$ for $t \in (0,T)$. Denote by G_1, respectively G_2, the distribution, defined by formula (4.3) with $\psi = \psi_1$, respectively $\psi = \psi_2$. Since $\operatorname{supp}(\psi_1 - \psi_2)$ is a compact subset of Q_T and $\int_\Omega [\psi_1(\,.\,,t) - \psi_2(\,.\,,t)]\ \mathrm{d}\mathbf{x} = 0$ for all $t \in (0,T)$, there exists a function $\boldsymbol{\phi} \in \mathbf{C}_0^\infty(Q_T)$ such that $\operatorname{div} \boldsymbol{\phi} = \psi_1 - \psi_2$ in Q_T. (See e.g. [10, Sec. III.3] or [5] for the construction of function $\boldsymbol{\phi}$.) Then

$$\big\langle G_1 - G_2, h \big\rangle_{(0,T)} \ := \ \langle\!\langle g, \psi_1 - \psi_2 \rangle\!\rangle_{Q_T} \ = \ \langle\!\langle g, \operatorname{div} \boldsymbol{\phi} \rangle\!\rangle_{Q_T},$$

which is equal to zero due to (4.2). Formula (4.3) and the identity $h(t) = \int_\Omega \psi(\mathbf{x},t)\ \mathrm{d}\mathbf{x}$ show that the distribution g has the form (4.1).

We have proven the theorem:

Theorem 4.1. *The pressure, associated with a weak solution to the IBVP (1.1)–(1.4), is unique up to an additive distribution of the form (4.1).*

4.3. Projections $E^{1,2}_\tau$ and $E^{-1,2}_\tau$. In this subsection, we introduce orthogonal projections $E^{1,2}_\tau$ and $E^{-1,2}_\tau$ in $\mathbf{W}^{1,2}_\tau(\Omega)$ and $\mathbf{W}^{-1,2}_\tau(\Omega)$, respectively, which further play an important role in the proof of the existence of an associated pressure.

$\mathbf{W}^{1,2}_\tau(\Omega)$ is a Hilbert space with the scalar product $(\,.\,,\,.\,)_{1,2} = \big\langle (\mathcal{A}_0 + I)\,.\,,\,.\,\big\rangle_\tau$, where $\mathcal{A}_0$ is the operator $\mathcal{A}$ from paragraph 3.2, corresponding to $\nu = 1$ and $\gamma = 0$. Similarly, $\mathbf{W}^{-1,2}_\tau(\Omega)$ is a Hilbert space with the scalar product

$$(\mathbf{g}, \mathbf{h})_{-1,2} \ := \ \big\langle \mathbf{g}, (\mathcal{A}_0 + I)^{-1}\mathbf{h} \big\rangle_\tau \ = \ \big((\mathcal{A}_0 + I)^{-1}\mathbf{g}, (\mathcal{A}_0 + I)^{-1}\mathbf{h} \big)_{1,2}. \tag{4.4}$$

Denote by $E^{1,2}_\tau$ the orthogonal projection in $\mathbf{W}^{1,2}_\tau(\Omega)$ that vanishes just on $\mathbf{W}^{1,2}_{\tau,\sigma}(\Omega)$, which means that

$$\ker E^{1,2}_\tau \ = \ \mathbf{W}^{1,2}_{\tau,\sigma}(\Omega). \tag{4.5}$$

Denote by $E_\tau^{-1,2}$ the adjoint projection in $\mathbf{W}_\tau^{-1,2}(\Omega)$. Applying (4.5), one can verify that the range of $E_\tau^{-1,2}$ is $\mathbf{W}_{\tau,\sigma}^{1,2}(\Omega)^\perp$.

Let $\mathbf{g} \in \mathbf{W}_\tau^{-1,2}(\Omega)$ and $\boldsymbol{\psi} \in \mathbf{W}_\tau^{1,2}(\Omega)$. Then, due to (4.4) and the orthogonality of $E_\tau^{1,2}$, we have

$$\big\langle \mathbf{g}, E_\tau^{1,2}\boldsymbol{\psi}\big\rangle_\tau \;=\; \big((\mathcal{A}_0+I)^{-1}\mathbf{g}, E_\tau^{1,2}\boldsymbol{\psi}\big)_{1,2} \;=\; \big(E_\tau^{1,2}(\mathcal{A}_0+I)^{-1}\mathbf{g}, \boldsymbol{\psi}\big)_{1,2}.$$

However, the duality on the left hand side can also be expressed in another way: using again (4.4) and the fact that $E_\tau^{-1,2}$ is adjoint to $E_\tau^{1,2}$, we get

$$\big\langle \mathbf{g}, E_\tau^{1,2}\boldsymbol{\psi}\big\rangle_\tau \;=\; \big\langle E_\tau^{-1,2}\mathbf{g}, \boldsymbol{\psi}\big\rangle_\tau \;=\; \big((\mathcal{A}_0+I)^{-1}E_\tau^{-1,2}\mathbf{g}, \boldsymbol{\psi}\big)_{1,2}.$$

Thus, we obtain the important identity

$$E_\tau^{1,2}(\mathcal{A}_0+I)^{-1} \;=\; (\mathcal{A}_0+I)^{-1}E_\tau^{-1,2}. \tag{4.6}$$

Applying (4.6), we can now show that the projection $E_\tau^{-1,2}$ is orthogonal in $\mathbf{W}_\tau^{-1,2}(\Omega)$. Indeed, if $\mathbf{g}, \mathbf{h} \in \mathbf{W}_\tau^{-1,2}(\Omega)$ then

$$\begin{aligned}
\big(E_\tau^{-1,2}\mathbf{g}, \mathbf{h}\big)_{-1,2} \;&=\; \big((\mathcal{A}_0+I)^{-1}E_\tau^{-1,2}\mathbf{g}, (\mathcal{A}_0+I)^{-1}\mathbf{h}\big)_{1,2} \\
&= \big(E_\tau^{1,2}(\mathcal{A}_0+I)^{-1}\mathbf{g}, (\mathcal{A}_0+I)^{-1}\mathbf{h}\big)_{1,2} \;=\; \big((\mathcal{A}_0+I)^{-1}\mathbf{g}, E_\tau^{1,2}(\mathcal{A}_0+I)^{-1}\mathbf{h}\big)_{1,2} \\
&= \; \big((\mathcal{A}_0+I)^{-1}\mathbf{g}, (\mathcal{A}_0+I)^{-1}E_\tau^{-1,2}\mathbf{h}\big)_{1,2} \;=\; \big(\mathbf{g}, E_\tau^{-1,2}\mathbf{h}\big)_{-1,2}.
\end{aligned}$$

This verifies the orthogonality of projection $E_\tau^{-1,2}$.

Finally, we will show that if $\phi \in C_0^\infty(\Omega)$ then

$$E_\tau^{1,2}\nabla\phi \;=\; \nabla\phi \qquad \text{for all } \phi \in C_0^\infty(\Omega). \tag{4.7}$$

Thus, let $\phi \in C_0^\infty(\Omega)$. Then $\nabla\phi \in \mathbf{W}_\tau^{1,2}(\Omega)$ and $(\mathcal{A}_0+I)\nabla\phi \equiv \nabla(-\Delta+I)\phi \in \mathbf{W}_{\tau,\sigma}^{1,2}(\Omega)^\perp$. Hence

$$E_\tau^{-1,2}(\mathcal{A}_0+I)\nabla\phi \;=\; (\mathcal{A}_0+I)\nabla\phi.$$

Applying (4.6), we also get

$$E_\tau^{-1,2}(\mathcal{A}_0+I)\nabla\phi \;=\; (\mathcal{A}_0+I)E_\tau^{1,2}\nabla\phi.$$

Since $\mathcal{A}_0+I$ is a one-to-one operator from $\mathbf{W}_\tau^{1,2}(\Omega)$ to $\mathbf{W}_\tau^{-1,2}(\Omega)$, the last two identities show that (4.7) holds.

4.4. Existence of an associated pressure. In this paragraph, we show that to every weak solution of the IBVP (1.1)–(1.4), an associated pressure exists and has a certain structure.

Let $\mathbf{u}$ be a weak solution to the IBVP (1.1)–(1.4). Due to [34, Lemma III.1.1], equation (3.9) is equivalent to

$$\mathbf{u}(t) - \mathbf{u}(0) + \int_0^t \mathcal{P}_2\big[\mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) - \mathbf{f}\big]\,\mathrm{d}\tau \;=\; \mathbf{0}$$

for a.a. $t \in (0,T)$. (As usually, we identify $\mathbf{u}(\,.\,,t)$ and $\mathbf{u}(t)$.) Since $\mathbf{u}(t)$ and $\mathbf{u}(0)$ are in $\mathbf{L}_{\tau,\sigma}^2(\Omega)$, they coincide with $\mathcal{P}_2\mathbf{u}(t)$ and $\mathcal{P}_2\mathbf{u}(0)$, respectively. (See paragraph 2.4.) Hence

$$\mathcal{P}_2\bigg(\mathbf{u}(t) - \mathbf{u}(0) + \int_0^t \big[\mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) - \mathbf{f}\big]\,\mathrm{d}\tau\bigg) \;=\; \mathbf{0}.$$

Define $\mathbf{F}(t) \in \mathbf{W}^{-1,2}_{\tau}(\Omega)$ by the formula

$$\mathbf{F}(t) \; := \; \mathbf{u}(t) - \mathbf{u}(0) + \int_0^t \big[\mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) - \mathbf{f}\big] \, \mathrm{d}\tau. \tag{4.8}$$

Since $\langle \mathbf{F}(t), \boldsymbol{\psi} \rangle_{\tau} \; = \; \langle \mathcal{P}_2 \mathbf{F}(t), \boldsymbol{\psi} \rangle_{\tau,\sigma} \; = \; 0$ for all $\boldsymbol{\psi} \in \mathbf{W}^{1,2}_{\tau,\sigma}(\Omega)$, $\mathbf{F}(t)$ belongs to $\mathbf{W}^{1,2}_{\tau,\sigma}(\Omega)^{\perp}$. Hence $E^{-1,2}_{\tau} \mathbf{F}(t) = \mathbf{F}(t)$ and $(I - E^{-1,2}_{\tau}) \mathbf{F}(t) = \mathbf{0}$. Thus,

$$\begin{aligned} &(I - E^{-1,2}_{\tau}) \mathbf{u}(t) - (I - E^{-1,2}_{\tau}) \mathbf{u}(0) \\ &\qquad\qquad + \int_0^t (I - E^{-1,2}_{\tau}) \big[\mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) - \mathbf{f}\big] \, \mathrm{d}\tau \; = \; \mathbf{0} \end{aligned}$$

holds as an equation in $\mathbf{W}^{-1,2}_{\tau}(\Omega)$. Applying Lemma III.1.1 from [34], we deduce that

$$\big[(I - E^{-1,2}_{\tau}) \mathbf{u}\big]' + (I - E^{-1,2}_{\tau}) \big[\mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) - \mathbf{f}\big] \; = \; \mathbf{0}.$$

This yields

$$\begin{aligned} \mathbf{u}' + \mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) \; &= \; \mathbf{f} \\ &\quad + E^{-1,2}_{\tau} [\mathbf{u}' + \mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) - \mathbf{f}]. \end{aligned} \tag{4.9}$$

(Here, $[(I - E^{-1,2}_{\tau}) \mathbf{u}]'$ and $\mathbf{u}'$ are the distributional derivatives with respect to t of $(I - E^{-1,2}_{\tau}) \mathbf{u}$ and $\mathbf{u}$, respectively, as functions from $(0,T)$ to $\mathbf{W}^{-1,2}_{\tau}(\Omega)$.) Let $\Omega_0 \subset\subset \Omega$ be a non-empty domain. By Lemma 2.2, there exist unique $p_1(t)$, $p_{21}(t)$, $p_{22}(t)$, $p_{23}(t)$ in $L^2_{loc}(\Omega)$ such that

$$\begin{aligned} \big\langle -E^{-1,2}_{\tau} \mathbf{u}(t), \boldsymbol{\psi} \big\rangle_{\tau} \; &= \; - \int_{\Omega} p_1(t) \, \mathrm{div}\, \boldsymbol{\psi} \, \mathrm{d}\mathbf{x}, \\ \big\langle -E^{-1,2}_{\tau} \mathcal{A}\mathbf{u}(t), \boldsymbol{\psi} \big\rangle_{\tau} \; &= \; - \int_{\Omega} p_{21}(t) \, \mathrm{div}\, \boldsymbol{\psi} \, \mathrm{d}\mathbf{x}, \\ \big\langle -E^{-1,2}_{\tau} \mathcal{B}(\mathbf{u}(t), \mathbf{u}(t)), \boldsymbol{\psi} \big\rangle_{\tau} \; &= \; - \int_{\Omega} p_{22}(t) \, \mathrm{div}\, \boldsymbol{\psi} \, \mathrm{d}\mathbf{x}, \\ \big\langle -E^{-1,2}_{\tau} \mathbf{f}(t), \boldsymbol{\psi} \big\rangle_{\tau} \; &= \; - \int_{\Omega} p_{23}(t) \, \mathrm{div}\, \boldsymbol{\psi} \, \mathrm{d}\mathbf{x} \end{aligned} \tag{4.10}$$

for a.a. $t \in (0,T)$ and all $\boldsymbol{\psi} \in \mathbf{W}^{1,2}_{\tau,c}(\Omega)$ and the inequalities

$$\begin{aligned} \|p_1(t)\|_{2;\, \Omega_R} \; &\leq \; c(R) \, \|E^{-1,2}_{\tau} \mathbf{u}(t)\|_{-1,2} && \leq \; c(R) \, \|\mathbf{u}(t)\|_{-1,2}, \\ \|p_{21}(t)\|_{2;\, \Omega_R} \; &\leq \; c(R) \, \|E^{-1,2}_{\tau} \mathcal{A}\mathbf{u}(t)\|_{-1,2} && \leq \; c(R) \, \|\mathcal{A}\mathbf{u}(t)\|_{-1,2}, \\ \|p_{22}(t)\|_{2;\, \Omega_R} \; &\leq \; c(R) \, \|E^{-1,2}_{\tau} \mathcal{B}(\mathbf{u}(t), \mathbf{u}(t))\|_{-1,2} && \leq \; c(R) \, \|\mathcal{B}(\mathbf{u}(t), \mathbf{u}(t))\|_{-1,2}, \\ \|p_{23}(t)\|_{2;\, \Omega_R} \; &\leq \; c(R) \, \|E^{-1,2}_{\tau} \mathbf{f}(t)\|_{-1,2} && \leq \; c(R) \, \|\mathbf{f}(t)\|_{-1,2} \end{aligned} \tag{4.11}$$

hold for all $R > 0$ and a.a. $t \in (0,T)$. Moreover, $\int_{\Omega_0} p_1(t) \, \mathrm{d}\mathbf{x} = \int_{\Omega_0} p_{2i}(t) \, \mathrm{d}\mathbf{x} = 0$ ($i = 1, 2, 3$) for a.a. $t \in (0,T)$. Using the inequality $\|\mathbf{u}(t)\|_{-1,2} \leq \|\mathbf{u}(t)\|_2$ and estimates (3.5), we get

$$\begin{aligned} &p_1 \; \in L^{\infty}(0,T;\, L^2(\Omega_R)), && p_{21} \in L^2(0,T;\, L^2(\Omega_R)), \\ &p_{22} \in L^{4/3}(0,T;\, L^2(\Omega_R)), && p_{23} \in L^2(0,T;\, L^2(\Omega_R)) \end{aligned} \tag{4.12}$$

for all $R>0$.

For a.a. $t\in(0,T)$, the functions $p_1(t)$ and $p_{21}(t)$ are harmonic in Ω. This follows from the identities

$$\int_\Omega p_1(t)\,\Delta\phi\;\mathrm{d}\mathbf{x} \;=\; -\big\langle\nabla p_1(t),\nabla\phi\big\rangle_\tau \;=\; \big\langle E_\tau^{-1,2}\mathbf{u}(t),\nabla\phi\big\rangle_\tau \;=\; \big\langle\mathbf{u}(t),E_\tau^{1,2}\nabla\phi\big\rangle_\tau$$
$$=\; \big\langle\mathbf{u}(t),\nabla\phi\big\rangle_\tau \;=\; \int_\Omega \mathbf{u}(t)\cdot\nabla\phi\;\mathrm{d}\mathbf{x} \;=\; 0 \quad \text{(for all } \phi\in C_0^\infty(\Omega)\text{)}.$$

(We have used (4.7).) Hence, by Weyl's lemma, $p_1(t)$ is a harmonic function in Ω. The fact that $p_{21}(t)$ is harmonic can be proved similarly.

Equation (4.9) is an equation in $\mathbf{W}_\tau^{-1,2}(\Omega)$. Applying successively each term in (4.9) to the function of the type $\boldsymbol{\varphi}(\mathbf{x})\,\eta(t)$, where $\boldsymbol{\varphi}\in\mathbf{W}_{\tau,c}(\Omega)$ and $\eta\in C_0^\infty(0,T)$, using formulas (4.10), and denoting $p_2:=p_{21}+p_{22}+p_{23}$, we obtain

$$\int_0^T\int_\Omega\big[-\mathbf{u}\cdot\boldsymbol{\varphi}\,\eta'(t)+\nu\nabla\mathbf{u}:\nabla\boldsymbol{\varphi}\,\eta(t)+\mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\varphi}\,\eta(t)\big]\;\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_0^T\int_\Omega\gamma\,\mathbf{u}\cdot\boldsymbol{\varphi}\,\eta(t)\;\mathrm{d}S\,\mathrm{d}t$$
$$=\int_0^T\langle\mathbf{f},\boldsymbol{\varphi}\rangle_\tau\,\eta(t)\;\mathrm{d}t-\int_0^T\int_\Omega p_1\,\mathrm{div}\,\boldsymbol{\varphi}\,\eta'(t)\;\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_0^T\int_\Omega p_2\,\mathrm{div}\,\boldsymbol{\varphi}\,\eta(t)\;\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

for all functions $\boldsymbol{\varphi}\in\mathbf{W}_{\tau,c}^{1,2}(\Omega)$ and $\eta\in C_0^\infty((0,T))$. Since the set of all finite linear combinations of functions of the type $\boldsymbol{\varphi}(\mathbf{x})\,\eta(t)$, where $\boldsymbol{\varphi}\in\mathbf{W}_{\tau,c}^{1,2}(\Omega)$ and $\eta\in C_0^\infty((0,T))$, is dense in $C_0^\infty\big((0,T);\,\mathbf{W}_{\tau,c}^{1,2}(\Omega)\big)$ in the norm of $W_0^{1,2}(0,T;\,\mathbf{W}_\tau^{1,2}(\Omega))$, we also obtain the equation

$$\int_0^T\int_\Omega\big[-\mathbf{u}\cdot\partial_t\boldsymbol{\phi}+\nu\nabla\mathbf{u}:\nabla\boldsymbol{\phi}+\mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\phi}\big]\;\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_0^T\int_\Omega\gamma\,\mathbf{u}\cdot\boldsymbol{\phi}\;\mathrm{d}S\,\mathrm{d}t$$
$$=\int_0^T\langle\mathbf{f},\boldsymbol{\phi}\rangle_\tau\;\mathrm{d}t-\int_0^T\int_\Omega p_1\,\mathrm{div}\,\partial_t\boldsymbol{\phi}\;\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_0^T\int_\Omega p_2\,\mathrm{div}\,\boldsymbol{\phi}\;\mathrm{d}\mathbf{x}\,\mathrm{d}t \tag{4.13}$$

for all $\boldsymbol{\phi}\in C_0^\infty\big((0,T);\,\mathbf{W}_{\tau,c}^{1,2}(\Omega)\big)$. Choosing particularly $\boldsymbol{\phi}\in\mathbf{C}_0^\infty(Q_T)$ and putting

$$p\;:=\;\partial_t p_1+p_2\;\equiv\;\partial_t p_1+p_{21}+p_{22}+p_{23} \tag{4.14}$$

(where $\partial_t p_1$ is the derivative in the sense of distributions), we observe that $(\mathbf{u},p)$ is a distributional solution of the system (1.1), (1.2) in Q_T.

The next theorem summarizes the results of this subsection:

Theorem 4.2. *Let $T>0$ and Ω be a locally Lipschitz domain in $\mathbb{R}^3$, satisfying condition (i) from subsection 1.1. Let $\mathbf{u}$ be a weak solution to the Navier-Stokes IBVP (1.1)–(1.4). Then there exists an associated pressure p in the form (4.14), where p_1, p_{21}, p_{22}, p_{23} satisfy (4.10)–(4.12). Moreover,*

1) *if $\Omega_0\subset\subset\Omega$ then the functions $p_1(t)$, $p_{21}(t)$, $p_{22}(t)$, $p_{32}(t)$ can be chosen so that they satisfy the additional conditions*

$$\int_{\Omega_0}p_1(t)\;\mathrm{d}\mathbf{x}\;=\;\int_{\Omega_0}p_{21}(t)\;\mathrm{d}\mathbf{x}\;=\;\int_{\Omega_0}p_{23}(t)\;\mathrm{d}\mathbf{x}\;=\;\int_{\Omega_0}p_{23}(t)\;\mathrm{d}\mathbf{x}\;=\;0,$$

2) *the functions $p_1(t)$ and $p_{21}(t)$ are harmonic in Ω for a.a. $t\in(0,T)$,*

3) *the functions* $\mathbf{u}$*,* p_1 *and* $p_2 \equiv p_{21} + p_{22} + p_{23}$ *satisfy the integral equation (4.13) for all test functions* $\boldsymbol{\phi} \in C_0^\infty\big((0,T;\ \mathbf{W}^{1,2}_{\tau,c}(\Omega)\big)$.

Note that if Ω is a bounded Lipschitz domain then the choice $\Omega_0 = \Omega$ is also permitted in statement 1) of Theorem 4.2.

5 The case of a smooth bounded domain Ω

5.1. Some results from paper [1]. In this section, we assume that Ω is a bounded domain in $\mathbb{R}^3$ with the boundary of the class C^2. We denote by A_q (for $1 < q < \infty$) the linear operator in $\mathbf{L}^q_{\tau,\sigma}(\Omega)$ with the domain defined by the equation

$$A_q\mathbf{v} \ := \ -\nu\, P_q \Delta\mathbf{v}$$

for $\mathbf{v} \in D(A_q)$, where

$$D(A_q) \ := \ \big\{\mathbf{v} \in \mathbf{W}^{2,q}(\Omega) \cap \mathbf{W}^{1,q}_{\tau,\sigma}(\Omega);\ [\mathbb{T}_{\mathrm{d}}(\mathbf{v}) \cdot \mathbf{n}]_\tau + \gamma\mathbf{v}_\tau = \mathbf{0} \text{ on } \partial\Omega\big\}$$

is the domain of operator A_q. Recall that $\mathbb{T}_{\mathrm{d}}(\mathbf{v}) \equiv 2\nu\mathbb{D}(\mathbf{v})$ is the dynamic stress tensor, induced by the vector field $\mathbf{v}$, and P_q is the Helmholtz projection in $\mathbf{L}^q(\Omega)$. Operator A_q is usually called the *Stokes operator* in $\mathbf{L}^q_{\tau,\sigma}(\Omega)$. Particularly, if $q = 2$ then A_2 coincides with the restriction of operator $\mathcal{A}$, defined in subsection 3.2, to $D(A_2)$. It is shown in the paper [1] by Ch. Amrouche, M. Escobedo and A. Ghosh that $(-A_q)$ generates a bounded analytic semigroup $\mathrm{e}^{-A_q t}$ in $\mathbf{L}^q_{\tau,\sigma}(\Omega)$. The next lemma also comes from [1], see [1, Theorem 1.3]. It concerns the solution of the inhomogeneous non–steady Stokes problem, given by the equations

$$\partial_t\mathbf{u} + \nabla\pi \ = \ \nu\Delta\mathbf{u} + \mathbf{g} \tag{5.1}$$

and (1.2) (in Q_T), by the boundary conditions (1.3) and by the initial condition (1.4). The initial velocity $\mathbf{u}_0$ is supposed to be from the space $\mathbf{E}^q_r(\Omega)$, which is defined to be the real interpolation space $[D(A_q),\ \mathbf{L}^q_{\tau,\sigma}(\Omega)]_{1/r,r}$. The problem (5.1), (1.2)–(1.3) can also be equivalently written in the form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + A_q\mathbf{u} \ = \ \mathbf{g}, \qquad \mathbf{u}(0) = \mathbf{u}_0, \tag{5.2}$$

which is the initial–value problem in $\mathbf{L}^q_{\tau,\sigma}(\Omega)$. Although the pressure π does not explicitly appear in (5.2), it can be always reconstructed in the way described in section 4.) The lemma says:

Lemma 5.1. *Let* $r, q \in (1,\infty)$*,* $T > 0$*,* $\mathbf{g} \in L^r(0,T;\ \mathbf{L}^q_{\tau,\sigma}(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{E}^q_r(\Omega)$*. Then the Stokes problem (5.1), (1.2), (1.3), (1.4) has a unique solution* $(\mathbf{u},\pi)$ *in* $\big[W^{1,r}(0,T;\ \mathbf{L}^q_{\tau,\sigma}(\Omega)) \cap L^r(0,T;\ \mathbf{W}^{2,q}(\Omega))\big] \times L^r(0,T;\ W^{1,q}(\Omega)/\mathbb{R})$*. The solution satisfies the estimate*

$$\int_0^T \|\partial_t\mathbf{u}\|_q^r\, \mathrm{d}t + \int_0^T \|\mathbf{u}\|_{2,q}^r\, \mathrm{d}t + \int_0^T \|\pi\|_{1,q}^r\, \mathrm{d}t \ \leq \ C\left(\int_0^T \|\mathbf{g}\|_q^r\, \mathrm{d}t + \|\mathbf{u}_0\|^r_{\mathbf{E}^q_r(\Omega)}\right). \tag{5.3}$$

The proof is based on a more general theorem from the paper [13] by Y. Giga and H. Sohr.

5.2. Application of Lemma 5.1. If $\mathbf{u}$ is a weak solution to the problem (1.1)–(1.4) then, since $\mathbf{u} \in L^\infty(0,T;\ \mathbf{L}^2_{\tau,\sigma}(\Omega)) \cap L^2(0,T;\ \mathbf{W}^{1,2}_{\tau,\sigma}(\Omega))$, one can verify that $\mathbf{u}\cdot\nabla\mathbf{u} \in L^r(0,T;\ \mathbf{L}^q(\Omega))$ for all $1 \le r \le 2$, $1 \le q \le \frac{3}{2}$, satisfying $2/r+3/q = 4$. In order to be consistent with the assumptions of Lemma 5.1 regarding q and r, assume that $1 < q < \frac{3}{2}$, $1 < r < 2$ and $2/r + 3/q = 4$. Furthermore, assume that $\mathbf{u}_0 \in \mathbf{E}^q_r(\Omega) \cap \mathbf{L}^2_{\tau,\sigma}(\Omega)$ and function $\mathbf{f}$ on the right hand side of equation (1.1) is in $L^r(0,T;\, \mathbf{L}^q(\Omega)) \cap L^2(0,T;\, \mathbf{W}^{-1,2}_\tau(\Omega))$. Put $\mathbf{g} := P_q\mathbf{f} - P_q(\mathbf{u}\cdot\nabla\mathbf{u})$. Then, due to the boundedness of projection P_q in $\mathbf{L}^q(\Omega)$, $\mathbf{g} \in L^r(0,T;\, \mathbf{L}^q_{\tau,\sigma}(\Omega))$. Assume, moreover, that $\mathbf{u}_0 \in \mathbf{E}^q_r(\Omega)$. Now, we are in a position that we can apply Lemma 5.1 and deduce that the linear Stokes problem (5.1), (1.2)–(1.4) has a unique solution $(\mathbf{U},\pi) \in \big[W^{1,r}(0,T;\, \mathbf{L}^q_{\tau,\sigma}(\Omega)) \cap L^r(0,T;\, \mathbf{W}^{2,q}(\Omega))\big] \times L^r(0,T;\, W^{1,q}(\Omega)/\mathbb{R})$, satisfying estimate (5.3) with $\mathbf{U}$ instead of $\mathbf{u}$. In order to show that the weak solution $\mathbf{u}$ of the nonlinear Navier–Stokes problem (1.1)–(1.4) satisfies the same estimate, too, we need to identify $\mathbf{u}$ with $\mathbf{U}$.

5.3. The identification of U and u. It is not obvious at the first sight that $\mathbf{U} = \mathbf{u}$, because while $\mathbf{U}$ is a unique solution of the problem (5.1), (1.2)–(1.4) in the class $W^{1,r}(0,T;\, \mathbf{L}^q_{\tau,\sigma}(\Omega)) \cap L^r(0,T;\, \mathbf{W}^{2,q}(\Omega))$, $\mathbf{u}$ is only known to be in $L^\infty(0,T;\, \mathbf{L}^2_{\tau,\sigma}(\Omega)) \cap L^2(0,T;\, \mathbf{W}^{1,2}_{\tau,\sigma}(\Omega))$. Nevertheless, applying the so called Yosida approximation of the identity operator in $\mathbf{L}^q_{\tau,\sigma}(\Omega)$, defined by the formula $J^{(k)}_q := (I + k^{-1}A_q)^{-1}$ (for $k \in \mathbb{N}$), in the same spirit as in [13] or [31], the equality $\mathbf{U} = \mathbf{u}$ can be established. We explain the main steps of the procedure in greater detail in the rest of this subsection.

At first, one can deduce from [1, Section 3] that the spectrum of A_q is a subset of the interval $(0,\infty)$ on the real axis, which implies that $J^{(k)}_q$ is a bounded operator on $\mathbf{L}^q_{\tau,\sigma}(\Omega)$ with values in $D(A_q)$. Obviously, $J^{(k)}_q$ commutes with A_q and with $J^{(m)}_q$ (for $k,m \in \mathbb{N}$, $k \ne m$) and $J^{(k)}_q = J^{(k)}_s$ on $\mathbf{L}^q_{\tau,\sigma}(\Omega) \cap \mathbf{L}^s_{\tau,\sigma}(\Omega)$ (for $1 < s < \infty$). If $q = 2$ then A_2 is a positive selfadjoint operator in $\mathbf{L}^2_{\tau,\sigma}$, see [4]. Consequently, $J^{(k)}_2$ is a selfadjoint operator in $\mathbf{L}^2_{\tau,\sigma}(\Omega)$, too. Finally, it is proven in [36, p. 246] that $J^{(k)}_q\mathbf{v} \to \mathbf{v}$ strongly in $\mathbf{L}^q_{\tau,\sigma}(\Omega)$ for all $\mathbf{v} \in \mathbf{L}^q_{\tau,\sigma}(\Omega)$ and $k \to \infty$.

Consider (3.1) with $\boldsymbol{\phi}(\mathbf{x},t) = [J^{(k)}_q\mathbf{w}](\mathbf{x})\,\vartheta(t)$, where $k \in \mathbb{N}$, $\mathbf{w} \in \mathbf{C}^\infty_{0,\sigma}(\Omega)$ and $\vartheta \in C^\infty_0\big([0,T)\big)$. In this case, (3.1) yields

$$\begin{aligned}
\int_0^T \int_\Omega \big[-\mathbf{u}\cdot J^{(k)}_q\mathbf{w}\,\vartheta' + (\mathbf{u}\cdot\nabla\mathbf{u})\cdot J^{(k)}_q\mathbf{w}\,\vartheta + 2\nu(\nabla\mathbf{u})_s : (\nabla J^{(k)}_q\mathbf{w})_s\big]\,\vartheta\,\mathrm{d}\mathbf{x}\,\mathrm{d}t& \\
+ \int_0^T \int_{\partial\Omega} \gamma\mathbf{u}\cdot J^{(k)}_q\mathbf{w}\,\vartheta\,\mathrm{d}S\,\mathrm{d}t& \\
= \int_0^T \int_\Omega \mathbf{f}\cdot J^{(k)}_q\mathbf{w}\,\vartheta\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_\Omega \mathbf{u}_0\cdot J^{(k)}_q\mathbf{w}\,\vartheta(0)\,\mathrm{d}\mathbf{x}.& \qquad (5.4)
\end{aligned}$$

The integral of $(\mathbf{u}\cdot\nabla\mathbf{u})\cdot J^{(k)}_q\mathbf{w}$ in Ω can be rewritten as follows:

$$\begin{aligned}
\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot J^{(k)}_q\mathbf{w}\,\mathrm{d}\mathbf{x} &= \int_\Omega P_q(\mathbf{u}\cdot\nabla\mathbf{u})\cdot J^{(k)}_q\mathbf{w}\,\mathrm{d}\mathbf{x} = \int_\Omega P_q(\mathbf{u}\cdot\nabla\mathbf{u})\cdot J^{(k)}_2\mathbf{w}\,\mathrm{d}\mathbf{x} \\
&= \lim_{m\to\infty} \int_\Omega J^{(m)}_q P_q(\mathbf{u}\cdot\nabla\mathbf{u})\cdot J^{(k)}_2\mathbf{w}\,\mathrm{d}\mathbf{x} = \lim_{m\to\infty} \int_\Omega J^{(k)}_2 J^{(m)}_q P_q(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{w}\,\mathrm{d}\mathbf{x} \\
&= \lim_{m\to\infty} \int_\Omega J^{(k)}_q J^{(m)}_q P_q(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{w}\,\mathrm{d}\mathbf{x} = \lim_{m\to\infty} \int_\Omega J^{(m)}_q J^{(k)}_q P_q(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{w}\,\mathrm{d}\mathbf{x} \\
&= \int_\Omega J^{(k)}_q P_q(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{w}\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

This shows, except others, that the integrals of $\mathbf{v}_1\cdot J^{(k)}_q\mathbf{v}_2$ and $J^{(k)}_q\mathbf{v}_1\cdot\mathbf{v}_2$ in Ω are equal for $\mathbf{v}_1,\,\mathbf{v}_2 \in \mathbf{L}^q_{\tau,\sigma}(\Omega)$. The integrals of $2\nu(\nabla\mathbf{u})_s : (\nabla J^{(k)}_q\mathbf{w})_s$ and $\gamma\mathbf{u}\cdot J^{(k)}_q\mathbf{w}$ over Ω and $\partial\Omega$, respectively, can be modified by means of the identities:

$$\begin{aligned}
&\int_\Omega 2\nu(\nabla\mathbf{u})_s : (\nabla J_q^{(k)}\mathbf{w})_s \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \gamma\mathbf{u}\cdot J_q^{(k)}\mathbf{w} \, \mathrm{d}S \\
&\quad = \int_\Omega 2\nu\nabla\mathbf{u} : (\nabla J_q^{(k)}\mathbf{w})_s \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \gamma\mathbf{u}\cdot J_q^{(k)}\mathbf{w} \, \mathrm{d}S \\
&\quad = \int_{\partial\Omega} 2\nu\mathbf{u}\cdot[(\nabla J_q^{(k)}\mathbf{w})_s\cdot\mathbf{n}] \, \mathrm{d}S - \int_\Omega \nu\mathbf{u}\cdot\Delta J_q^{(k)}\mathbf{w} \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \gamma\mathbf{u}\cdot J_q^{(k)}\mathbf{w} \, \mathrm{d}S \\
&\quad = -\int_\Omega \nu\mathbf{u}\cdot\Delta J_q^{(k)}\mathbf{w} \, \mathrm{d}\mathbf{x} \;=\; \int_\Omega \mathbf{u}\cdot A_q J_q^{(k)}\mathbf{w} \, \mathrm{d}\mathbf{x} \;=\; \int_\Omega A_q\mathbf{u}\cdot J_q^{(k)}\mathbf{w} \, \mathrm{d}\mathbf{x} \\
&\quad = -\int_\Omega J^{(k)} A_q\mathbf{u}\cdot\mathbf{w} \, \mathrm{d}\mathbf{x} \;=\; -\int_\Omega A_q J^{(k)}\mathbf{u}\cdot\mathbf{w} \, \mathrm{d}\mathbf{x}.
\end{aligned}$$

Thus, we obtain from (5.4):

$$\begin{aligned}
&\int_0^T \int_\Omega \big[-J_q^{(k)}\mathbf{u}\cdot\mathbf{w}\,\vartheta' + J_q^{(k)} P_q(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{w}\,\vartheta - \nu A_q J_q^{(k)}\mathbf{u}\cdot\mathbf{w}\big]\,\vartheta \, \mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\qquad\qquad = \int_0^T \int_\Omega J_q^{(k)}\mathbf{f}\cdot\mathbf{w}\,\vartheta \, \mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_\Omega J_q^{(k)}\mathbf{u}_0\cdot\mathbf{w}\,\vartheta(0) \, \mathrm{d}\mathbf{x}.
\end{aligned}$$

As $\mathbf{w}$ and ϑ are arbitrary functions from $\mathbf{C}^\infty_{0,\sigma}(\Omega)$ and $C_0^\infty\big([0,T)\big)$, respectively, this shows that $J_q^{(k)}\mathbf{u}$ is a solution of the initial–value problem

$$(J_q^{(k)}\mathbf{u})' + A_q J_q^{(k)}\mathbf{u} \;=\; J_q^{(k)}\mathbf{g}, \qquad J_q^{(k)}\mathbf{u}(\,.\,,0) = J_q^{(k)}\mathbf{u}_0 \tag{5.5}$$

(which is a problem in $\mathbf{L}^q_{\tau,\sigma}(\Omega)$) in the class $W^{1,r}(0,T;\,\mathbf{L}^q_{\tau,\sigma}(\Omega)) \cap L^r(0,T;\,\mathbf{W}^{2,q}(\Omega))$. Since $J_q^{(k)}\mathbf{U}$ solves the same problem and belongs to the same class, we obtain the identity $J_q^{(k)}\mathbf{U}(t) = J_q^{(k)}\mathbf{u}(t)$ for a.a. $t\in(0,T)$. Consequently, $\mathbf{U}(t) = \mathbf{u}(t)$ for a.a. $t\in(0,T)$.

5.4. The estimate of $\mathbf{u}$ and an associated pressure p. Since $\mathbf{g} = P_q\mathbf{f} - P_q(\mathbf{u}\cdot\nabla\mathbf{u})$, we can also write equation (5.1) in the form

$$\begin{aligned}
\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} \;&=\; -\nabla\pi + \nu\Delta\mathbf{u} + \mathbf{f} + (I-P_q)(-\mathbf{f} + \mathbf{u}\cdot\nabla\mathbf{u}) \\
&=\; -\nabla(\pi+\zeta) + \nu\Delta\mathbf{u} + \mathbf{f},
\end{aligned}$$

where $\nabla\zeta = (I-P_q)(\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{f})$. (The fact that $(I-P_q)(\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{f})$ can be expressed in the form $\nabla\zeta$ follows e.g. from [10, section III.1].) We observe that $p := \pi+\zeta$ is a pressure, associated with the weak solution $\mathbf{u}$. Since the pair $(\mathbf{U},\pi)$ satisfies (5.3), $\mathbf{u}$ and p satisfy the analogous estimate

$$\begin{aligned}
&\int_0^T \|\partial_t\mathbf{u}\|_q^r \, \mathrm{d}t + \int_0^T \|\mathbf{u}\|_{2,q}^r \, \mathrm{d}t + \int_0^T \|p\|_{1,q}^r \, \mathrm{d}t \\
&\qquad \le\; C\int_0^T \big(\|\mathbf{f}\|_q^r + \|P_q(\mathbf{u}\cdot\nabla\mathbf{u})\|_q^r\big) \, \mathrm{d}t + C\,\|\mathbf{u}_0\|^r_{\mathbf{E}^q_r(\Omega)}.
\end{aligned} \tag{5.6}$$

We have proven the theorem:

Theorem 5.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with the boundary of the class C^2 and $T>0$. Let $1<q<\frac{3}{2}$, $1<r<2$, $2/r+3/q=4$, $\mathbf{u}_0\in\mathbf{E}^q_r(\Omega)\cap\mathbf{L}^2_{\tau,\sigma}(\Omega)$ and $\mathbf{f}\in L^r(0,T;\,\mathbf{L}^q(\Omega))\cap L^2(0,T;\,\mathbf{L}^2(\Omega))$. Let $\mathbf{u}$ be a weak solution to the Navier-Stokes IBVP (1.1)–(1.4) and p be an associated pressure. Then $\mathbf{u}\in L^r(0,T;\;\mathbf{W}^{2,q}(\Omega))\cap W^{1,r}(0,T;\,\mathbf{L}^q(\Omega))$ and p can be identified with a function from $L^r(0,T;\,L^{3q/(3-q)}(\Omega))$. The functions $\mathbf{u}$, p satisfy equations (1.1), (1.2) a.e. in Q_T and the boundary conditions (1.3) a.e. in Γ_T. Moreover, they also satisfy estimate (5.6).*

6 An interior regularity of the associated pressure

6.1. On previous results on the interior regularity of velocity and pressure. The next lemma recalls the well known Serrin's result on the interior regularity of weak solutions to the system (1.1), (1.2). (See e.g. [23], [27] or [11].) It concerns weak solutions in $\Omega_1 \times (t_1, t_2)$, where Ω_1 is a sub-domain of Ω, independently of boundary conditions on
$Gamma_T$.

Lemma 6.1. *Let Ω_1 be a sub-domain of Ω, $0 \leq t_1 < t_2 \leq T$ and let $\mathbf{u}$ be a weak solution to the system (1.1), (1.2) with $\mathbf{f} = \mathbf{0}$ in $\Omega_1 \times (t_1, t_2)$. Let $\mathbf{u} \in L^r(t_1, t_2; \mathbf{L}^s(\Omega_1))$, where $r \in [2, \infty)$, $s \in (3, \infty]$ and $2/r + 3/s = 1$. Then, if $\Omega_2 \subset\subset \Omega_1$ and $0 < 2\epsilon < t_2 - t_1$, solution $\mathbf{u}$ has all spatial derivatives (of all orders) bounded in $\Omega_2 \times (t_1 + \epsilon, t_2 - \epsilon)$.*

Note that Lemma 6.1 uses no assumptions on boundary conditions, satisfied by $\mathbf{u}$ on $\partial\Omega \times (0, T)$. The assumption that $\mathbf{u}$ is a weak solution to the system (1.1), (1.2) in $\Omega_1 \times (t_1, t_2)$ means that $\mathbf{u} \in L^\infty(t_1, t_2; \mathbf{L}^\infty(\Omega_1)) \cap L^2(t_1, t_2; \mathbf{W}^{1,2}(\Omega_1))$, $\operatorname{div} \mathbf{u} = 0$ holds in the sense of distributions in $\Omega_1 \times (t_1, t_2)$ and $\mathbf{u}$ satisfies (3.1) for all infinitely differentiable divergence–free test functions $\boldsymbol{\phi}$ that have a compact support in $\Omega_1 \times (t_1, t_2)$. (Then the last integral on the left hand side and both integrals on the right hand side are equal to zero.) Also note that applying the results of [26], one can add to the conclusions of Lemma 6.1 that $\mathbf{u}$ is Hölder–continuous in $\Omega_2 \times (t_1 + \epsilon, t_2 - \epsilon)$. Lemma 6.1 provides no information on the associated pressure p or the time derivative $\partial_t \mathbf{u}$ in $\Omega_2 \times (t_1 + \epsilon, t_2 - \epsilon)$. The known results on the regularity of $\mathbf{u}$ and ∂_t in $\Omega_2 \times (t_1 + \epsilon, t_2 - \epsilon)$, under the assumptions that $\mathbf{u}$ is a weak solution of (1.1), (1.2) in $\Omega \times (t_1, t_2)$ satisfying the conditions formulated in Lemma 6.1 in $\Omega_1 \times (t_1, t_2)$, say:

a) If $\Omega = \mathbb{R}^3$ then p, $\partial_t \mathbf{u}$ and all their spatial derivatives (of all orders) are in $L^\infty(\Omega_2 \times (t_1 + \epsilon, t_2 - \epsilon)$, see [18], [22] [30].

b) If Ω is a bounded or exterior domain $\mathbb{R}^3$ with the boundary of the class $C^{2+(h)}$ for some $h > 0$ and $\mathbf{u}$ satisfies the no–slip boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega \times (0, T)$ then p and $\partial_t \mathbf{u}$ have all spatial derivatives (of all orders) in $L^q(t_1 + \epsilon, t_2 - \epsilon;\, L^\infty(\Omega_2))$ for any $q \in (1, 2)$, see [19], [18], [22] or [30].

c) If Ω is a bounded domain $\mathbb{R}^3$ with the boundary of the class $C^{2+(h)}$ for some $h > 0$ and $\mathbf{u}$ satisfies the Navier–type boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = 0, \qquad \mathbf{curl}\; \mathbf{u} \times \mathbf{n} = \mathbf{0} \qquad \text{on } \partial\Omega \times (t_1, t_2)$$

then p and $\partial_t \mathbf{u}$ have the same regularity in $\Omega_2 \times (t_1 + \epsilon, t_2 - \epsilon)$ as stated in item a), see [21].

In the proofs, it is always sufficient to show that the aforementioned statements hold for p. The same statements on $\partial_t \mathbf{u}$ follow from the fact that ∇p and $\partial_t \mathbf{u}$ are interconnected through the Navier–Stokes equation (1.1).

6.2. An interior regularity of p in case of Navier's boundary conditions. We further assume that Ω and T are as in Theorem 5.1 and $\mathbf{f} = \mathbf{0}$. The main result of this section says:

Theorem 6.1. *Let Ω and T be as in Theorem 5.1 and $\mathbf{f} = \mathbf{0}$. Let $\mathbf{u}$ be a weak solution to the problem (1.1)–(1.4). Let Ω_1 be a sub-domain of Ω, $0 < t_1 < t_2 \leq T$ and let $\mathbf{u} \in L^r(t_1, t_2; \mathbf{L}^s(\Omega_1))$, where $r \in [2, \infty)$, $s \in (3, \infty]$ and $2/r + 3/s = 1$. Finally, let $\Omega_3 \subset\subset \Omega_1$ and $0 < \epsilon < t_2 - t_1$. Then p can be chosen so that all its spatial derivatives (of all orders) are in $L^4(t_1 + \epsilon, t_2 - \epsilon;\, L^\infty(\Omega_3))$. Similarly, $\partial_t \mathbf{u}$ and all its spatial derivatives (of all orders) are in $L^4(t_1 + \epsilon, t_2 - \epsilon;\, \mathbf{L}^\infty(\Omega_3))$.*

Proof. There exists $t_* \in (0, t_1)$ such that $\mathbf{u}(\,.\,, t_*) \in \mathbf{W}^{1,2}_{\tau,\sigma}(\Omega) \subset \mathbf{E}^q_r(\Omega)$ for all r and q, considered in Theorem 5.1. Hence $\mathbf{u} \in L^r(t_*, T;\ \mathbf{W}^{2,q}(\Omega)) \cap W^{1,r}(t_*, T;\ \mathbf{L}^q(\Omega))$ and p can be chosen so that $p \in L^r(t_*, T;\ L^{3q/(3-q)}(\Omega))$. Let ϵ and Ω_2 be the number and domain, respectively, given by Lemma 6.1. We may assume that Ω_2 and Ω_3 are chosen so that $\emptyset \neq \Omega_3 \subset\subset \Omega_2 \subset\subset \Omega$.

Applying the operator of divergence to equation (1.1), we obtain the equation

$$\Delta p \ = \ -\nabla\mathbf{u} : (\nabla\mathbf{u})^T, \tag{6.1}$$

which holds in the sense of of distributions in Q_T. Taking into account that p is at least locally integrable in $\Omega_1 \times (t_1, t_2)$, we obtain from (6.1) that

$$\int_{t_1}^{t_2} \theta(t) \int_{\Omega_1} \big[p\,\Delta\varphi(\mathbf{x}) + \nabla\mathbf{u} : (\nabla\mathbf{u})^T\,\varphi(\mathbf{x})\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \ = \ 0$$

for all $\theta \in C_0^\infty((t_1, t_2))$ and $\varphi \in C_0^\infty(\Omega_1)$. From this, we deduce that equation (6.1) holds in Ω_1 in the sense of distributions at a.a. fixed time instants $t \in (t_1 + \epsilon, t_2 - \epsilon)$. Let further t be one of these time instants and let t be also chosen so that $\mathbf{u}(\,.\,, t) \in \mathbf{W}^{2,q}(\Omega)$, $\partial_t\mathbf{u}(\,.\,, t) \in \mathbf{L}^q(\Omega)$ and $p(\,.\,, t) \in L^{3q/(3-q)}(\Omega)$. As $p(\,.\,, t) \in L^1_{loc}(\Omega_1)$ and the right hand side of (6.1) (at the fixed time t) is infinitely differentiable in the spatial variable in Ω_2, the function $p(\,.\,, t)$ is also infinitely differentiable in Ω_2, see e.g. [8].

Let $\mathbf{x}_0 \in \Omega_3$ and $0 < \rho_1 < \rho_2$ be so small that $B_{\rho_2}(\mathbf{x}_0) \subset \Omega_2$. Define an infinitely differentiable non-increasing cut–off function η in $[0, \infty)$ by the formula

$$\eta(\sigma) \begin{cases} = 1 & \text{for } 0 \leq \sigma \leq \rho_1, \\ \in (0,1) & \text{for } \rho_1 < \sigma < \rho_2, \\ = 0 & \text{for } \rho_2 \leq \sigma. \end{cases}$$

Let $\mathbf{x} \in B_{\rho_1}(\mathbf{x}_0)$ and $\mathbf{e}$ be a constant unit vector in $\mathbb{R}^3$. Then

$$\begin{aligned} \nabla_{\mathbf{x}} p(\mathbf{x}, t) \cdot \mathbf{e} \ &= \ \eta\big(|\mathbf{x} - \mathbf{x}_0|\big)\, \nabla_{\mathbf{x}} p(\mathbf{x}, t) \cdot \mathbf{e} \\ &= \ -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{y} - \mathbf{x}|}\, \Delta_{\mathbf{y}} \big[\eta\big(|\mathbf{y} - \mathbf{x}_0|\big)\, \nabla_{\mathbf{y}} p(\mathbf{y}, t) \cdot \mathbf{e}\big]\, \mathrm{d}\mathbf{y}. \end{aligned}$$

Particularly, this also holds for $\mathbf{x} = \mathbf{x}_0$:

$$\begin{aligned} \nabla_{\mathbf{x}} p(\mathbf{x}, t) \cdot \mathbf{e}\,\big|_{\mathbf{x}=\mathbf{x}_0} \ &= \ -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{y} - \mathbf{x}_0|}\, \Delta_{\mathbf{y}} \big[\eta\big(|\mathbf{y} - \mathbf{x}_0|\big)\, \nabla_{\mathbf{y}} p(\mathbf{y}, t) \cdot \mathbf{e}\big]\, \mathrm{d}\mathbf{y} \\ &= \ -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{y}|}\, \Delta_{\mathbf{y}} \big[\eta\big(|\mathbf{y}|\big)\, \nabla_{\mathbf{y}} p(\mathbf{x}_0 + \mathbf{y}, t) \cdot \mathbf{e}\big]\, \mathrm{d}\mathbf{y} \\ &= \ -\frac{1}{4\pi} \big[P^{(1)}(\mathbf{x}_0) + 2P^{(2)}(\mathbf{x}_0) + P^{(3)}(\mathbf{x}_0)\big], \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} P^{(1)}(\mathbf{x}_0) \ &= \ \int_{B_{\rho_2}(\mathbf{0})} \frac{1}{|\mathbf{y}|}\, \Delta_{\mathbf{y}} \eta\big(|\mathbf{y}|\big)\, \big[\nabla_{\mathbf{y}} p(\mathbf{x}_0 + \mathbf{y}, t) \cdot \mathbf{e}\big]\, \mathrm{d}\mathbf{y}, \\ P^{(2)}(\mathbf{x}_0) \ &= \ \int_{B_{\rho_2}(\mathbf{0})} \frac{1}{|\mathbf{y}|}\, \nabla_{\mathbf{y}} \eta\big(|\mathbf{y}|\big) \cdot \nabla_{\mathbf{y}} \big[\nabla_{\mathbf{y}} p(\mathbf{x}_0 + \mathbf{y}, t) \cdot \mathbf{e}\big]\, \mathrm{d}\mathbf{y}, \end{aligned}$$

$$P^{(3)}(\mathbf{x}_0) \ = \ \int_{B_{\rho_2}(\mathbf{0})} \frac{\eta\bigl(|\mathbf{y}|\bigr)}{|\mathbf{y}|}\, \Delta_{\mathbf{y}} \bigl[\nabla_{\mathbf{y}} p(\mathbf{x}_0+\mathbf{y},t)\cdot\mathbf{e}\bigr]\; \mathrm{d}\mathbf{y}.$$

The estimate of $P^{(3)}(\mathbf{x}_0)$. The estimate of the last term is easy:

$$\begin{aligned} \bigl|P^{(3)}(\mathbf{x}_0)\bigr| \ &= \ \biggl| \int_{B_{\rho_2}(\mathbf{0})} \Bigl(\nabla_{\mathbf{y}}\, \frac{\eta\bigl(|\mathbf{y}|\bigr)}{|\mathbf{y}|}\cdot\mathbf{e}\Bigr)\, \Delta_{\mathbf{y}} p(\mathbf{x}_0+\mathbf{y},t)\; \mathrm{d}\mathbf{y}\biggr| \\ &= \ \biggl| \int_{B_{\rho_2}(\mathbf{0})} \Bigl(\nabla_{\mathbf{y}}\, \frac{\eta\bigl(|\mathbf{y}|\bigr)}{|\mathbf{y}|}\cdot\mathbf{e}\Bigr)\, \bigl[\nabla_{\mathbf{y}}\mathbf{u}(\mathbf{x}_0+\mathbf{y},t) : \bigl(\nabla_{\mathbf{y}}\mathbf{u}(\mathbf{x}_0+\mathbf{y},t)\bigr)^T\bigr]\; \mathrm{d}\mathbf{y}\biggr| \\ &\leq \ c \int_{B_{\rho_2}(\mathbf{0})} \biggl|\nabla_{\mathbf{y}}\, \frac{\eta\bigl(|\mathbf{y}|\bigr)}{|\mathbf{y}|}\cdot\mathbf{e}\biggr|\; \mathrm{d}\mathbf{y} \ \leq \ c. \end{aligned} \tag{6.3}$$

The estimate of $P^{(2)}(\mathbf{x}_0)$. We can write

$$\frac{1}{|\mathbf{y}|}\, \nabla_{\mathbf{y}} \eta\bigl(|\mathbf{y}|\bigr) \ = \ \nabla_{\mathbf{y}} \mathcal{F}\bigl(|\mathbf{y}|\bigr),$$

where $\mathcal{F}(s) := -\int_s^\infty \eta'(\sigma)/\sigma \ \mathrm{d}\sigma$ for $s \geq 0$. We observe that $\mathcal{F}$ is constant on $[0,\rho_1]$, equal to zero on $[\rho_2,\infty)$ and $\mathcal{F}'(s) = \eta'(s)/s$ for $s > 0$. Thus, we have

$$\begin{aligned} \bigl|P^{(2)}(\mathbf{x}_0)\bigr| \ &= \ \biggl| \int_{B_{\rho_2}(\mathbf{0})} \nabla_{\mathbf{y}} \mathcal{F}\bigl(|\mathbf{y}|\bigr)\cdot \nabla_{\mathbf{y}} \bigl[\nabla_{\mathbf{y}} p(\mathbf{x}_0+\mathbf{y},t)\cdot\mathbf{e}\bigr]\; \mathrm{d}\mathbf{y}\biggr| \\ &= \ \biggl| \int_{B_{\rho_2}(\mathbf{0})} \Delta_{\mathbf{y}} \mathcal{F}\bigl(|\mathbf{y}|\bigr)\, \mathbf{e}\cdot \nabla_{\mathbf{y}} p(\mathbf{x}_0+\mathbf{y},t)\; \mathrm{d}\mathbf{y}\biggr|. \end{aligned} \tag{6.4}$$

The vector function $\Delta_{\mathbf{y}} \mathcal{F}\bigl(|\mathbf{y}|\bigr)\, \mathbf{e}$ can be written in the form

$$\Delta_{\mathbf{y}} \mathcal{F}\bigl(|\mathbf{y}|\bigr)\, \mathbf{e} \ = \ \nabla_{\mathbf{y}} \varphi(\mathbf{y}) + \mathbf{w}(\mathbf{y}), \tag{6.5}$$

where

$$\varphi(\mathbf{y}) = \nabla_{\mathbf{y}} \mathcal{F}\bigl(|\mathbf{y}|\bigr)\cdot\mathbf{e}, \qquad \mathbf{w}(\mathbf{y}) = \Delta_{\mathbf{y}} \mathcal{F}\bigl(|\mathbf{y}|\bigr)\, \mathbf{e} - \nabla_{\mathbf{y}} \bigl[\nabla_{\mathbf{y}} \mathcal{F}\bigl(|\mathbf{y}|\bigr)\cdot\mathbf{e}\bigr].$$

The functions φ and $\mathbf{w}$ are infinitely differentiable in $\mathbb{R}^3$ and $\varphi = 0$, $\mathbf{w} = \mathbf{0}$ in $\mathbb{R}^3 \smallsetminus B_{\rho_2}(\mathbf{0})$. Since

$$\operatorname{div} \mathbf{w} \ = \ \nabla_{\mathbf{y}} \Delta_{\mathbf{y}} \mathcal{F}\bigl(|\mathbf{y}|\bigr)\cdot\mathbf{e} - \Delta_{\mathbf{y}} \bigl[\nabla_{\mathbf{y}} \mathcal{F}\bigl(|\mathbf{y}|\bigr)\cdot\mathbf{e}\bigr] \ = \ 0,$$

(6.5) in fact represents the Helmholtz decomposition of $\Delta_{\mathbf{y}} \mathcal{F}\bigl(|\mathbf{y}|\bigr)\, \mathbf{e}$ in $B_{\rho_2}(\mathbf{0})$. Substituting from (6.5) to (6.4), we obtain

$$\begin{aligned} \bigl|P^{(2)}(\mathbf{x}_0)\bigr| \ &= \ \biggl| \int_{B_{\rho_2}(\mathbf{0})} \bigl[\nabla_{\mathbf{y}} \varphi(\mathbf{y}) + \mathbf{w}(\mathbf{y})\bigr]\cdot \nabla_{\mathbf{y}} p(\mathbf{x}_0+\mathbf{y},t)\; \mathrm{d}\mathbf{y}\biggr| \\ &= \ \biggl| \int_{B_{\rho_2}(\mathbf{0})} \nabla_{\mathbf{y}} \varphi(\mathbf{y})\cdot \nabla_{\mathbf{y}} p(\mathbf{x}_0+\mathbf{y},t)\; \mathrm{d}\mathbf{y}\biggr| \\ &= \ \biggl| \int_{B_{\rho_2}(\mathbf{0})} \varphi(\mathbf{y})\, \Delta_{\mathbf{y}} p(\mathbf{x}_0+\mathbf{y},t)\; \mathrm{d}\mathbf{y}\biggr| \\ &= \ \biggl| \int_{B_{\rho_2}(\mathbf{0})} \varphi(\mathbf{y})\, \bigl[\nabla_{\mathbf{y}}\mathbf{u}(\mathbf{x}_0+\mathbf{y},t) : \bigl(\nabla_{\mathbf{y}}\mathbf{u}(\mathbf{x}_0+\mathbf{y},t)\bigr)^T\bigr]\; \mathrm{d}\mathbf{y}\biggr| \end{aligned}$$

$$\leq \int_{B_{\rho_2}(\mathbf{0})} |\varphi(\mathbf{y})| \, \mathrm{d}\mathbf{y} \;\leq\; c. \tag{6.6}$$

The estimate of $P^{(1)}(\mathbf{x}_0)$. Finally, we have

$$\begin{aligned} P^{(1)}(\mathbf{x}_0) \;&=\; \int_{B_{\rho_2}(\mathbf{0})} \frac{1}{|\mathbf{y}|} \nabla_{\mathbf{y}} \eta\big(|\mathbf{y}|\big) \cdot \nabla_{\mathbf{y}} \left[\nabla_{\mathbf{y}} p(\mathbf{x}_0+\mathbf{y},t) \cdot \mathbf{e}\right] \mathrm{d}\mathbf{y} \\ &\quad - \int_{B_{\rho_2}(\mathbf{0})} \Big[\frac{\mathbf{y}}{|\mathbf{y}|^3} \cdot \nabla_{\mathbf{y}} \eta\big(|\mathbf{y}|\big)\Big] \left[\nabla_{\mathbf{y}} p(\mathbf{x}_0+\mathbf{y},t) \cdot \mathbf{e}\right] \mathrm{d}\mathbf{y}. \end{aligned} \tag{6.7}$$

The first integral coincides with the integral in the formula for $P^{(2)}(\mathbf{x}_0)$ and it can be therefore treated in the same way. The second integral on the right hand side of (6.7) - let us denote it by $P_2^{(1)}(\mathbf{x}_0)$ - represents the main obstacle, which finally causes that p and all its spatial derivatives are only in $L^4(t_1+\epsilon, t_2-\epsilon;\, L^\infty(\Omega_3))$ and not in $L^\infty(t_1+\epsilon, t_2-\epsilon;\, L^\infty(\Omega_3))$, as in the cases from items a) and c) in subsection 6.1. The integral can be written in the form

$$\begin{aligned} P_2^{(1)}(\mathbf{x}_0) \;&=\; \int_{B_{\rho_2}(\mathbf{0})} \frac{\eta'\big(|\mathbf{y}|\big)}{|\mathbf{y}|^2} \, \mathbf{e} \cdot \nabla_{\mathbf{y}} p(\mathbf{x}_0+\mathbf{y},t) \, \mathrm{d}\mathbf{y} \\ &=\; \int_{\Omega} \frac{\eta'\big(|\mathbf{y}-\mathbf{x}_0|\big)}{|\mathbf{y}-\mathbf{x}_0|^2} \, \mathbf{e} \cdot \nabla_{\mathbf{y}} p(\mathbf{y},t) \, \mathrm{d}\mathbf{y}. \end{aligned} \tag{6.8}$$

Now, we use the Helmholtz decomposition

$$\frac{\eta'\big(|\mathbf{y}-\mathbf{x}_0|\big)}{|\mathbf{y}-\mathbf{x}_0|^2} \, \mathbf{e} \;=\; \nabla_{\mathbf{y}} \psi(\mathbf{y}) + \mathbf{z}(\mathbf{y}), \tag{6.9}$$

in the whole domain Ω, where

$$\begin{aligned} \Delta_{\mathbf{y}} \psi(\mathbf{y}) = \operatorname{div} \Big(\frac{\eta'\big(|\mathbf{y}-\mathbf{x}_0|\big)}{|\mathbf{y}-\mathbf{x}_0|^2} \, \mathbf{e}\Big) &= \Big(\frac{\eta''\big(|\mathbf{y}-\mathbf{x}_0|\big)}{|\mathbf{y}-\mathbf{x}_0|^3} - \frac{\eta'\big(|\mathbf{y}-\mathbf{x}_0|\big)}{|\mathbf{y}-\mathbf{x}_0|^4}\Big) (\mathbf{y}-\mathbf{x}_0) \cdot \mathbf{e} \quad \text{for } \mathbf{y} \in \Omega, \\ \frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{y}) \;&=\; 0 \qquad \text{for } \mathbf{y} \in \partial\Omega. \end{aligned}$$

As $\mathbf{z}$ is divergence–free and its normal component on $\partial\Omega$ is zero, and the integral of $\nabla\psi \cdot \partial_t \mathbf{u}$ is zero, we get

$$\begin{aligned} P_2^{(1)}(\mathbf{x}_0) \;&=\; \int_\Omega \big[\nabla_{\mathbf{y}} \psi(\mathbf{y}) + \mathbf{z}(\mathbf{y})\big] \cdot \nabla_{\mathbf{y}} p(\mathbf{y},t) \, \mathrm{d}\mathbf{y} \;=\; \int_\Omega \nabla_{\mathbf{y}} \psi(\mathbf{y}) \cdot \nabla_{\mathbf{y}} p(\mathbf{y},t) \, \mathrm{d}\mathbf{y} \\ &=\; \int_\Omega \nabla_{\mathbf{y}} \psi(\mathbf{y}) \cdot \big[\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u}\big](\mathbf{y},t) \, \mathrm{d}\mathbf{y} \\ &=\; \int_\Omega \nabla_{\mathbf{y}} \psi(\mathbf{y}) \cdot \big[\mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u}\big](\mathbf{y},t) \, \mathrm{d}\mathbf{y} \end{aligned} \tag{6.10}$$

We have

$$\begin{aligned} \bigg|\int_\Omega \nabla_{\mathbf{y}} \psi \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \, \mathrm{d}\mathbf{y}\bigg| \;&=\; \bigg|\int_\Omega \nabla_{\mathbf{y}}^2 \psi : (\mathbf{u} \otimes \mathbf{u}) \, \mathrm{d}\mathbf{y}\bigg| \;\leq\; c \int_\Omega |\mathbf{u}|^2 \, \mathrm{d}\mathbf{y} \;\leq\; c, \qquad (6.11) \\ \bigg|\int_\Omega \nabla_{\mathbf{y}} \psi \cdot \nu \Delta \mathbf{u} \, \mathrm{d}\mathbf{y}\bigg| \;&=\; \bigg|\int_\Omega \nabla_{\mathbf{y}} \psi \cdot \operatorname{div} \mathbb{T}_{\mathrm{d}}(\mathbf{u}) \, \mathrm{d}\mathbf{y}\bigg| \end{aligned}$$

$$
\begin{aligned}
&= \left| \int_{\partial\Omega} \nabla_{\mathbf{y}}\psi \cdot [\mathbb{T}_{\rm d}(\mathbf{u}) \cdot \mathbf{n}] \, {\rm d}S - \int_{\Omega} \nabla^2_{\mathbf{y}}\psi : \mathbb{T}_{\rm d}(\mathbf{u}) \, {\rm d}\mathbf{y} \right| \\
&= \left| - \int_{\partial\Omega} \nabla_{\mathbf{y}}\psi \cdot \gamma\mathbf{u} \, {\rm d}S - \int_{\Omega} \nabla^2_{\mathbf{y}}\psi : \nu(\nabla\mathbf{u})_s \, {\rm d}\mathbf{y} \right| \\
&\leq c \int_{\partial\Omega} |\mathbf{u}| \, {\rm d}S + \left| \int_{\Omega} \nabla^2_{\mathbf{y}}\psi : \nu\nabla\mathbf{u} \, {\rm d}\mathbf{y} \right| = c \int_{\partial\Omega} |\mathbf{u}| \, {\rm d}S + \left| \int_{\Omega} (\partial_i\partial_j\psi) \, \nu \, (\partial_j u_i) \, {\rm d}\mathbf{y} \right| \\
&= c \int_{\partial\Omega} |\mathbf{u}| \, {\rm d}S + \left| \int_{\partial\Omega} (\partial_j\psi) \, n_i \, \nu \, (\partial_j u_i) \, {\rm d}\mathbf{y} \right| \\
&= c \int_{\partial\Omega} |\mathbf{u}| \, {\rm d}S + \nu \left| \int_{\partial\Omega} (\partial_j\psi) \, [\partial_j(n_i u_i) - (\partial_j n_i) \, u_i] \, {\rm d}\mathbf{y} \right| \\
&= c \int_{\partial\Omega} |\mathbf{u}| \, {\rm d}S + \nu \left| \int_{\partial\Omega} (\partial_j\psi) \, (\partial_j n_i) \, u_i \, {\rm d}\mathbf{y} \right| \\
&\leq c \int_{\partial\Omega} |\mathbf{u}| \, {\rm d}S \leq c \left(\int_{\partial\Omega} |\mathbf{u}|^2 \, {\rm d}S \right)^{1/2} \leq c \left(\|\mathbf{u}\|_2 + \|\mathbf{u}\|_2^{1/2} \, \|\mathbf{u}\|_{1,2}^{1/2} \right) \\
&\leq c + c \, \|\mathbf{u}\|_{1,2}^{1/2}. \qquad (6.12)
\end{aligned}
$$

The right hand side is in $L^4(t_1 + \epsilon, t_2 - \epsilon)$. We have used the estimate

$$
\big| \nabla\psi \big|_{1+(h)} \leq c \left| \left(\frac{\eta''(|\mathbf{y} - \mathbf{x}_0|)}{|\mathbf{y} - \mathbf{x}_0|^3} - \frac{\eta'(|\mathbf{y} - \mathbf{x}_0|)}{|\mathbf{y} - \mathbf{x}_0|^4} \right) (\mathbf{y} - \mathbf{x}_0) \cdot \mathbf{e} \right|_{0+(h)} \leq c,
$$

where $| \, . \, |_{1+(h)}$ and $| \, . \, |_{0+(h)}$ are the norms in the Hölder spaces $\mathbf{C}^{1+(h)}(\overline{\Omega})$ and $C^{0+(h)}(\overline{\Omega})$, respectively, see [17]. The integral of $|\mathbf{u}|^2$ on $\partial\Omega$ has been estimated by means of [10, Theorem II.4.1].

We have shown that the norm of $\nabla_{\mathbf{x}} p(\mathbf{x}, t)|_{\mathbf{x}=\mathbf{x}_0} \cdot \mathbf{e}$ in $L^4(t_1+\epsilon, t_2-\epsilon)$ is finite and independent of vector $\mathbf{e}$ and a concrete position of point $\mathbf{x}_0$ in domain Ω_3. Hence $\nabla p \in L^4(0, T; \, \mathbf{L}^\infty(\Omega_3))$. From this, one can deduce that p can be chosen so that $p \in L^4(0, T; \, L^\infty(\Omega_3))$. Similarly, dealing with $D^\alpha_{\mathbf{x}} p(\mathbf{x}, t)$, where $\alpha \equiv (\alpha_1, \alpha_2, \alpha_3)$ is an arbitrary multi-index, instead of $p(\mathbf{x}, t)$, we show that $D^\alpha p \in L^4(0, T; \, L^\infty(\Omega_3))$, too. The proof is completed $\square$

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