Lectures on mappings of finite distortion

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Summary

Let $\Omega \subset \mathbf{R}^n$ be a domain. We study a certain class of mapping $f : \Omega \to \mathbf{R}^n$ that can serve as a class of deformations in nonlinear elasticity. These kind of problems were originally studied by Ball, Šverák, Müller, Spector, Fonseca and many others.

The natural problems studied in this area are the optimal condition that guarantee continuity (material cannot break and no cavities are created), null sets are mapped to null sets (material cannot be created from nothing), invertibility (interpenetration of matter), properties of the inverse mapping (backward deformation should be nice) and many others.

The monograph Lectures on Mappings Distortion written jointly with Pekka Koskela is an self-contained introduction to this field. The material is based on the graduate level courses that the authors have given at the University of Michigan, University of Jyväskylä and at Charles University in Prague and on short courses by the authors at summer schools in Ischia and at the de Giorgi center in Pisa.

The book can serve as a reading material on the graduate level or as a reference book to the researchers in the area. It contains also recent results in the field and some of them were obtained by the author of the thesis together with his coauthors.

Resumé

Nechť $\Omega \subset \mathbf{R}^n$ je oblast. Studujeme třídu zobrazení $f : \Omega \to \mathbf{R}^n$, která může sloužit jako třída deformací v teorii nelinární elasticity. Tyto problémy byly původně studovány matematiky jako Ball, Šverák, Müller, Spector, Fonseca a mnoha jinými.

Přirozené problémy v této oblasti jsou nalezení optimálních podmínek, které zaručí, že zobrazení je spojité (materiál se netrhá a uvnitř nevznikají dutiny), množiny nulové míry se zobrazí na množiny nulové míry (nový materiál nemůže vzniknout z ničeho), existence inverzního zobrazení (dvě části tělěsa se nemohou zobrazit do téhož místa), vlastnosti inverzního zobrazení (zpětná deformace by měla být hezká) a mnoho jiných.

Monografie Lectures on Mappings Distortion napsaná spolu s Pekkou Koskelou tvoří samostatný ú samostatný a nezávislý úvod do této problematiky. Materiál vznikl na základě přednášek obou autorů na University of Michigan, University of Jyväskylä, na Karlově Univerzitě v Praze a na krátkých minikurzech, které autoři přednášeli na letní škole na Ischii a na de Giorgi center v Pise.

Kniha může sloužit jako samostatná četba na úrovni PhD studentů, nebo i jako referenční kniha pro jiné odborníky v dané oblasti. Obsahuje také mnoho nedávných výsledků a některé z nich byly dosaženy autorem spolu se spoluautory,

1. Introduction

In 1981, J.M.Ball [3] established a class of mappings that can serve as a class of deformations in nonlinear elasticity. Nowadays the whole theory is very rich and we recommend the monographs [21] and [1] for history, references and further motivation.

We can view a domain $\Omega \subset \mathbf{R}^n$ as a solid body in space and our mapping $f : \Omega \to \mathbf{R}^n$ as a deformation of the body Ω to $f(\Omega)$. There are several natural questions one can ask.

- Is f continuous?
- Does f map sets of zero measure to sets of zero measure?
- Is the mapping one-to-one? Does the inverse map f^{-1} exist?
- What are the properties of f^{-1}? (Is the reverse deformation reasonable?)

It would be possible to consider many other questions, but we would like to concentrate on these in detail. The first question of continuity means from the physical point of view that the material does not break down into pieces and that no holes are created inside the material during our deformation.

The second question can be interpreted that the new material cannot be created or lost during our deformation. There are continuous and invertible mappings that map a set of measure zero (i.e. zero volume) to a set of positive measure or a set of positive measure to a set of measure zero. However these mappings are not suitable for developing a reasonable theory because it does not seem to be natural that something negligible can be mapped to something big. From a mathematical point of view this assumption is also crucial, because it is closely related with the validity of the substitution formula for the integral and it is one the main tools in this area.

A basic requirement of continuum mechanics is that interpenetration of matter does not occur, i.e. the mapping $f(x)$ giving the position of a particle is invertible. It means that it is not possible that two different points are mapped to the same point. Therefore in any reasonable theory we expect that the inverse of our mapping exists. The inverse map can be viewed as a backward deformation to the original state and it is natural to expect that if the original map is nice that the inverse map will posses also some good properties.

These questions of course make sense also in other dimensions than $n = 3$ and they are usually studied for mappings $f : \Omega \to \mathbf{R}^n$ where $\Omega \subset \mathbf{R}^n$. The situation in $\mathbf{R}^2$ is sometimes simpler and can be used as model example.

Let us now introduce the class of mappings that is usually used now in this context. We assume that the reader is familiar with Sobolev space $W^{1,1}(\Omega, \mathbf{R}^n)$, i.e. mappings in L^1 whose distributional derivatives are also in L^1.

Definition 1.1. *We say that a mapping $f : \Omega \to \mathbf{R}^n$ on an open connected set $\Omega \subset \mathbf{R}^n$ has finite distortion if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$, $J_f \in L^1_{\mathrm{loc}}(\Omega)$ and there is a function $K : \Omega \to [1, \infty]$ with $K(x) < \infty$ almost everywhere such that*

$$|Df(x)|^n \le K(x) J_f(x) \qquad \text{for almost all } x \in \Omega. \tag{1}$$

For mappings of finite distortion we can define the optimal distortion function as

$$K_f(x) := \begin{cases} \frac{|Df(x)|^n}{J_f(x)} & \text{for all } x \in \{J_f > 0\}, \\ 1 & \text{for all } x \in \{J_f = 0\}. \end{cases}$$

Let us note that that the definition (1) is equivalent to the assumptions that $J_f(x) \ge 0$ a.e. (mapping does not change orientation) and to the assumption that $|Df(x)|$ vanishes a.e. in the zero set of the Jacobian $\{x : \; J_f(x) = 0\}$.

The condition (1) is a relaxation of the definition of quasiregularity (or quasiconformality) that requires that infinitesimal circles be mapped to infinitesimal ellipses whose eccentricities $K_f(x)$ are uniformly bounded. Thus the study of mappings as in previous definition can be viewed as a generalization of the study of quasiregular mappings, also called mappings of bounded distortion. In a sense, one relaxes the boundedness of the distortion to integrability of the distortion. Mappings of bounded distortion are continuous, map sets of measure zero to sets of measure zero, and they are either constant or locally bounded to one. Also, a mapping of bounded distortion that is injective close to the boundary is necessarily a homeomorphism and the inverse is also of bounded distortion. Thus this class of mappings has the properties that are desirable from the point of view of nonlinear elasticity. Te class of quasiregular mappings was introduced by Yu.Reshetnyak in 1967 [35]. We recommend the monographs [35], [36] and [27] for an interested reader.

In the next chapters we relax the assumption $K \in L^\infty$ and we show that mappings of finite distortion have properties similar to those of mappings of bounded distortion. We usually have two kinds of positive results. We assume that f is in the nice Sobolev space $W^{1,n}$ and then we require some mild assumptions on the distortion like integrability or only finiteness almost everywhere. Alternatively, we assume only that $f \in W^{1,1}$ but then we usually need much stronger assumptions on the distortion, like $\exp(\lambda K_f) \in L^1$ for some $\lambda > 0$.

In the case of bounded distortion in the complex plane, one has the associated Beltrami equation

$$\overline{\partial} f(z) = \mu(z) \partial f(z),$$

where one assumes that $\|\mu\|_{L^\infty} < 1$. Each mapping of bounded distortion is a solution to this equation and each such an equation with $\|\mu\|_{L^\infty} < 1$ has a homeomorphic solution of bounded distortion. It is possible to show the existence of solutions under weaker assumptions (see [1] for exact statement and proofs), like for those compactly supported μ with $\exp(\frac{p}{1-|\mu(z)|})$ integrable for some p; this corresponds to the class of mappings whose distortion is not necessarily bounded but $\exp(\lambda K_f(x)) \in L^1$ for some $\lambda > 0$.

2. Summary of the book

In this section we give the summary of the book. This section is divided into six subsections, each corresponds to some chapter of the monograph. In each Subsection we will point out the original results that were obtained by the author in this area. For the detailed reference of the sources we point the readers attention to the Remarks at the end of the Chapters in the book. There he may also find 26 Open problems in the area.

2.1. Continuity.

It is a well-known fact that each function in the Sobolev space $W^{1,p}$ has a continuous representative if $p > n$ but not necessarily for any $p \leq n$.

For mappings of finite distortion we can consider the following example:

Example 2.1. For $x \in B(0,1) \setminus \{0\}$ let us set

$$f(x) = x + \frac{x}{|x|}$$

(and define $f(0) = 0$). Then f is a mapping of finite distortion such that $f \in W^{1,p}(B(0,1), \mathbf{R}^n)$ for all $p < n$, but f is not continuous at the origin, i.e. no representative of f is continuous at the origin.

Proof. Clearly $f(x) = \frac{x}{|x|}(|x| + 1)$ maps spheres centered at the origin onto similar spheres and it is a diffeomorphism from $B(0,1) \setminus \{0\}$ onto $B(0,2) \setminus \overline{B(0,1)}$. Using elementary computations it is not difficult to check that

$$|Df(x)| = 1 + \frac{1}{|x|} \text{ and } J_f(x) = \Big(1 + \frac{1}{|x|}\Big)^{n-1}.$$

It follows that $f \in W^{1,p}$ for all $p < n$, $J_f \in L^1(B(0,1))$ and, since $J_f > 0$ almost everywhere, that f has finite distortion. On the other hand, we cannot extend f continuously to the origin since the sphere $S^{n-1}(0,\varepsilon)$ is mapped to the sphere $S^{n-1}(0, 1+\varepsilon)$. □

From the previous example we know that we cannot hope for a positive result if we only know that a mapping f of finite distortion belongs to $W^{1,p}$ for some $p < n$. The following result shows that in the limiting situation $f \in W^{1,n}$ mappings of finite distortion have better properties than general Sobolev mappings.

Theorem 2.2. *Let $\Omega \subset \mathbf{R}^n$ be open and let $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ be a mapping of finite distortion. Then f has a continuous representative.*

Moreover, we can relax the regularity assumptions on f if we require additional restrictions on the integrability of the distortion function. Later in this chapter it is shown that under the same assumption we obtain also differentiability almost everywhere for the continuous representative of these mappings.

Theorem 2.3. *Let $\Omega \subset \mathbf{R}^n$ be open and let $f : \Omega \to \mathbf{R}^n$ be a mapping of finite distortion. Suppose that there is $\lambda > 0$ such that $\exp(\lambda K) \in L^1_{\mathrm{loc}}(\Omega)$. Then f has a continuous representative.*

Again the assumption $\exp(\lambda K) \in L^1$ cannot be essentially relaxed and there is a simple example given in the book.

Now we give some brief idea of the proof of these two main theorems in this Section. First it is necessary to show that mappings of finite distortion in this class are in some sense monotone, i.e. the nonnegativity of the Jacobian implies that some sort of $\operatorname{diam} f(B) \leq \operatorname{diam} f(\partial B)$ for all balls $B \subset \Omega$. In fact we show that they are weakly monotone, which is some weak version of this inequality suitable for Sobolev mappings. In order to achieve that we first need to show that under our regularity assumption we know that the Jacobian is integrable and equals to the distributional Jacobian, i.e.

$$\int_\Omega \varphi J_f = -\int_\Omega f_1 J(\varphi, f_2, ..., f_n) \qquad \text{for all } \varphi \in \mathcal{D}(\Omega) \ .$$

This crucial identity is the main place where we need our regularity assumptions and it is the main tool for establishing many other results in this area. This identity is due to Greco [10] and Iwaniec and Sbordone [22]. These results are inspired by the earlier result of Müller [32] who showed that $J_f \in L^1 \log L$ if $f \in W^{1,n}$ satisfies $J_f \geq 0$.

The results of this Chapter were established mostly before the author started to work on this topic. There are later results about the optimal modulus of continuity for mappings of finite distortion with $\exp(\lambda K) \in L^1$. In [5] he showed with his student D. Campbell that this modulus of continuity is indeed the optimal one (see Theorem 5.19 in the monograph) which was a bit surprising.

2.2. Openness and Discreteness.

One of the crucial properties in the models on nonlinear elasticity is that there is no interpenetration of matter. This corresponds to the fact that two parts of the body cannot be mapped to the same place. From the mathematical point of view this means that the map should be one-to-one and thus invertible.

Let us consider the conformal mapping $f(z) = z^2$ in the complex plane which can be identified with $\mathbf{R}^2$. We know that $f \in C^\infty$ is conformal and hence its distortion satisfies $K \equiv 1$. On the other hand each nonzero point has two preimages and this mapping is not invertible. This shows that even for analytically very nice mappings we cannot conclude that the inverse exists without some extra information.

As a first step one usually attempts to conclude that the mapping in question is open and discrete. Note that for example homeomorphisms are automatically open and discrete.

Definition 2.4. *Let $\Omega \subset \mathbf{R}^n$ be a domain. We say that the mapping $f : \Omega \to \mathbf{R}^n$ is open if $f(U)$ is open for each open set $U \subset \Omega$. The mapping f is called discrete if the preimage of each point $f^{-1}(y)$ is a discrete set, i.e. it does not have an accumulation point in Ω.*

Each open and discrete map which equals to a homeomorphism close to the boundary is necessarily a homeomorphism. Moreover, an open and discrete mapping is locally invertible in neighborhoods of most of the points by the following result of Chernavskii [6]. Recall that the branch set of a map is the set of points where it fails to be locally injective.

Theorem 2.5. *Let $\Omega \subset \mathbf{R}^n$ be a domain and let $f : \Omega \to \mathbf{R}^n$ be a discrete and open mapping. Then the topological dimension of the branch set B_f satisfies*

$$\dim B_f = \dim f(B_f) \leq n-2\ .$$

The following examples show that openness and discreteness may fail even for Lipschitz mappings if the degree of integrability of the distortion is not high enough.

Example 2.6 (Ball)**.** *Let $f : (-1,1)^2 \to \mathbf{R}^2$ be defined by*

$$f(x,y) = [x, |x|y]\ .$$

Then f is not open and discrete since $f^{-1}([0,0]) = \{0\} \times (-1,1)$. The derivative of f is

$$Df(x,y) = \begin{pmatrix} 1 & 0 \\ \pm y & |x| \end{pmatrix}$$

for $x \neq 0$ and therefore it is easy to see that f is Lipschitz and $J_f(x,y) = |x| \geq 0$. Hence it is a mapping of finite distortion and its distortion for small enough $||[x,y]||$ equals to

$$K_f(x,y) = \frac{1}{|x|}$$

and it is integrable with any power strictly less than 1.

Analogously, the mapping $f : (-1,1)^n \to \mathbf{R}^n$ defined as

$$f([x_1,\dots,x_n]) = [x_1,\dots,x_{n-1}, \sqrt{x_1^2+\ldots+x_{n-1}^2}\,x_n]$$

is a Lipschitz mapping of finite distortion and its distortion for small enough $|x|$ satisfies

$$K_f(x) = \frac{1}{\sqrt{x_1^2 + \ldots + x_{n-1}^2}}$$

and thus $K_f(x) \in L^p$ for every $p < n-1$. However, f is not open and discrete since $f^{-1}([0,\ldots,0]) = \{0\}^{n-1}\times(-1,1)$. Moreover, it is possible to extend this mapping to a Lipschitz mapping $\hat{f} : (-2,2)^n \to \mathbf{R}^n$ so that the restriction of $\hat{f}$ to a neighborhood of the boundary, $\hat{f}|_{(-2,2)^n\setminus[-1,1]^n}$, is a homeomorphism.

The following positive results for continuous mappings of finite distortion are the main result of the chapter. Recall that the existence of continuous representative in this setting follows by Theorem 2.2. and Theorem 2.3.

Theorem 2.7. *Let $\Omega \subset \mathbf{R}^n$ be open and let $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbf{R}^n)$ be a continuous mapping of finite distortion such that $K_f \in L^p(\Omega)$ for some $p > n-1$ or $K_f \in L^1(\Omega)$ for $n = 2$. Then f is either constant or both open and discrete.*

Theorem 2.8. *Let $\Omega \subset \mathbf{R}^n$ be open and let $f : \Omega \to \mathbf{R}^n$ be a continuous mapping of finite distortion. Suppose that there is $\lambda > 0$ such that $\exp(\lambda K_f) \in L^1_{\text{loc}}(\Omega)$. Then f is either constant or both open and discrete.*

In order to show these results one must use the theory of the topological degree. This chapter contains an independent introduction to the theory of topological degree suitable for people in mathematical analysis, i.e. avoiding Algebraic Topology and using properties of integrals and Jacobians. In the last part of the chapter we also show basic properties of open and discrete mappings like local boundedness of the multiplicity (= number of preimages) or the fact that each open and discrete mapping which equals to a homeomorphism close to the boundary is in fact a global homeomorphism.

It has been an open problem posed already in [23] if the positive conclusion of Theorem 2.7 holds in higher dimensions also in the borderline case $K \in L^{n-1}$. If we moreover know that f equals to a homeomorphism close to the boundary (or that the multiplicity is essentially bounded), then the positive answer was established by Hencl and Malý [16] and Hencl and Koskela [13]. On the other hand, in a recent result of Hencl and Rajala [19] it was shown that there is even a Lipschitz mapping with $K \in L^{n-1}$ which fails to be discrete and thus the result fails in general. It is still not known if this assumption guarantees openness.

2.3. Images and preimages of null sets.

In this chapter we mainly study when the null sets are mapped to null sets.

Definition 2.9. *Let $\Omega \subset \mathbf{R}^n$ be open. We say that $f : \Omega \to \mathbf{R}^n$ satisfies the Lusin (N) condition if*

$$\text{for each } E \subset \Omega \text{ such that } |E| = 0 \text{ we have } |f(E)| = 0\ .$$

There are two major motivations for the study of this property. From the physical point of view this property corresponds to the fact that our deformation f of the body in $\mathbf{R}^n$ cannot create new material from 'nothing'. This would be unnatural in any physically relevant model and hence we would like to know conditions which exclude such pathological behavior. To study conditions like that one needs to know obstacles and the natural counterexamples.

From the mathematical point of view this property is crucial for the validity of the change of variables formula which is an essential tool in this area. In fact, the Area Formula holds for Sobolev mappings if and only if f satisfies condition (N).

We study the validity of the Lusin (N) condition both in the setting of general Sobolev mappings and in the setting of mappings of finite distortion. We show that the Lusin (N) condition is satisfied for general Sobolev mapping in the supercritical case $p > n$.

Theorem 2.10. *Let $\Omega \subset \mathbf{R}^n$ and $p > n$. Suppose that $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ is continuous. Then f satisfies the Lusin (N) condition.*

On the other hand we construct a mapping $f \in W^{1,n}((0,1)^n, (0,1)^n)$ which maps a line segment to the whole cube $(0,1)^n$. The situation is slightly better for a homeomorphisms in the Sobolev space.

Theorem 2.11. *Let $\Omega \subset \mathbf{R}^n$ and suppose that $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ is a homeomorphism. Then f satisfies the Lusin (N) condition.*

We show the sharpness of this result by the construction of the homeomorphism of finite distortion in $W^{1,p}$, $p < n$, which maps a null set to a set of positive measure.

For mappings of finite distortion we give the following positive results.

Theorem 2.12. *Let $\Omega \subset \mathbf{R}^n$ be open and let $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbf{R}^n)$ be a mapping of finite distortion. Then the continuous representative of f satisfies the Lusin (N) condition.*

Theorem 2.13. *Let $\Omega \subset \mathbf{R}^n$ be open and let $f : \Omega \to \mathbf{R}^n$ be a mapping of finite distortion. Suppose that there is $\lambda > 0$ such that $\exp(\lambda K_f) \in L^1_{\text{loc}}(\Omega)$. Then the continuous representative of f satisfies the Lusin (N) condition.*

Again we give a general construction of a mapping which maps a certain product of Cantor type sets to a product of Cantor type sets to show the sharpness of our assumptions.

In the last part of this chapter we study condition that guarantee that preimages of sets of measure zero have zero measure. This corresponds to the fact that no material can be lost during our deformation.

Theorem 2.14. *Let a continuous mapping $f \in W^{1,1}(\Omega, \mathbf{R}^n)$ be a mapping of finite distortion with $K_f^{\frac{1}{n-1}} \in L^1(\Omega)$. If the multiplicity of f is essentially bounded by a constant N and f is not constant, then $J_f(x) > 0$ a.e. in Ω and hence f satisfies the Lusin (N^{-1}) condition, i.e.*

$$\text{for each } E \subset f(\Omega) \text{ such that } |E| = 0 \text{ we have } |f^{-1}(E)| = 0\ .$$

By similar construction as before with different Cantor type sets we show the sharpness of the assumption $K_f^{\frac{1}{n-1}} \in L^1(\Omega)$.

Lately it has been shown in Hencl [11] that for every $p < n$ it is possible to construct a homeomorphism $f \in W^{1,p}((0,1)^n, (0,1)^n)$ which maps a set of measure zero to a set of measure one and the remaining set of measure one to a set of measure zero. This shows that the Lusin (N) and (N^{-1}) condition may fail in a terrible way even for a Sobolev homeomorphisms. This was improved in D'Onofrio, Hencl and Schiattarella [8] where it was shown that this homeomorphism in dimension $n \geq 3$ may even satisfy $f^{-1} \in W^{1,1}$.

2.4. Homeomorphisms of finite distortion.

In this chapter we study various properties of homeomorphisms of finite distortion like the regularity of the inverse mapping, regularity of the composition, sharp moduli of continuity, sense preservation (i.e. the Jacobian does not change sign) and approximation of Sobolev homeomorphism.

The results about the regularity of the inverse in the planar case were established by Hencl and Koskela [12] and they are the main content in the first chapter of the book. The main result states that the inverse of a planar homeomorphism of finite distortion is a Sobolev mapping and it is a mapping of finite distortion as well.

Theorem 2.15. *Let $\Omega \subset \mathbf{R}^2$ be a domain and let $f : \Omega \to f(\Omega) \subset \mathbf{R}^2$ be a homeomorphism with $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbf{R}^2)$ and assume that $J_f(x) \geq 0$ almost everywhere. Then the following conditions are equivalent:*

(i) $f^{-1} \in W^{1,1}_{\text{loc}}(f(\Omega), \mathbf{R}^2)$,

(ii) f has finite distortion,

(iii) f^{-1} has finite distortion.

For the higher dimensional counterpart we need to assume that $f \in W^{1,n-1}$. These results were established in Hencl, Koskela, Malý [17] and Csörnyei, Hencl and Malý [7].

Theorem 2.16. *Let $\Omega \subset \mathbf{R}^n$ be an open set. Suppose that $f \in W^{1,n-1}(\Omega, \mathbf{R}^n)$ is a homeomorphism of finite distortion. Then $f^{-1} \in W^{1,1}_{\text{loc}}(f(\Omega), \mathbf{R}^n)$ and has finite distortion.*

There are counterexamples of finite distortion if one only assumes that $f \in W^{1,p}$ for $p < n-1$. Without the assumption that f has finite distortion the inverse of even Lipschitz homeomorphism in the plane may fail to be Sobolev.

Example 2.17. *There is a homeomorphism $f : \mathbf{R}^2 \to \mathbf{R}^2$ such that f is Lipschitz, but $f^{-1} \notin W^{1,1}_{\text{loc}}(\mathbf{R}^2, \mathbf{R}^2)$.*

Proof. Indeed, let u be the usual Cantor ternary function on the interval $(0, 1)$. Then u is continuous, non-decreasing, constant on each complementary interval of the ternary Cantor set and fails to be absolutely continuous. Let now $v(x) = x + u(x)$ on $(0, 1)$ and extend v to negative reals as $v(x) = x$ and to $x \geq 1$ as $v(x) = x + 1$. Then also v fails to be absolutely continuous but v^{-1} is Lipschitz. The mapping g defined simply by $g([x_1, x_2]) = [v(x_1), x_2]$ is clearly a homeomorphism, but it is not absolutely continuous on almost all lines parallel to coordinate axes as v is not absolutely continuous. It follows that g does not satisfy the ACL-condition and hence $g \notin W^{1,1}_{\text{loc}}(\mathbf{R}^2, \mathbf{R}^2)$. It is easy to check that $f = g^{-1}$ is Lipschitz continuous and thus f has the desired properties. $\square$

On the other hand we still get the weaker regularity of the inverse without the assumption that f has finite distortion.

Theorem 2.18. *Let $\Omega \subset \mathbf{R}^n$ be an open set. Suppose that $f \in W^{1,n-1}(\Omega, \mathbf{R}^n)$ is a homeomorphism. Then $f^{-1} \in BV_{\text{loc}}(f(\Omega), \mathbf{R}^n)$.*

Assuming that f has finite distortion for the regularity of the inverse is not artificial. Every homeomorphism with $f \in W^{1,1}$ and f^{-1} in $W^{1,1}$ is necessarily a mapping of finite inner distortion - this was shown in Hencl, Moscariello, Passarelli di Napoli, Sbordone [18] . Let us note that Theorem 2.16 holds even with the weaker assumption that f has finite inner distortion.

In the subsection about the regularity of the composition we study optimal conditions on a homeomorphism f that guarantee that $u \circ f \in W^{1,q}$ for every $u \in W^{1,p}$, $p \geq q$. These results are based on a paper by Kleprlík [26] which was written as a Master Thesis under the supervision of Hencl.

In the next part, we address the following problem, originally asked by P. Hajlasz. Suppose that $\Omega \subset \mathbf{R}^n$ is a domain and that $f : \Omega \to \mathbf{R}^n$ is a homeomorphism of the Sobolev class $W^{1,1}_{\text{loc}}(\Omega, \mathbf{R}^n)$. Is it true that the Jacobian J_f is either non-negative almost everywhere or non-positive almost everywhere? It is well-known that every homeomorphism defined on a domain Ω is either sense-preserving or sense-reversing and therefore we can ask whether each sense-preserving homeomorphism in the Sobolev space $W^{1,s}_{\text{loc}}$ satisfies $J_f \geq 0$ almost everywhere. Roughly speaking, we are interested in the question whether topological and analytical definitions of orientation lead to the same result. We give a detailed proof under the assumption that $p > n - 1$ and we recall the result of Hencl and Malý [15].

Theorem 2.19. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a domain, $p = 1$ for $n = 2, 3$ and $p > [n/2]$ for $n \geq 4$. Suppose that $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbf{R}^n)$ is homeomorphism. Then either $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e.*

It is still not known if this result holds for example in dimension $n = 4$ under the assumption that $f \in W^{1,1}$.

In the last part we give an overview of one of the most interesting and important problems in this area, originally posed by Evans and later promoted by Ball. Let $\Omega \subset \mathbf{R}^n$ be a domain and $1 \leq p < \infty$. Suppose that $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ is a homeomorphism. Is it possible to find a sequence of piecewise affine homeomorphisms $\tilde{f}_k$ such that $\|\tilde{f}_k - f\|_{W^{1,p}} \to 0$? Is it possible to find a sequence of smooth homeomorphisms f_k such that $\|f_k - f\|_{W^{1,p}} \to 0$? Partial motivation for this problem comes from regularity of models in nonlinear elasticity. Also, piecewise affine approximation would be nice for numerical approximation.

2.5. Integrability of J_f and other results.

It is well-known that for each quasiregular mapping $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ (and $K \in L^\infty$) there is $p > n$ such that $|Df| \in L^p_{\text{loc}}(\Omega)$ and hence $J_f \in L^{\frac{p}{n}}_{\text{loc}}(\Omega)$. This remarkable self-improving regularity result, which is based on a reverse Hölder inequality, is important for many other properties of quasiregular mappings.

In this chapter we give a generalization of this fact for mappings with exponentially integrable distortion.

Theorem 2.20. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a domain and let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbf{R}^n)$ be a mapping of finite distortion. Assume that $\exp(\beta K_f) \in L^1_{\text{loc}}(\Omega)$, for some $\beta > 0$. Then*

$$J_f \log^\alpha(e + J_f) \in L^1_{\text{loc}}(\Omega), \text{ and } |Df|^n \log^{\alpha-1}(e + |Df|) \in L^1_{\text{loc}}(\Omega),$$

where $\alpha = C_1\beta$ and $C_1 = C_1(n) > 0$.

Next we study the optimal integrability of $\frac{1}{J_f}$ for homeomorphisms of finite distortion. As an application of our estimates we show that sets of corresponding zero capacity are removable singularities for mappings with exponentially integrable distortion.

2.6. Final comments and Appendix.

In chapter Final Comments we briefly discuss the inner distortion function. For mappings of finite distortion we have defined the distortion function K_f. It is often called the outer distortion function and referred to by

$$K_O(x) := \begin{cases} \frac{|Df(x)|^n}{J_f(x)} & \text{for } J_f(x) > 0 \ , \\ 1 & \text{for } J_f(x) = 0 \ . \end{cases}$$

It is possible to define also other distortion functions such as the inner distortion function

$$K_I(x) := \begin{cases} \frac{|\operatorname{adj} Df(x)|^n}{J_f(x)^{n-1}} & \text{for } J_f(x) > 0 \ , \\ 1 & \text{for } J_f(x) = 0 \ , \end{cases}$$

where $\operatorname{adj} Df(x)$ denotes the adjugate matrix of $Df(x)$, i.e. the matrix of the $(n-1) \times (n-1)$ subdeterminants. These distortion functions coincide for $n = 2$ but they are different for $n \geq 3$.

We have the following geometrical interpretation. Let E be the ellipsoid defined as $E = \{Df(x)z \in \mathbf{R}^n : |z| \leq 1\}$. Then K_O corresponds to the ratio of the longest axis of E to power n divided by the volume of E and K_I corresponds to the ratio of the $(n-1)$-dimensional volume of the largest intersection of E with $(n-1)$-dimensional hyperplane to power n divided by the volume of E to power $n-1$. Roughly speaking the outer distortion corresponds to the deformation of lengths of segments and the inner distortion corresponds to the deformation of the $(n-1)$-dimensional volumes of intersection with hyperplanes.

We also give the connection in the plane between mappings of finite distortion and solutions to a degenerate Beltrami equation. Moreover, we study the shape of the image of the unit disk under a mapping of finite distortion. It is possible to characterize the image of a disc under a quasiconformal mapping and we aim at some characterization for mappings in our class. We give some first results in this direction and we ask many open problems in this area. Finally, we show that certain families of mappings with exponentially integrable distortion are closed under weak convergence

In the Appendix we have included many classical results from Real Analysis, Sobolev spaces and Geometric Measure Theory. Most of these results contain proofs for the convenience of the reader as it may serve as a material for the graduate students.

This chapter contains for example the proof of the approximative differentiability of Sobolev mappings and the approximation of Sobolev function by the smooth functions given by mollification. Later we show that Sobolev function can be approximated by Lipschitz function that agree with the given Sobolev function on a big set and using this we prove the Area Formula and Change of Variables formula for Sobolev mappings assuming that the reader is familiar with Area Formula for Lipschitz mappings.

3. List of articles included in the monograph

The thesis is formed by the monograph Lectures on mappings of finite distortion written jointly with Pekka Koskela. It contains many new results in the field and six of them were obtained by the author of the thesis jointly with his coauthors. We give citations of the original papers included in this monograph. The symbol IF denotes the value of the impact factor of the corresponding journal in the year 2012. The list of citations of each article is typed in a small font.

(C12): Stanislav Hencl, Pekka Koskela: *Regularity of the inverse of a planar Sobolev homeomorphism*, Arch. Rational Mech. Anal. **180** (2006), 75–95. (IF=2.292)

(a) Cited in impact factor journals

(Q1) Jani Onninen: *Regularity of the inverse of spatial mappings with finite distotion*, Calc. Var. Partial Differential Equations **26** (2006), no. 3, 331–341.

(Q2) A. Clop: *Removable singularities for Hölder continuous quasiregular mappings in the plane*, Ann. Acad. Sci. Fenn. Math. **32** (2007), no. 1, 171–178.

(Q3) T. Iwaniec, G. Martin: *The Beltrami equation*, Mem. Amer. Math. Soc. **191** (2008), no. 893, 92 pp.

(Q4) N. Fusco, G. Moscariello, C. Sbordone: *The limit of $W^{1,1}$ homeomorphisms with finite distortion*, Calc. Var., **12** no. 3 (2008), 377-390.

(Q5) S. K. Vodopyanov: *On regularity of Mappings Inverse to Sobolev Mappings*, Doklady Mathematics, **423** no. 5 (2008), 592-596.

(Q6) J. T. Gill: *Integrability of derivatives of inverses of maps of exponentially integrable distortion in the plane*, J. Math. Anal. Appl. **352** no. 2 (2009), 762-766.

(Q7) T. Iwaniec, J. Onninen: *Hyperelastic deformations of smallest total energy*, Arch. Rational Mech. Anal. **194** no. 3 (2009), 927—986.

(Q8) T. Iwaniec, J. Onninen: *Deformations of finite conformal energy: Existence, and Removability of Singularities*, Proc. London Math. Soc. **100** (2010), 1–23.

(Q9) K. Astala, T. Iwaniec, G. Martin: *Deformations of Annuli with Smallest Mean Distortion*, Arch. Rational Mech. Anal., **195** (2010), no. 3, 899-921.

(Q10) T. Iwaniec, J. Onninen: *An invitation to n-harmonic hyperelasticity*, Pure Appl. Math. Quarterly **7** (2011), no. 2, 319–343.

(Q11) F. Giannetti, A. Passarelli di Napoli: *Bi-Sobolev mappings with differential matrices in Orlicz Zygmund classes*, J. Math. Anal. Appl. **369** (2010), no. 1, 346-356

(Q12) T. Iwaniec, J. Onninen: *Neohookean deformations of annuli, existence, uniqueness and radial symmetry*, Math. Ann. **348** (2010), no. 1., 35–55.

(Q13) D. Henao, C. Mora-Corral: *Fracture surfaces and the regularity of inverses for BV deformations*, Arch. Rational Mech. Anal. **201** (2011), 575—629.

(Q14) T. Iwaniec, J. Onninen: *Deformations of finite conformal energy: boundary behavior and limit theorems*, Trans. Amer. Math. Soc. **363** (2011), no. 11, 5605–5648.

(Q15) T. Iwaniec, L. Kovalev, J. Onninen: *Diffeomorphic approximation of Sobolev homeomorphisms*, Arch. Rational Mech. Anal. **201** (2011), no. 3, 1047–1067.

(Q16) P. di Gironimo, L. D'onofrio, C. Sbordone, R. Schiattarella : *Anisotropic Sobolev homeomorhisms*, Ann. Acad. Sci. Fenn. Math. **36** (2011), 593–602.

(Q17) B. Bojarski, V. Gutlyanski, V. Ryazanov: *On integral conditions for the general Beltrami equations*, Complex Analysis and Operator Theory **5** (2011), no.3, 835–845.

(Q18) T. V. Lomako: *On the theory of convergence and compactness for Beltrami equations*, Ukrainian mathematical journal **63** (2011), no.3, 393–402.

(Q19) T. Iwaniec, N.-T. Koh, L. Kovalev, J. Onninen: *Existence of energy-minimal diffeomorphisms between doubly connected domains*, Inventiones Mathematicae **186** no. 3 (2011), 667–707.

(Q20) S. K. Vodopyanov: *Mappings of finite codistortion and Sobolev classes of functions*, Doklady Mathematics, **84** no. 2 (2011), 640-644.

(Q21) L. Greco, C. Sbordone, R. Schiattarella : *Composition of bi-Sobolev homeomorphims*, Proc. Roy. Soc. Edinburgh Sect. A. **142** (2012), no. 1, 61-80.

(Q22) T. V. Lomako: *On the theory of convergence and compactness for Beltrami equations with constraints of set-theoretic type*, Ukrainian Mathematical Journal **63** (2012), no. 9, 1400–1414.

(Q23) T. Iwaniec, J. Onninen: *n-Harmonis mappings between annuli: The Art of Integrating Free Lagrangians* , Memoirs of the American Mathematical Society **218** (2012).

(Q24) Farroni F., Giova R.: *Quasiconformal mappings and sharp estimates for the distance to L^∞ in some function spaces*, J. Math. Anal. Appl. **395** (2012), no. 2, 694–704.

(Q25) T. Iwaniec, L. Kovalev, J. Onninen: *Hopf differentials and smoothing Sobolev homeomorphisms*, International Mathematics Research Notices **14** (2012), 3256–3277.

(Q26) V. Ryazanov, U. Srebro and E. Yakubov: *Integral conditions in the theory of the Beltrami equations*, Complex Variables and Elliptic Equations **57** (2012), no. 12, 1247–1270

(Q27) L. D'Onofrio, C. Sbordone, R. Schiattarella : *The Grand Sobolev Homeomorphisms and Their Measurability Properties*, Adv. Nonlinear Stud. **12** (2012), no. 4, 767-782.

(Q28) D. Henao, C. Mora-Corral: *Lusin's condition and the distributional determinant for deformations with finite energy*, Advances in Calculus of Variations **5** (2012), no. 4, 355-409.

(Q29) S. K. Vodopyanov: *Regularity of mappings inverse to Sobolev mappings*, Sbornik: Mathematics, **203** no. 10 (2012), 1383-1410.

(Q30) A. Passarelli di Napoli: *Bisobolev mappings and homeomorphisms with finite distortion*, Rend. Lincei Mat. Appl. **23** (2012), 437–454.

(Q31) L. D'onofrio, R. Schiattarella : *On the total variation for the inverse of a BV-homeomorphism*, Adv. Calc. Var. **6** (2013), no.3., 321–338.

(Q32) C. Capone, M. R. Formica, R. Giova and R. Schiattarella: *On the regularity theory of bi-Sobolev mappings*, Rend. Lincei Mat. Appl. **24** (2013), 527–548.

(Q33) L. D'onofrio, C. Sbordone ,R. Schiattarella : *Grand Sobolev spaces and their applications in geometric function theory and PDEs*, Journal of Fixed Point Theory and Applications **13** (2013), no.2., 309–340.

(b) Cited in non-impact factor journals and monographs

(Q34) G. Moscariello, C. Sbordone: *A note on weak convergence in $L^1_{\rm loc}(\mathbb{R})$*, J. Fixed point theory appl. **1** (2007), 337-350.

(Q35) C. Capozzoli, M. Carozza: *On Γ-Convergence of Quadratic Functionals in the Plane*, Ricerche Mat. **57** (2008), 283–300.

(Q36) L. Greco, C. Sbordone, C. Trombetti: *A note on planar homeomorphisms*, Rend. Accad. Sci. Fis. Mat. Napoli (4) **75** (2008), 53–59.

(Q37) K. Astala, T. Iwaniec, G. Martin: *Elliptic Partial Differential Equations and Quasiconformal mappings in the plane*, Princeton University Press, 2009.

(Q38) C. Capozzoli: *Sufficient conditions for integrability of distortion function $K_{f^{-1}}$*, Bolletino dell Unione Matematica Italiana **2** (2009), no. 3, 699–710.

(Q39) B. Bojarski, V. Gutlyanski, V. Ryazanov: *On the Beltrami equations with two characteristics*, Complex Variables and Elliptic Equations **54** (2009), no. 10, 935—-950

(Q40) V. Ryazanov, U. Srebro and E. Yakubov: *On strong solutions of the Beltrami equations*, Complex Variables and Elliptic Equations **55** (2010), no. 1-3, 219–236

(Q41) G. Moscariello, A. Passarelli di Napoli, C. Sbordone: *ACL-homeomorphisms in the plane*, Operator Theory: Advances and Applications **193** (2009), 215–225.

(Q42) V. Goldstein, A. Ukhlov: *Sobolev Homeomorphisms and Composition Operators*, International Mathematical Series **11**, Around the Research of Vladimir Maz'ya I (2009), Springer, 207-220.

(Q43) G. Moscariello, A. Passarelli di Napoli, C. Sbordone: *Planar ACL-homeomorphisms: critical points of their components*, Comm. Pure Appl. Anal. **9** (2010), 1391–1397.
(Q44) V. Gutlyanski, V. Ryazanov, U. Srebro, E. Yakubov: *On recent advances in the Beltrami equations*, Journal of Mathematical Sciences **175** (2011), no.4., 413–449.
(Q45) Y. Dybov: *On regular solutions of the Dirichlet problem for the Beltrami equations*, Complex Variables and Elliptic Equations **55** (2010), no. 12, 1099–1116.
(Q46) R. Schiattarella : *Composition of bi-Sobolev mappings*, Rend. Acc. Sc. fis. mat. Napoli **77** (2010), 7–14.
(Q47) V. Gutlyanski, T. Lomako, V. Ryazanov: *To the theory of variational method for Beltrami equations*, Journal of Mathematical Sciences **182** (2012), no.1., 37–54.
(Q48) V. Gutlyanski, V. Ryazanov, U. Srebro, E. Yakubov: *The Beltrami Equation A Geometric Approach*, Springer, Developments in Mathematics 26, 2012.
(Q49) J. R. Hussan, G. J. Martin: *Non-linear Models for the Deformation of Cellular Structures*, Journal of Modern Mathematics Frontier **2** (2013), no.2.
(Q50) Farroni F., Giova R., Sbordone C.: *Regularity points of ACL-homeomorphisms in the plane*, Contemporary Mathematics **594** (2013), 167–178.
(Q51) Capogna L.: *L^∞-Extremal Mappings in AMLE and Teichmüller Theory* in *Fully Nonlinear PDEs in Real and Complex Geometry*, Lecture Notes in Mathematics 2087, Springer (2014).

(C14): Stanislav Hencl, Pekka Koskela, Jan Malý: *Regularity of the inverse of a Sobolev homeomorphism in space*, Proc. Roy. Soc. Edinburgh Sect. A. **136A** no.6 (2006), 1267–1285. (IF=0.637)

(a) Cited in impact factor journals

(Q52) N. Fusco, G. Moscariello, C. Sbordone: *The limit of $W^{1,1}$ homeomorphisms with finite distortion*, Calc. Var., **12** no. 3 (2008), 377-390.
(Q53) S. K. Vodopyanov: *On regularity of Mappings Inverse to Sobolev Mappings*, Doklady Mathematics, **423** no. 5 (2008), 592-596.
(Q54) K. Astala, T. Iwaniec, G. Martin: *Deformations of Annuli with Smallest Mean Distortion*, Arch. Rational Mech. Anal., **195** (2010), no. 3, 899-921.
(Q55) D. Henao, C. Mora-Corral: *Fracture surfaces and the regularity of inverses for BV deformations*, Arch. Rational Mech. Anal. **201** (2011), 575—629.
(Q56) T. Rajala, A. Zapadinskaya, T. Zürcher : *Generalized Hausdorff dimension distortion in euclidean spaces under Sobolev mappings*, J. Math. Anal. Appl. **384** no. 2 (2011), 468–477.
(Q57) T. Rajala, A. Zapadinskaya, T. Zürcher : *Generalized dimension distortion under mappings of sub-exponentially integrable distortion*, Ann. Acad. Sci. Fenn. Math. **36** (2011), 553–566
(Q58) S. K. Vodopyanov: *Mappings of finite codistortion and Sobolev classes of functions*, Doklady Mathematics, **84** no. 2 (2011), 640-644.
(Q59) L. Greco, C. Sbordone, R. Schiattarella : *Composition of bi-Sobolev homeomorphims*, Proc. Roy. Soc. Edinburgh Sect. A. **142** (2012), no. 1, 61-80.
(Q60) K. Rajala: *Quantitative isoperimetric inequalities and homeomorphisms with finite distortion*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **11** (2012), no.1, 177-196.
(Q61) Farroni F., Giova R.: *Quasiconformal mappings and sharp estimates for the distance to L^∞ in some function spaces*, J. Math. Anal. Appl. **395** (2012), no. 2, 694–704.
(Q62) D. Henao, C. Mora-Corral: *Lusin's condition and the distributional determinant for deformations with finite energy*, Advances in Calculus of Variations **5** (2012), no. 4, 355-409.
(Q63) S. K. Vodopyanov: *Regularity of mappings inverse to Sobolev mappings*, Sbornik: Mathematics, **203** no. 10 (2012), 1383-1410.
(Q64) A. Passarelli di Napoli: *Bisobolev mappings and homeomorphisms with finite distortion*, Rend. Lincei Mat. Appl. **23** (2012), 437–454.
(Q65) L. D'onofrio, R. Schiattarella : *On the total variation for the inverse of a BV-homeomorphism*, Adv. Calc. Var. **6** (2013), no.3., 321–338.

(Q66) K. Rajala, G. Moscariello, A. Passarelli di Napoli: *Mappings of finite distortion and asymmetry of domains*, Ann. Acad. Sci. Fenn. Math. **38** (2013), no. 1, 367–375.

(Q67) A. Lorent: *On functions whose symmetric part of gradient agree and a generalization of Reshetnyak's compactness theorem*, Calc. Var. and PDE **48** (2013), no 3-4., 625–665.

(Q68) L. D'onofrio, C. Sbordone ,R. Schiattarella : *Grand Sobolev spaces and their applications in geometric function theory and PDEs*, Journal of Fixed Point Theory and Applications **13** (2013), no.2., 309–340.

(b) Cited in non-impact factor journals

(Q69) L. Greco, C. Sbordone, C. Trombetti: *A note on planar homeomorphisms*, Rend. Accad. Sci. Fis. Mat. Napoli (4) **75** (2008), 53–59.

(Q70) A. Ukhlov: *Sobolev homeomorphisms and H^1-rectifiable curves on Carnot groups*, Far East Journal of Mathematical Sciences, **33** no. 2 (2009), 151-166.

(Q71) C. Capozzoli: *Sufficient conditions for integrability of distortion function $K_{f^{-1}}$*, Bolletino dell Unione Matematica Italiana **2** (2009), no. 3, 699–710.

(Q72) V. Goldstein, A. Ukhlov: *Sobolev Homeomorphisms and Composition Operators*, International Mathematical Series **11**, Around the Research of Vladimir Maz'ya I (2009), Springer, 207-220.

(Q73) V. Goldstein, A. Ukhlov: *About homeomorphisms that induce composition operators on Sobolev spaces*, Complex Variables and Elliptic Equations **55** (2010), no. 8-10, 833-845.

(Q74) R. Schiattarella : *Composition of bi-Sobolev mappings*, Rend. Acc. Sc. fis. mat. Napoli **77** (2010), 7–14.

(Q75) C. Sbordone, R. Schiattarella : *Critical points for Sobolev homeomorphisms*, Rendiconti lincei-matematica e applicazioni **22** (2011), no.2., 207–222.

(Q76) Farroni F., Giova R., Sbordone C.: *Regularity points of ACL-homeomorphisms in the plane*, Contemporary Mathematics **594** (2013), 167–178.

(C19): Stanislav Hencl, G. Moscariello, A. Passarelli di Napoli and C. Sbordone: *Bi-Sobolev mappings and elliptic equations in the plane*, J. Math. Anal. Appl. **355** (2009), 22–32. (IF=1.050)

(a) Cited in impact factor journals

(Q77) T. Iwaniec, L. Kovalev, J. Onninen: *Diffeomorphic approximation of Sobolev homeomorphisms*, Arch. Rational Mech. Anal. **201** (2011), no. 3, 1047–1067.

(Q78) L. D'onofrio, R. Schiattarella : *On the total variation for the inverse of a BV-homeomorphism*, Adv. Calc. Var. **6** (2013), no.3., 321–338.

(Q79) F. Farroni, R. Giova: *The distance to L^∞ in the grand Orlicz spaces*, Journal of Function Spaces and its Applications, Article ID 658527 (2013), 7 pages.

(Q80) C. Capone, M. R. Formica, R. Giova and R. Schiattarella: *On the regularity theory of bi-Sobolev mappings*, Rend. Lincei Mat. Appl. **24** (2013), 527–548.

(b) Cited in non-impact factor journals

(Q81) C. Capozzoli: *Sufficient conditions for integrability of distortion function $K_{f^{-1}}$*, Bolletino dell Unione Matematica Italiana **2** (2009), no. 3, 699–710.

(Q82) F. Farroni, R. Giova, T. Ricciardi: *Best constants and extremals for a vector Poincaré inequality with weights*, Scientiae Mathematicae Japonicae online, e-2010, 53–68.

(Q83) R. Schiattarella : *Composition of bi-Sobolev mappings*, Rend. Acc. Sc. fis. mat. Napoli **77** (2010), 7–14.

(C28): Daniel Campbell, Stanislav Hencl: *A note on mappings of finite distortion: Examples for the sharp modulus of continuity*, Ann. Acad. Sci. Fenn. Math. **36** (2011), 531–536. (IF=0.602)

(a) Cited in impact factor journals

(Q84) T. Ricciardi, G. Zecca: *On the continuity of solutions to degenerate elliptic equations in two dimensions*, Potential Analysis **37** (2012), no. 2, 171–185.

(Q85) P. Koskela: *Planar Mappings of Finite Distortion* , Computational methods and function theory **10** (2010), no. 2, 663–678.

(C21): Marianna Csörnyei, Stanislav Hencl, Jan Malý: *Homeomorphisms in the Sobolev space* $W^{1,n-1}$, J. Reine Angew. Math. **644** (2010), 221–235. (IF=1.083)

(a) Cited in impact factor journals

(Q86) N. Fusco, G. Moscariello, C. Sbordone: *The limit of* $W^{1,1}$ *homeomorphisms with finite distortion*, Calc. Var., **12** no. 3 (2008), 377-390.

(Q87) S. K. Vodopyanov: *On regularity of Mappings Inverse to Sobolev Mappings*, Doklady Mathematics, **423** no. 5 (2008), 592-596.

(Q88) T. Iwaniec, J. Onninen: *Deformations of finite conformal energy: Existence, and Removability of Singularities*, Proc. London Math. Soc. **100** (2010), 1–23.

(Q89) F. Giannetti, A. Passarelli di Napoli: *Bi-Sobolev mappings with differential matrices in Orlicz Zygmund classes*, J. Math. Anal. Appl. **369** (2010), no. 1, 346-356

(Q90) D. Henao, C. Mora-Corral: *Fracture surfaces and the regularity of inverses for BV deformations*, Arch. Rational Mech. Anal. **201** (2011), 575—629.

(Q91) T. Iwaniec, J. Onninen: *Deformations of finite conformal energy: boundary behavior and limit theorems*, Trans. Amer. Math. Soc. **363** (2011), no. 11, 5605–5648.

(Q92) P. di Gironimo, L. D'onofrio, C. Sbordone, R. Schiattarella : *Anisotropic Sobolev homeomorhisms*, Ann. Acad. Sci. Fenn. Math. **36** (2011), 593–602.

(Q93) S. K. Vodopyanov: *Mappings of finite codistortion and Sobolev classes of functions*, Doklady Mathematics, **84** no. 2 (2011), 640-644.

(Q94) L. Greco, C. Sbordone, R. Schiattarella : *Composition of bi-Sobolev homeomorphims*, Proc. Roy. Soc. Edinburgh Sect. A. **142** (2012), no. 1, 61-80.

(Q95) T. Iwaniec, J. Onninen: *n-Harmonis mappings between annuli: The Art of Integrating Free Lagrangians* , Memoirs of the American Mathematical Society **218** (2012).

(Q96) L. D'Onofrio, C. Sbordone, R. Schiattarella : *The Grand Sobolev Homeomorphisms and Their Measurability Properties*, Adv. Nonlinear Stud. **12** (2012), no. 4, 767-782.

(Q97) D. Henao, C. Mora-Corral: *Lusin's condition and the distributional determinant for deformations with finite energy*, Advances in Calculus of Variations **5** (2012), no. 4, 355-409.

(Q98) S. K. Vodopyanov: *Regularity of mappings inverse to Sobolev mappings*, Sbornik: Mathematics **203** no. 10 (2012), 1383-1410.

(Q99) G. Moscariello, A. Passarelli di Napoli, K. Rajala: *Mappings of finite distortion and asymmetry of domains*, Ann. Acad. Sci. Fenn. Math. **38** no. 1 (2013), 367-375.

(Q100) L. D'onofrio, R. Schiattarella : *On the total variation for the inverse of a BV-homeomorphism*, Adv. Calc. Var. **6** (2013), no.3., 321–338.

(Q101) A. Lorent: *On functions whose symmetric part of gradient agree and a generalization of Reshetnyak's compactness theorem*, Calc. Var. and PDE **48** (2013), no 3-4., 625–665.

(Q102) L. D'onofrio, C. Sbordone ,R. Schiattarella : *Grand Sobolev spaces and their applications in geometric function theory and PDEs*, Journal of Fixed Point Theory and Applications **13** (2013), no.2., 309–340.

(b) Cited in non-impact factor journals and monographs

(Q103) C. Capozzoli, M. Carozza: *On* Γ*-Convergence of Quadratic Functionals in the Plane*, Ricerche Mat. **57** (2008), 283–300.

(Q104) Martio, O.; Ryazanov, V.; Srebro, U.; Yakubov, E.: *Moduli in Modern Mapping Theory*, Springer, 2009.

(Q105) A. Ukhlov: *Sobolev homeomorphisms and* H^1*-rectifiable curves on Carnot groups*, Far East Journal of Mathematical Sciences, **33** no. 2 (2009), 151-166.

(Q106) G. Moscariello, A. Passarelli di Napoli, C. Sbordone: *ACL-homeomorphisms in the plane*, Operator Theory: Advances and Applications **193** (2009), 215–225.

(Q107) G. Moscariello, A. Passarelli di Napoli, C. Sbordone: *Planar ACL-homeomorphisms: critical points of their components*, Comm. Pure Appl. Anal. **9** (2010), 1391–1397.

(Q108) V. Goldstein, A. Ukhlov: *About homeomorphisms that induce composition operators on Sobolev spaces*, Complex Variables and Elliptic Equations **55** (2010), no. 8-10, 833-845.

(Q109) Farroni F., Giova R., Sbordone C.: *Regularity points of ACL-homeomorphisms in the plane*, Contemporary Mathematics **594** (2013), 167–178.

(Q110) Capogna L.: *L^∞-Extremal Mappings in AMLE and Teichmüller Theory* in *Fully Nonlinear PDEs in Real and Complex Geometry*, Lecture Notes in Mathematics 2087, Springer (2014).

(Q111) V. I. Ryazanov, R. R. Salimov, E. A. Sevostyanov: *On convergence analysis of space homeomorphisms*, Siberian Advances in Mathematics **23** (2013), no.4., 263–293.

(C22): Stanislav Hencl, Jan Malý: *Jacobians of Sobolev homeomorphisms*, Calc. Var. Partial Differential Equations **38** (2010), 233—242. (IF=1.236)

(a) Cited in impact factor journals

(Q112) P. di Gironimo, L. D'onofrio, C. Sbordone, R. Schiattarella : *Anisotropic Sobolev homeomorhisms*, Ann. Acad. Sci. Fenn. Math. **36** (2011), 593–602.

(Q113) T. Rajala, A. Zapadinskaya, T. Zürcher : *Generalized Hausdorff dimension distortion in euclidean spaces under Sobolev mappings*, J. Math. Anal. Appl. **384** no. 2 (2011), 468–477.

(Q114) T. Iwaniec, N.-T. Koh, L. Kovalev, J. Onninen: *Existence of energy-minimal diffeomorphisms between doubly connected domains*, Inventiones Mathematicae **186** no. 3 (2011), 667–707.

(Q115) L. D'Onofrio, C. Sbordone, R. Schiattarella : *The Grand Sobolev Homeomorphisms and Their Measurability Properties*, Adv. Nonlinear Stud. **12** (2012), no. 4, 767-782.

(Q116) L. D'onofrio, R. Schiattarella : *On the total variation for the inverse of a BV-homeomorphism*, Adv. Calc. Var. **6** (2013), no.3., 321–338.

(b) Cited in non-impact factor journals

(Q117) R. Schiattarella : *Composition of bi-Sobolev mappings*, Rend. Acc. Sc. fis. mat. Napoli **77** (2010), 7–14

(Q118) C. Sbordone, R. Schiattarella : *Critical points for Sobolev homeomorphisms*, Rendiconti lincei-matematica e applicazioni **22** (2011), no.2., 207–222.

(Q119) Farroni F., Giova R., Sbordone C.: *Regularity points of ACL-homeomorphisms in the plane*, Contemporary Mathematics **594** (2013), 167–178.

Bibliography

[1] K. Astala, T. Iwaniec and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton Mathematical Series, 48. Princeton University Press, Princeton, NJ, 2009.

[2] J. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1978), 337–403.

[3] J. Ball, *Global invertibility of Sobolev functions and the interpenetration of matter*, Proc. Roy. Soc. Edinburgh Sect. A **88** (1981), 315–328.

[4] L. Cesari, *Sulle transformazioni continue*, Ann. Math. Pura Appl. **21** (1942), 157–188.

[5] D. Campbell and S. Hencl, *A note on mappings of finite distortion: Examples for the sharp modulus of continuity*, Ann. Acad. Sci. Fenn. Math. **36** (2011), 531–536.

[6] A.V. Chernavskii, *Finite to one open mappings of manifolds*, Mat. Sb. **65** (1964), 357–369.

[7] M. Csörnyei, S. Hencl and J. Malý, *Homeomorphisms in the Sobolev space $W^{1,n-1}$*, J. Reine Angew. Math **644** (2010), 221–235.

[8] L. D'onofrio, S. Hencl and R. Schiattarella, *Bi-Sobolev homeomorphism with zero Jacobian almost everywhere*, to appear in Calc. Var. Partial Differential Equations.

[9] D. Faraco, P. Koskela and X. Zhong, *Mappings of finite distortion: The degree of regularity*, Adv. Math. **90** (2005), 300–318.

[10] L. Greco, *A remark on the equality* $\det Df = \operatorname{Det} Df$, Diff. Integral Equations **6** (1993), 1089–1100.

[11] S. Hencl, *Sobolev homeomorphism with zero jacobian almost everywhere*, J. Math. Pures Appl. **95** (2011), 444–458.

[12] S. Hencl and P. Koskela, *Regularity of the inverse of a planar Sobolev homeomorphism*, Arch. Rational Mech. Anal **180** (2006), 75–95.

[13] S. Hencl and P. Koskela, *Mappings of finite distortion: Discreteness and openness for quasi-light mappings*, Ann. Inst. H. Poincaré Anal. Non Linéaire. **22** (2005), 331–342.

[14] S. Hencl, P. Koskela and J. Onninen, *Homeomorphisms of bounded variation*, Arch. Rational Mech. Anal **186** (2007), 351–360.

[15] S. Hencl and J. Malý, *Jacobians of Sobolev homeomorphisms*, Calc. Var. Partial Differential Equations **38** (2010), 233—242.

[16] S. Hencl and J. Malý, *Mappings of finite distortion: Hausdorff measure of zero sets*, Math. Ann. **324** (2002), 451–464.

[17] S. Hencl, P. Koskela and J. Malý, *Regularity of the inverse of a Sobolev homeomorphism in space*, Proc. Roy. Soc. Edinburgh Sect. A **136** (2006), no. 6, 1267–1285.

[18] S. Hencl, G. Moscariello, A. Passarelli di Napoli and C. Sbordone, *Bi-Sobolev mappings and elliptic equations in the plane*, J. Math. Anal. Appl. **355** (2009), 22–32.

[19] S. Hencl and K. Rajala, *Optimal assumptions for discreteness*, to appear in Arch. Rational Mech. Anal **207** (2013), no. 3, 775–783.

[20] T. Iwaniec, P. Koskela and J. Onninen, *Mappings of finite distortion: Monotonicity and continuity*, Invent. Math. **144** (2001), 507–531.

[21] T. Iwaniec and G. Martin, *Geometric function theory and nonlinear analysis*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 2001.

[22] T. Iwaniec and C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Rational Mech. Anal. **119** (1992), 129–143.

[23] T. Iwaniec and V. Šverák, *On mappings with integrable dilatation*, Proc. Amer. Math. Soc. **118** (1993), 181–188.

[24] J. Kauhanen, P. Koskela and J. Malý, *Mappings of finite distortion: Condition N*, Michigan Math. J. **49** (2001), 169–181.

[25] J. Kauhanen, P. Koskela and J. Malý, *Mappings of finite distortion: Discreteness and openness*, Arch. Ration. Mech. Anal. **160** (2001), 135–151.

[26] L. Kleprlík, *The zero set of the Jacobian and composition of mappings*, J. Math. Anal. Appl. **386** (2012), 870–881.

[27] P. Koskela, *Lectures on Quasiconformal and Quasisymmetric Mappings*, University of Jyväskylä, to appear.

[28] P. Koskela and J. Malý, *Mappings of finite distortion: the zero set of the Jacobian*, J. Eur. Math. Soc. **5** (2003), 95–105.

[29] J. Malý and O. Martio, *Lusin's condition (N) and mappings of the class $W^{1,n}$*, J. Reine Angew. Math. **458** (1995), 19–36.

[30] J. Manfredi, *Weakly Monotone functions*, J. Geom. Anal. **4** (1994), 393—402.

[31] J. Manfredi and E. Villamor, *An extension of Reshetnyak's theorem*, Indiana Univ. Math. J. **47** (1998), 1131–1145.

[32] S. Müller, *Higher integrability of determinants and weak convergence in L^1*, J. Reine Angew. Math. **412** (1990), 20–34.

[33] J. Onninen and X. Zhong, *Mappings of finite distortion: a new proof for discreteness and openness*, Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), 1097–1102.

[34] S. Ponomarev, *Examples of homeomorphisms in the class $ACTL^p$ which do not satisfy the absolute continuity condition of Banach* (Russian), Dokl. Akad. Nauk USSR **201** (1971), 1053-1054.
[35] Yu. G. Reshetnyak, *Space Mappings with Bounded Distortion*, Trans. of Mathematical Monographs, Amer. Math. Soc, vol. 73, 1989.
[36] S. Rickman, *Quasiregular Mappings*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 26. Springer-Verlag, Berlin, 1993.