Dissipative solutions to the compressible Euler system

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Euler system of gas dynamics

Leonhard Paul Euler 1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

Momentum equation – Newton's second law

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0, \ p(\varrho) = a \varrho^\gamma$$

Boundary and/or far field conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0, \ \mathbf{u} \to \mathbf{u}_\infty, \ \varrho \to \varrho_\infty \text{ as } |x| \to \infty$$

Initial state

$$\varrho(0, \cdot) = \varrho_0, \ \varrho \mathbf{u}(0, \cdot) = \mathbf{m}_0$$

Classical solutions

- **Local existence.** Classical solutions exist locally in time as long as the initial data are regular and the initial density strictly positive
- **Finite time blow–up.** Classical solutions develop singularity (become discontinuous) in a *finite* time for a fairly generic class of initial data

Weak (distributional) solutions

Jacques Hadamard 1865–1963

Laurent Schwartz 1915–2002

Mass conservation

$$\int_B \Big[\varrho(t_2,\cdot) - \varrho(t_1,\cdot)\Big] \mathrm{d}x = -\int_{t_1}^{t_2}\int_{\partial B} \varrho \mathbf{u}\cdot\mathbf{n}\ \mathrm{dS}_x \mathrm{d}t$$

$$\left[\int_\Omega \varrho\varphi\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big[\varrho\partial_t\varphi + \mathbf{m}\cdot\nabla_x\varphi\Big]\ \mathrm{d}x\mathrm{d}t,\ \ \mathbf{m}\equiv\varrho\mathbf{u}$$

Momentum balance

$$\int_B \Big[\mathbf{m}(t_2,\cdot) - \mathbf{m}(t_1,\cdot)\Big]\ \mathrm{d}x$$

$$= -\int_{t_1}^{t_2}\int_{\partial B}\Big[\mathbf{m}\otimes\mathbf{u}\cdot\mathbf{n} + p(\varrho)\mathbf{n}\Big]\ \mathrm{dS}_x\ \mathrm{d}t$$

$$\left[\int_\Omega \mathbf{m}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right]_{t=0}^{t=\tau}$$

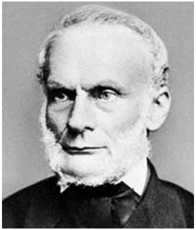
$$= \int_0^\tau\int_\Omega\left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}\right]\ \mathrm{d}x\mathrm{d}t$$

Time irreversibility – energy dissipation

Energy

$$\boxed{\mathcal{E} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)}$$

$$p' \geq 0 \Rightarrow [\varrho, \mathbf{m}] \mapsto \left\{ \begin{array}{l} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0 \\ P(\varrho) \text{ if } |\mathbf{m}| = 0,\ \ \varrho \geq 0 \\ \infty \text{ otherwise} \end{array} \right. \quad \text{is convex l.s.c}$$



Rudolf Clausius 1822–1888

Energy balance (conservation)

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E} \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) = 0$$

Energy dissipation

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E} \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) \boxed{\leq} 0$$

$$E = \int_\Omega \mathcal{E}\ \mathrm{d}x,\ \ \partial_t E \leq 0,\ \ E(0+) = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right]\ \mathrm{d}x$$

Wild solutions?

Charles Hermite [1822-1901]

In a letter to Stieltjes

I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives

- **Past:** What is not allowed is forbidden
- **Present:** What is not forbidden is allowed

Ill posedness

Anna Abbatiello (TU Berlin)

Theorem [A.Abbatiello, EF 2019]

Let $d = 2, 3$. Let ϱ_0, $\mathbf{m}_0$ be given such that

$$\varrho_0 \in \mathcal{R},\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R},\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R},\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^{\infty} \subset (0, T)$ be an arbitrary (countable dense) set of times.
Then the Euler problem admits infinitely many weak solutions ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that

$$\varrho \in C_{\rm weak}([0, T]; L^{\gamma}(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

but

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)]\ \boxed{\text{is not}}\ \text{strongly continuous at any } \tau_i$$

Consistent approximation

Equation of continuity

$$\int_0^T \int_\Omega \left[\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi \right] \mathrm{d}x\mathrm{d}t = e_{1,n}[\varphi]$$

Momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \varphi + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t = e_{2,n}[\boldsymbol{\varphi}]$$

Stability - bounded energy

$$\mathcal{E}(\varrho_n, \mathbf{m}_n) \equiv \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] \mathrm{d}x \lesssim 1$$

Consistency

$$e_{1,n}[\varphi] \to 0, \ \ e_{2,n}[\boldsymbol{\varphi}] \to 0 \text{ as } n \to \infty$$

Weak vs strong convergence

Weak convergence

$$(\varrho_n - \varrho) \to 0 \text{ weakly-(*) } L^\infty(0,T;L^\gamma(R^d))$$

$$(\mathbf{m}_n - \mathbf{m}) \to 0 \text{ weakly-(*) } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(R^d;R^d))$$

Strong convergence (Theorem EF, M.Hofmanová)

$$\Omega = R^d, \ \varrho \to \varrho_\infty, \ \mathbf{m} \to \mathbf{m}_\infty \text{ as } |x| \to \infty$$

$\varrho, \mathbf{m}$ weak solution to the Euler system

$$\Leftrightarrow$$

$$\varrho_n \to \varrho, \ \mathbf{m}_n \to \mathbf{m} \text{ strongly (pointwise) in } R^d$$

Martina Hofmanová (Bielefeld)

Dissipative solutions – limits of numerical schemes

Dominic Breit (Edinburgh)

Martina Hofmanová (Bielefeld)

Equation of continuity

$$\partial_t \boxed{\varrho} + \mathrm{div}_x \mathbf{m} = 0$$

Momentum balance

$$\partial_t \boxed{\mathbf{m}} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \left(\mathfrak{R}_v + \mathfrak{R}_p \mathbb{I} \right)$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq 0, \ E(t) \leq E_0, \ E_0 = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$

$$\boxed{E} \equiv \left(\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d} \frac{1}{2} \mathrm{trace}[\mathfrak{R}_v] + \int_{\overline{\Omega}} \mathrm{d} \frac{1}{\gamma - 1} \mathfrak{R}_p \right)$$

Turbulent defect measures

$$\mathfrak{R}_v \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\mathrm{sym}})), \ \mathfrak{R}_p \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}))$$

Basic properties of dissipative solutions

Well posedness, weak strong uniqueness

- **Existence.** Dissipative solutions exist globally in time for any finite energy initial data
- **Limits of consistent approximations** Limits of consistent approximations are dissipative solutions, in particular limits of consistent numerical schemes.
- **Compatibility.** Any C^1 dissipative solution $[\varrho, \mathbf{m}]$, $\varrho > 0$ is a classical solution of the Euler system
- **Weak–strong uniqueness.** If $[\widetilde{\varrho}, \widetilde{\mathbf{m}}]$ is a classical solution and $[\varrho, \mathbf{m}]$ a dissipative solution starting from the same initial data, then $\mathfrak{R}_v = \mathfrak{R}_p = 0$ and $\varrho = \widetilde{\varrho}$, $\mathbf{m} = \widetilde{\mathbf{m}}$.

Semiflow selection

Set of data

$$\mathcal{D} = \left\{ \varrho, \mathbf{m}, E \ \Big| \ \int_\Omega \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \, \mathrm{d}x \leq E \right\}$$

Set of trajectories

$$\mathcal{T} = \left\{ \varrho(t, \cdot), \mathbf{m}(t, \cdot), E(t-, \cdot) \Big| t \in (0, \infty) \right\}$$

Solution set

$$\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] = \Big\{ [\varrho, \mathbf{m}, E] \ \Big| [\varrho, \mathbf{m}, E] \text{ dissipative solution}$$

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0, \ E(0+) \leq E_0 \Big\}$$

Semiflow selection – semigroup

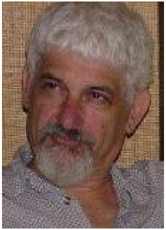
$$U[\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0], \ [\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{D}$$

$$U(t_1 + t_2)[\varrho_0, \mathbf{m}_0, E_0] = U(t_1) \circ \Big[U(t_2)[\varrho_0, \mathbf{m}_0, E_0] \Big], \ t_1, t_2 > 0$$

Andrej Markov (1856–1933)

N. V. Krylov

Strong instead of weak (numerics)



Janos Komlos (Ruthers Univ.)

Erich J. Balder (Utrecht)

Komlos theorem (a variant of Strong Law of Large Numbers)

$$\{U_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q)$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} U_{n_k} \to \overline{U} \text{ a.a. in } Q \text{ as } N \to \infty$$

Conclusion for the approximate solutions

$$\frac{1}{N}\sum_{k=1}^{N} \varrho_{n_k} \to \varrho \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{m}_{n_k} \to \mathbf{m} \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \left[\frac{1}{2}\frac{|\mathbf{m}_{n,k}|^2}{\varrho_{n,k}} + P(\varrho_{n,k})\right] \to \overline{\mathcal{E}} \in L^1((0,T)\times\Omega) \text{ a.a. in } (0,T)\times\Omega$$

Computing defect – Young measure

Young measure

$$\{U_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q) \approx \nu^n_{t,x} = \delta_{U_n(t,x)}$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} \nu^{n_k}_{t,x} \to \nu_{t,x} \text{ narrowly a.a. in } Q \text{ as } N \to \infty$$

Monge–Kantorowich (Wasserstein) distance

$$\left\| \mathrm{dist}\left(\frac{1}{N}\sum_{k=1}^{N} \nu^{n_k}_{t,x}; \nu_{t,x} \right) \right\|_{L^q(Q)} \to 0$$

for some $q > 1$

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Hana Mizerová (Bratislava)

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